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CONSTRUCTION OF TIME-RELAXED MINIMAL BROADCAST NETWORKS

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ABSTRACT

In broadcasting, or one-to-all communication, a message originally held in one node of the network must be transmitted to all the other nodes. A minimal broadcast network is a communication network that can transmit a message originated at any node to all other nodes of the network in minimum time. In this paper, we present a compound method to construct sparse, time-relaxed, minimal broadcast networks (t-mbn), in which broadcasting can be accomplished in slightly more than the minimum time. The proposed method generates a new network by connecting a subset of nodes from several copies of a t_1-mbn using the structure of another t_2-mbn. The objective is to construct a network as sparse as possible satisfying the desired broadcasting time constraint. Computational results illustrate the effectiveness of the proposed method.

1. Introduction

A communication network can be modeled as a connected graph $G = (V, E)$ without loops or parallel edges, consisting of a set of nodes $V = V(G)$ with cardinality $v(G) = |V|$, and a set of undirected edges $E = E(G)$ with cardinality $e(G) = |E|$.

In communication networks, broadcasting is a special type of information transmission in which a single message, originated at a node of the network, must be transmitted to all the other nodes. Broadcasting is usually required to be completed as rapidly as possible by a sequence of transmissions through the communication lines. It is assumed that broadcasting is carried out under the following three constraints [7,8]: (i) each transmission requires one time unit, (ii) a node can make at most one transmission in one time unit, and (iii) a node can only transmit the message to its adjacent nodes. Thus, in one time unit, the number of informed nodes can at most be doubled. This implies that after m time units the number of nodes that have received the message is bounded by 2^m, including the originator. As a result, the minimum number of time units required to broadcast a message in a network G is $\lceil \lg v(G) \rceil$. A *minimal broadcast network* (mbn) is defined to be a communication network in which a message, originated at any node, can be broadcast in minimum time.

Let the *receiving time* of a node be the time at which the node receives the message. Suppose that a node u in a network G is the originator of a message. Let $b(u, G)$ be the minimum time required to broadcast a message from node u to all other nodes of G. A *broadcast protocol* $P(u, G)$ is a rooted spanning tree in which the originator u is the root and all the nodes are labeled by their receiving times, all of which are at most $b(u, G)$. In a broadcast protocol, each edge is used exactly once and the message is always transmitted from parent to child, with the child's label being greater than the parent's and different from any of its siblings'. Let $b(G) = \max\{b(u, G) \mid u \in V(G)\}$ be the broadcast time of G. Thus, a network G is an mbn if and only if $b(G) = \lceil \lg v(G) \rceil$. The problem of recognizing whether an arbitrary network is an mbn is $\mathcal{NP}$-complete [7]. An *optimal broadcast network* (obn) is defined as an mbn with the minimum number of edges, and the broadcast function $B(n)$ is defined to be the number of edges of an obn with n nodes. There is no known efficient method for determining $B(n)$ for an arbitrary value of n. Farley et al. showed that hypercubes are obn's, i.e. $B(2^m) = m(2^{m-1})$, for $m \geq 1$ [8]. Khachatrian and Harutounian [11] and Dinneen et al. [5] proved independently that $B(2^m - 2) = (m - 1)(2^{m-1} - 1)$, for $m \geq 2$. Farley et al. also determined the values of $B(n)$, for $1 \leq n \leq 15$. Recently, Bermond et al. [1], Weng and Ventura [15] and Dinneen et al. [6] have reported known values of $B(n)$, for n up to 127. Figure 1 shows the Heawood graph, an example of an obn with 14 nodes, together with one of its broadcast protocols. Since this obn is vertex symmetric, all the nodes have similar originating broadcast protocols.

In 1979 Farley proposed a recursive algorithm to construct sparse mbn's with an arbitrary number of nodes [7]. Since then, several additional algorithms have been developed [3,9,4,10,2,14]. These algorithms use different strategies to produce larger mbn's by combining small known obn's or mbn's using as few edges as possible without violating the broadcasting time constraint. Recently, a new class of methods using graph compounding has been developed. The compound of a graph G into another graph H relative to a set $S \subseteq V(G)$, denoted by $G_S[H]$, is the graph

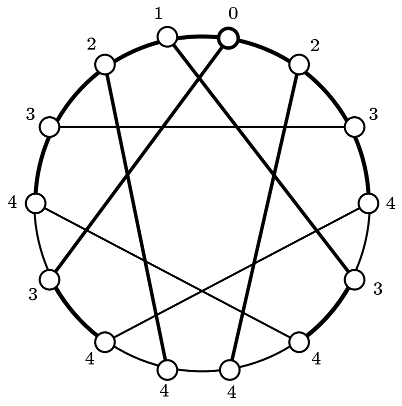
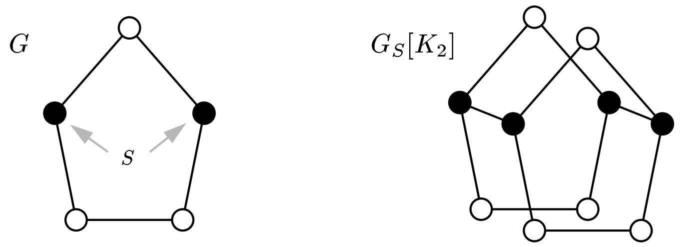


Figure 1: Obn with 14 nodes and its broadcast protocol.

obtained by replacing each node of H with a copy of G, and connecting together the $v(H)$ copies of each node $s \in S$ to form a copy of H. Here $V(G_S[H]) = V(G) \times V(H)$ and $E(G_S[H]) = \{(g_1h_1, g_2h_2) \mid (h_1{=}h_2 \text{ and } (g_1, g_2) \in E(G)) \text{ or } (g_1{=}g_2 \in S \text{ and } (h_1, h_2) \in E(H))\}$. Figure 2 illustrates the compound method by generating an obn with 10 nodes from an obn $G = C_5$ into $H = K_2$ (with respect to the black nodes S). Compound methods have been studied by Khachatrian and Harutounian [11] and Bermond et al. [1]. Weng and Ventura [15] and Dinneen et al. [6] have developed generalized compounding algorithms which allow for node deletion (i.e., the vertex set of the compound is a subset of $V(G) \times V(H)$).



(a) Obn with 5 nodes. (b) Obn with 10 nodes.

Figure 2: Constructing an obn with 10 nodes by compounding.

Shastri studied the problem of constructing the sparsest possible networks in which broadcasting can be accomplished in slightly more than $\lceil \lg v(G) \rceil$ time. A *t-relaxed minimal broadcast network* (t-mbn) G is a network in which broadcasting can be accomplished in $\lceil \lg v(G) \rceil + t$ time units from any node [12,13]. Define $B_t(n)$ to be the minimum number of edges of a network with n nodes in which broadcasting can be accomplished in $\lceil \lg n \rceil + t$ time units. A *t-relaxed optimal broadcast network* (t-obn) with n nodes is a t-mbn with $B_t(n)$ edges. Obviously, a 0-mbn is an mbn and a 0-obn is an obn by these definitions.

Shastri presented several elementary constructions of t-mbn's for $t \leq 4$, leading to upper bounds on $B_t(n)$ for $n \leq 65$. The main objective of this paper is to extend and utilize the compound methods in [15] and [6] to construct the sparsest known t-mbn's. The concepts of official broadcasting and center node sets defined in these two papers is also adapted to the new context. We present computational results that provide *all* of the best-known upper bounds on $B_t(n)$, for $1 \leq t \leq 3$ and $n \leq 128$. The proofs of our generated bounds are concisely specified by a table of graph compounds.

2. Official Broadcasting and Center Node Sets

The compound methods proposed in [15] and [6] are based on the concept of official broadcasting with respect to a center node set. This concept is not a new type of broadcasting, the only purpose for introducing official broadcasting is to clearly define center node sets which are utilized in network compounding. In official broadcasting, it is assumed that a message, originated at any node, must be officialized by a certain type of node in the network, and the official message must be transmitted to all the nodes of the network. Nodes that possess the ability to officialize a message are called *center nodes*. A transmitted message is *official* if it has been officialized by a center node; otherwise, it is *unofficial*. It is assumed that an unofficial message becomes official immediately after it arrives at a center node. In official broadcasting, where all the nodes must receive an official message, a center node will only receive one message, so that if the incoming message is unofficial, it will be officialized immediately after its arrival. A non-center node may receive one or two messages. In the first case, the message must be official. In the second case, the first message must be unofficial and the second one official. In addition, in official broadcasting, it is possible for a non-center node to send an unofficial message to a neighbor and receive an official message during the same time unit. This is allowed in the broadcast model since within the underlying broadcast protocol a node will only receive one message from one of its neighbors and then transmit the message to some of its other neighbors. Therefore, any subsequent simultaneous reception and transmission of messages in the official protocol do not violate the constraints of broadcasting. For official broadcasting in a network G, let $b(u, G; S)$ be the minimum time required to broadcast a message from node u with respect to a center node set $S \subseteq V(G)$ and $b(G; S) = \max\{b(u, G; S) \mid u \in V(G)\}$ be the *official broadcast time* of G with respect to S.

An *official broadcast protocol* for a node u with respect to a set $S \subseteq V(G)$ in a network G, denoted by $P(G, u; S)$, is a connected spanning subnetwork of G, in which the nodes are labeled by one or two receiving times, all of which are at most $b(G; S)$. If a node has two receiving times, it must not belong to S; the first receiving time is for the unofficial message and the second one for the official message. If $u \in S$, each node is labeled by one single receiving time, and the official broadcast protocol is simply a spanning-tree broadcast protocol rooted at u.

Given a network G with n nodes, official broadcasting with respect to a set

$S \subseteq V(G)$ can be completed in $b(G;S)$ time units regardless of the originator, where $b(G;S) \geq b(G)$. The set S is called a *t-relaxed center node set* (t-cns) of G, where $t = b(G;S) - \lceil \lg n \rceil$. In addition, if S has the smallest cardinality among all the t-cns's of G, then S is called a *t-relaxed optimal center node set* (t-ocns) of G. If $b(G;S) = b(G)$, then we simply call S a *center node set* (cns) of G. There may exist multiple t-ocns's for a given network, and the cardinality of the t-ocns's of non-isomorphic networks with n nodes and m edges with the same official broadcast time may be different. There is no known polynomial-time algorithm for computing (the cardinality of) a t-ocns for an arbitrary network G. Figure 3 shows the three different official broadcast protocols for a 1-obn with 5 nodes, where the two black nodes define a 1-ocns (ocns). Figure 4 shows official broadcast protocols with the 1-ocns of Figure 3 replaced with a single node 2-ocns.

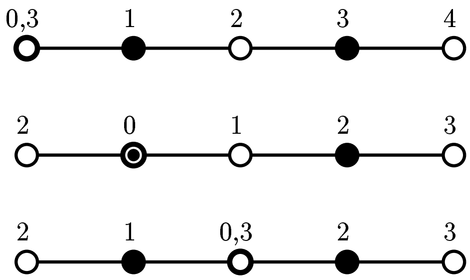


Figure 3: Official protocols for a 5 node 1-obn w.r.t. a 1-ocns.

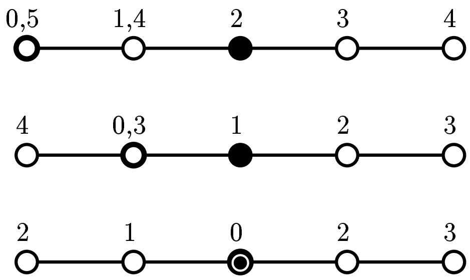


Figure 4: Official protocols for a 5 node 1-obn w.r.t. a 2-ocns.

3. The Generalized Compound Method

The proposed method for constructing t-mbn's includes two steps:

i. Let G be a network and fix a t_G-cns $S \subseteq V(G)$. If $S \subset V(G)$ choose a node $v \in V(G) \setminus S$ of minimum degree; otherwise, choose $v \in S$ of minimum degree. Construct a network G^v by deleting v and all its incident edges from G, and adding the required edges to form a clique among the neighbors of v. The subset of nodes $S^v = S \setminus \{v\}$ is a center node set for G^v such that $b(G^v, S^v) \leq b(G;S)$ (see [15]).

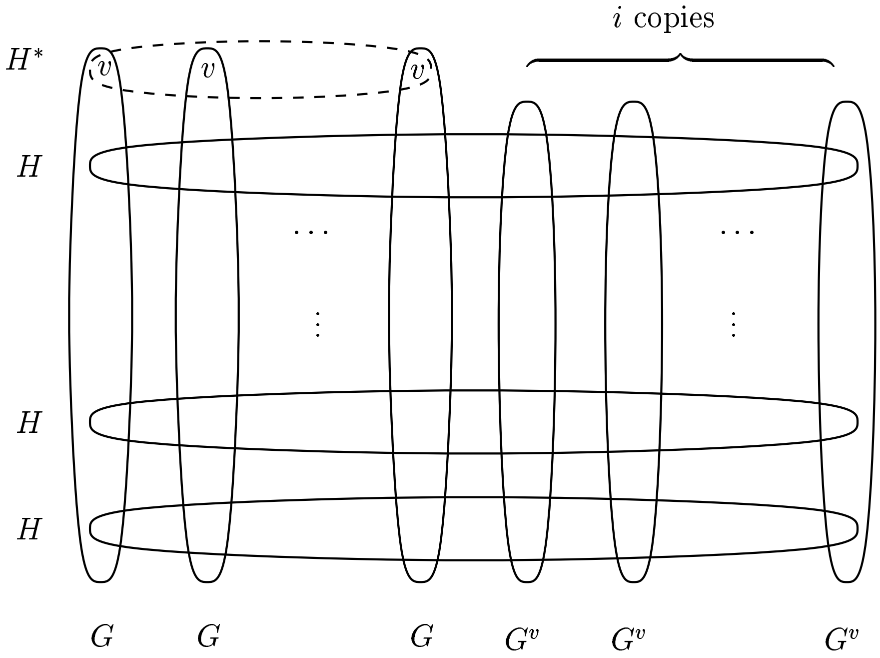


Figure 5: Illustrating the generalized compound method.

ii. Let H be a network and pick a t_H-cns $T \subseteq V(H)$. For a fixed integer i with $0 \leq i \leq v(H) - 1$, construct a network $\mathcal{G}$ by connecting $v(H) - i$ copies of G and i copies of G^v as follows. For each fixed $s \in S^v$, connect all $v(H)$ copies of s to form a copy H_s of H with corresponding copy T_s of T. If $v \in S$, the $v(H) - i$ copies of v are connected to form a network H^* with $v(H) - i$ nodes and $b(H^*) \leq b(H)$; otherwise, let H^* be an edgeless network. If $v \in S$ choose $U \subseteq V(H^*)$ so that $b(H^*; U) \leq b(H; T)$; otherwise, let $U = \emptyset$. Finally, define $\hat{\mathcal{S}} = \bigcup_{s \in S^v} T_s \cup U$ as the $t_{\mathcal{G}}$-cns.

The structure of the network $\mathcal{G}$ generated by the generalized compound method is illustrated in Figure 5. A concrete example is shown in Figure 6 with $G = C_5$, $H = P_3$ and $i = 1$. Note that given an official protocol P for G with respect to S, there is an appropriate modification P^v of P which is official for G^v with respect to S^v.

Theorem 1 The network $\mathcal{G}$ generated by the generalized compound method satisfies $b(\mathcal{G}) \leq b(G; S) + b(H)$ and $b(\mathcal{G}; \hat{\mathcal{S}}) \leq b(G; S) + b(H; T)$.

PROOF. Let x be a node of $\mathcal{G}$. We consider several cases depending on the location of x. First we recall some notation from [6], for which we refer for the definitions. For a given official broadcast protocol P for G, P_u, P_o denote respectively the unofficial and official parts of P, while $V_{cu}(P)$ denotes the set of center nodes which receive

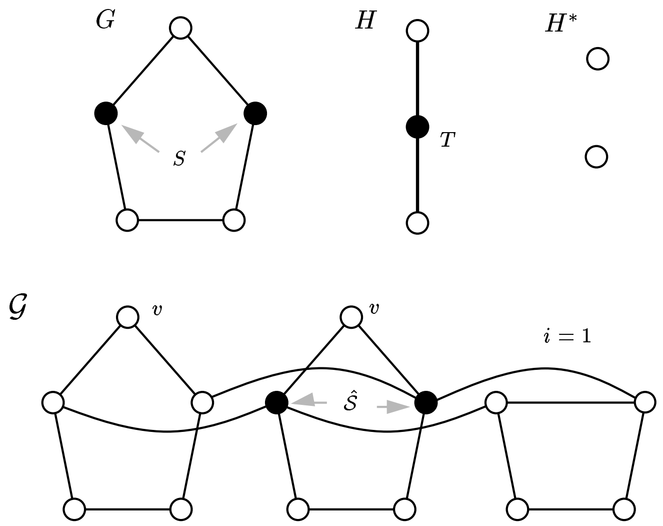


Figure 6: An example of a generalized compound.

an unofficial message in P.

First suppose that x belongs to some H_s. First broadcast in H_s which takes $b(H)$ time units. In the next $b(G)$ time units, each node in H_s broadcasts vertically inside its copy of G or G^v. By this time, all nodes in $\mathcal{G}$ are informed, so $b(G)+b(H)$ time units suffice for ordinary broadcasting. For official broadcasting, we need only ensure that the initial broadcasting in H_s is official with respect to T_s, which takes $b(H;T)$ time units, so $b(G) + b(H;T)$ time units suffice in this case.

Next suppose that x is a node of H^*. Then all nodes of G are center nodes and so all nodes in $\mathcal{G}$ belong to H^* or some H_s. Also x belongs to some copy G_i of G. First broadcast vertically in G_i, which takes $b(G) = b(G;S)$ time units. Now broadcast horizontally in the appropriate H_s or H^*. This can be done in $b(H)$ time units and officially in $b(H;T)$ time units and so all nodes are informed by time $b(G;S) + b(H)$ and officially informed by time $b(G;S) + b(H;T)$.

Suppose now that x is a node of some copy G_i of G but not a node of any H_s. Broadcast according to the unofficial part P_u of some official protocol P, ending at $V_{cu}(P)$. Each center node in $V_{cu}(P)$ broadcasts horizontally in its corresponding copy of H, taking $b(H)$ time units for ordinary broadcasting and $b(H;T)$ for official broadcasting. Once this is over, each copy w of a node in $V_{cu}(P)$ continues according to P_o or $(P^v)_o$ according as w belongs to a copy of G or G^v. Since it takes $b(G;S)$ time units to perform official broadcasting in G according to P_o and P_u, all nodes of $\mathcal{G}$ are informed by time $b(G;S) + b(H)$ and officially informed by time $b(G;S) + b(H;T)$.

Finally, suppose that x is a node of some G^v and not a node of any H_s. Note that in this case v is not a center node of G. Let P be an official protocol for x in G. Broadcast in G^v according to P^v, ending in $V_{cu}(P)$. Let Q be the subprotocol followed so far. Let V_1 (respectively V_2) be the subset of $V_{cu}(P)$ consisting of nodes which have (respectively do not have) v as an ancestor. For nodes in V_2 then, we have so far followed only $(P^v)_u$, and this is precisely P_u for such nodes. However, if $w \in V_1$ then w may have descendants in $V_{cu}(P)$ which are not in $V_{cu}(P^v)$. Thus Q will include a part of $(P^v)_o$ for nodes in V_1 and their descendants.

Now broadcast horizontally in the appropriate H_s, which takes $b(H)$ time units, or $b(H;T)$ if done officially. Each copy of each $w \in V_{cu}(P)$ then follows this with broadcasting until the end of P or P^v as appropriate. Now Q takes the same time as $(P^v)_u$ for nodes in V_2. This is the same as the time taken by P_u for such nodes. Also Q takes the same time as P_u for nodes in V_1. Hence Q takes the same time as P_u for all nodes in $V_{cu}(P)$. Since P^v takes at most $b(G,S)$ time units and P_u and P_o together also use this many time units, all nodes in $\mathcal{G}$ are informed by time $b(G;S) + b(H)$ and officially informed by time $b(G;S) + b(H;T)$.

In the above construction all official broadcast protocols are generated with respect to $\hat{\mathcal{S}}$ only. Thus, $\hat{\mathcal{S}}$ is a center node set of the resulting t-mbn $\mathcal{G}$. $\square$

The following hold for $\mathcal{G}$, where $d = \deg v$.

$$\begin{aligned} v(\mathcal{G}) &= v(G)v(H) - i \\ e(\mathcal{G}) &\leq v(H)e(G) + id(d-3)/2 + |S^v|e(H) + e(H^*) \\ |\hat{\mathcal{S}}| &= |S^v||T| + |U| \end{aligned}$$

Two examples of this compounding method are given in Figure 7, where $G = H$ and $i = 2$. The center node sets for G and H are taken from Figures 3 and 4. The left 3-mbn [right 4-mbn] of the figure is obtained by using the 1-ocns for G [for H] and the 2-ocns for H [for G]. The boxed nodes of these two compounds are both 4-cns's (i.e., the $\hat{\mathcal{S}}$ obtained by the compounding method). One generated official protocol, as dictated by the proof of Theorem 1, is also given for each case.

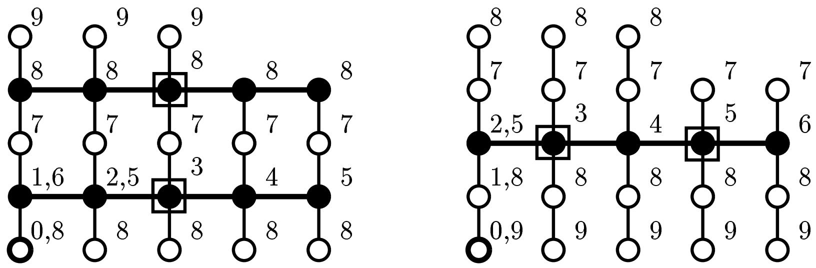


Figure 7: Two t-mbn's with 23 nodes obtained by compounding with different center node sets.

4. Computational Results

This section describes and justifies our computed bounds on $B_t(n)$, for $1 \leq t \leq 3$ and $n \leq 128$. For convenience we also include our bounds for $B_0(n) = B(n)$ that were presented in [6].

We have implemented the construction method given in Section 3. Our time-relaxed programs use the same input format that is discussed in [6]. That is, we do not actually store each t-mbn as a graph but only enough data (which is seven integer parameters) as a basis for obtaining edge and center node bounds of the compounds. For the $B_t(n)$ problem, an additional integer value t is kept for each input or generated t-mbn.

We now discuss our initial input data.

For $B_1(n)$, $B_2(n)$ and $B_3(n)$ we used the t-obn's (and t-cns's) of Figures 8, 9 and 10. Each optimal center node set of these t-obn's is displayed using black nodes. The exception being the two 3-obn's (which are also 2-obn's) of Figure 8. Here the two smaller center node sets are indicated by boxed nodes. To generate the $B_0(n)$ bounds we used a complete set of best-known mbn's, which is a superset of the obn's listed in the first column of Figure 8 and is completely documented in [6].

The tree t-obn's of Figures 9 and 10 are symmetric trees of odd diameter created by joining two roots of bounded-depth rooted maximal broadcast trees. We can verify that two center nodes are sufficient in these cases. Here any originating node needs at most 'depth'+1 time units to inform the two attached roots and then $\lceil \lg n/2 \rceil = \lceil \lg n \rceil - 1$ more time units to send the official message throughout the tree. Also one can easily see that at least two center nodes are required.

Table 2 at the end of this paper contains the compound constructions that yield the best-known upper bounds for $B_t(n)$. The entries in bold indicate when the bound $T_t(n)$ is the best possible. These bounds are achieved by the compound constructions that are specified under the columns labeled "compound". Each empty compound specification indicates that the bound was obtained from an input t-mbn. The notation $(v_G, v_H, i){:}t_G t_H$ represents how each t-mbn is constructed, with the node-deletion variable i being omitted if it is 0. Using the notation of Section 3, $v_G = v(G)$ and $v_H = v(H)$, where G is a t_G-mbn and H is a t_H-mbn. (Note $0 \le t_G, t_H \le 3$ for our computations.)

We now justify the optimal bounds of our main table. These proofs are based on the Shastri's broadcast tree bounds [12,13]. The smallest broadcast times $T(n)$ for various trees of order n up to 128 nodes are summarized in Table 1. (In this table we also added a column, |CNS|, that indicates the size of the smallest t-cns for the largest tree in each range.) By inspecting the column labeled t, we see that for instance $B_1(4), \ldots, B_1(6)$, $B_2(7)$, $B_2(8)$, $B_1(9)$ and $B_2(10), \ldots, B_2(14)$ are all optimal because there exist t-obn's which are trees. Since the minimum broadcast times are known for trees of these orders, we can conclude for each $n \ge 4$ that $T_{t-1}(n) \ge n$ where $t = T(n) - \lceil \lg n \rceil$. For example, $B_2(28) \ge 28$ since any tree with 28 nodes (27 edges) must be a t-mbn for $t \ge 3$; that is, the smallest broadcast time of any such tree is 8 while $\lceil \lg 28 \rceil = 5$ so the difference is $t = 3$. Besides the sharp $B_0(n)$ cases, these two simple observations show optimality for all but the five bold entries $B_1(13)$, $B_1(14)$, $B_1(19)$, $B_1(20)$ and $B_1(21)$, which are presented by Shastri in [13].

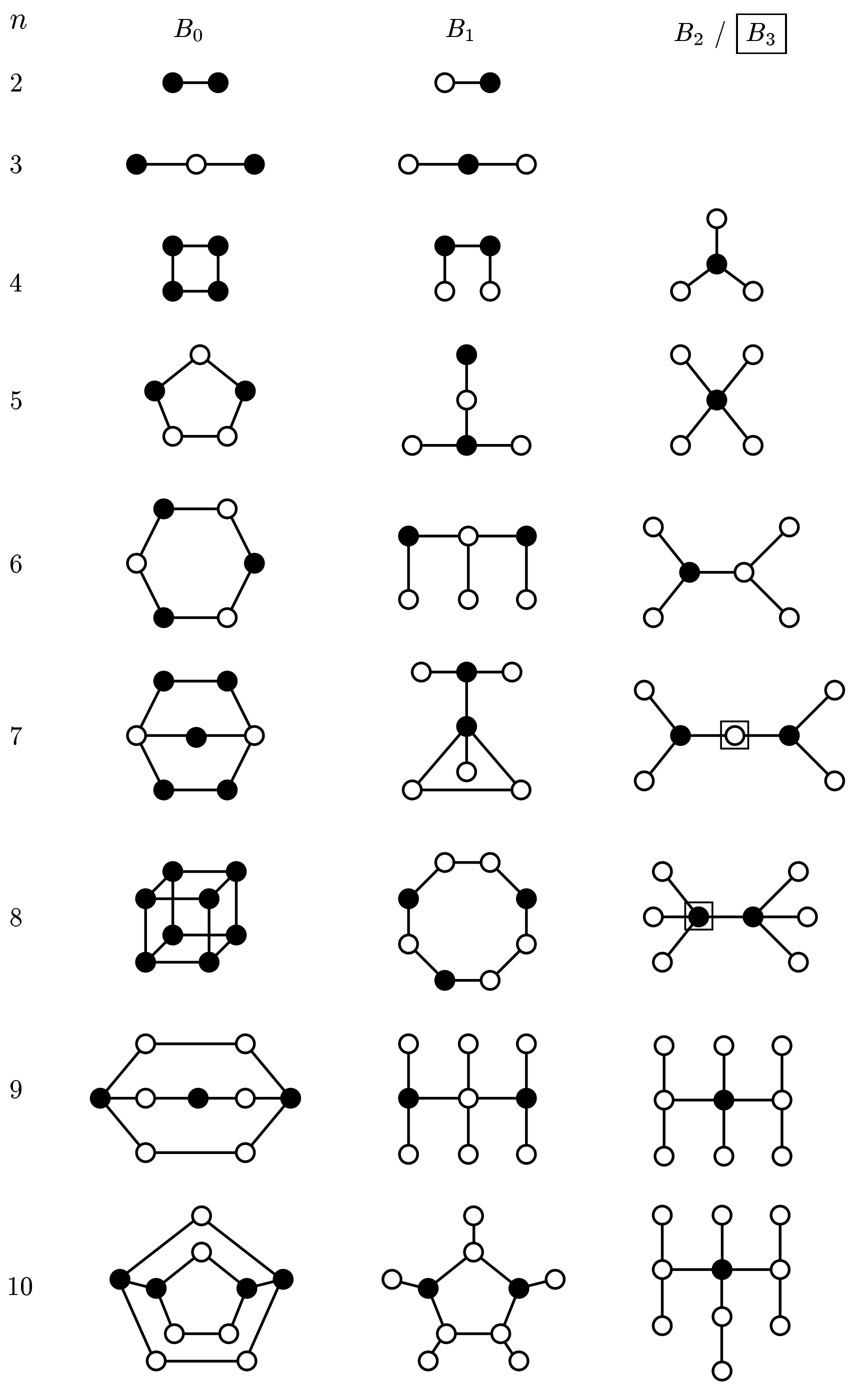


Figure 8: Some t-obn's (and their minimum center nodes sets) for n up to 10.

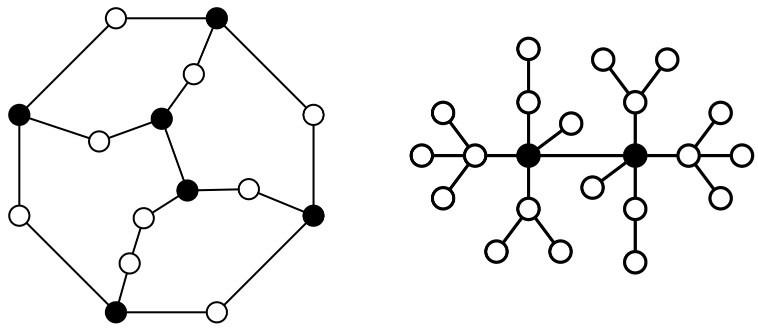


Figure 9: A planar 1-mbn with 15 nodes and a tree 2-obn with 22 nodes.

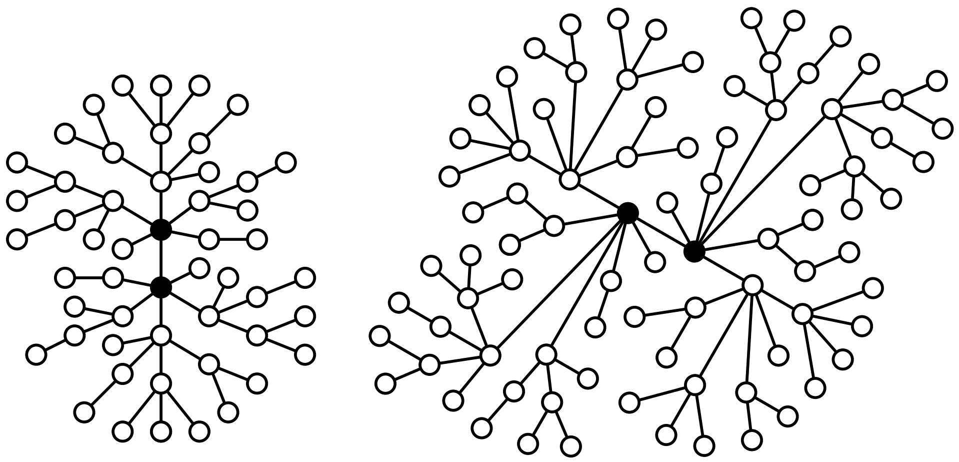


Figure 10: Two maximal tree 3-obn's (52 nodes and 84 nodes).

5. Conclusion

Although the time-relaxed minimal broadcast problem has not been studied in depth, we have shown that graph compound constructions provide the best-known upper bounds on $B_t(n)$ as the number of nodes n increases for each $t \geq 0$. About 40% of Shastri's original bounds for $1 \leq n \leq 65$ have been improved (usually by one), by compounding with only a small set of building-block t-mbn's. The main benefit of associating center node sets and official broadcasting protocols to a t-mbn is that official (and unofficial) broadcast protocols are known after the generalized compound method is applied, thus allowing for the compounding process to be repeated.

There are many open problems related to this paper, of which the following is a selection.

- Besides the tree t-obn families of Shastri [12], do there exist other direct constructions of t-obn's for $t > 0$?
- Is it beneficial to use relaxed center node sets (i.e., t-cns's for $t > b(G)$) in the construction of sparsest known t-mbn's by the compounding methods? (Using the two 3-cns's of Figure 8 did not improve any of our generated $B_t(n)$ bounds.)
- The conjecture that $B_1(15) = 18$ and $B_1(16) = 19$ is still open [13].
- Is it true that $B_t(n+1) > B_t(n)$ for all $t \geq 0$ and $n \neq 2^k$? Also, is it true that $B_t(n) > B_{t+1}(n)$ whenever $B_t(n) \neq n-1$?
- The computational complexity of deciding if a graph has a t-ocns of cardinality at most k is unknown (but believed to be $\mathcal{NP}$-hard for any fixed $t \geq 0$).

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Table 1: Smallest broadcast times $T(n)$ for trees of order n (see [12]).

n	$\lceil \lg n \rceil$	$T(n)$	$t = T(n) - \lceil \lg n \rceil$	$\lvert\text{CNS}\rvert$
1	0	0	0	1
2	1	1	0	2
3	2	2	0	2
4	2	3	1	1
5-6	3	4	1	1
7-8	3	5	2	2
9	4	5	1	1
10-11	4	6	2	1
12-14	4	6	2	2
15-16	4	7	3	2
17-22	5	7	2	2
23-32	5	8	3	2
33	6	8	2	2
34-52	6	9	3	2
53-64	6	10	4	2
65-84	7	10	3	2
85-128	7	11	4	2

Table 2: Some $B_t(n) \leq T_t(n)$ bounds (bold entries indicates optimal).

n	$T_0(n)$	compound (v_G, v_H, i)	$T_1(n)$	compound (v_G, v_H, i):$t_G t_H$	$T_2(n)$	compound (v_G, v_H, i):$t_G t_H$	$T_3(n)$	compound (v_G, v_H, i):$t_G t_H$
1	**0**							
2	**1**							
3	**2**							
4	**4**	(2,2)	**3**					
5	**5**		**4**					
6	**6**		**5**					
7	**8**		**7**		**6**			
8	**12**	(2,4)	**8**		**7**			
9	**10**		**8**	(3,3):10				
10	**12**	(5,2)	**10**	(2,5):10	**9**			
11	**13**		**11**	(2,6,1):10	**10**			
12	**15**	(6,2)	**12**	(2,6):10	**11**	(2,6):11		
13	**18**		**14**	(2,7,1):10	**12**			
14	**21**		**15**	(2,7):10	**13**			
15	**24**		18		**15**	(3,5):10	**14**	(3,5):11
16	**32**	(2,8)	19	(8,2):10	**16**	(2,8):11	**15**	(2,8):12
17	**22**		**17**	(3,6,1):10	**16**	(6,3,1):20		
18	**23**	(9,2)	**18**	(3,6):10	**17**			
19	**25**		**20**	(3,7,2):10	**18**			
20	**26**		**21**	(3,7,1):10	**19**			
21	**28**		**22**	(3,7):10	**20**			
22	**31**	(11,2)	24	(2,11):10	**21**			
23	34		26	(2,12,1):10	**23**	(2,12,1):11	**22**	(2,12,1):12
24	36		27	(2,12):10	**24**	(2,12):11	**23**	(2,12):12
25	40		30	(2,13,1):10	**25**	(13,2,1):20	**24**	(2,13,1):12
26	**42**		31	(2,13):10	**26**	(13,2):20	**25**	(2,13):12
27	**44**		34	(2,14,1):10	**27**	(14,2,1):20	**26**	(2,14,1):12
28	**48**		35	(2,14):10	**28**	(14,2):20	**27**	(2,14):12
29	**52**		38	(2,15,1):10	31	(3,10,1):10	**28**	(10,3,1):20
30	**60**		39	(2,15):10	32	(3,10):10	**29**	(10,3):20
31	**65**		43	(8,4,1):10	33	(3,11,2):10	**30**	(11,3,2):20
32	**80**	(2,16)	44	(8,4):10	34	(3,11,1):10	**31**	(11,3,1):20
33	48	(3,11)	35	(3,11):10	**32**	(11,3):20		
34	49	(17,2)	37	(3,12,2):10	**34**	(2,17):11	**33**	(2,17):12
35	51	(5,7)	38	(3,12,1):10	**35**	(2,18,1):11	**34**	(2,18,1):12
36	52	(9,4)	39	(3,12):10	**36**	(2,18):11	**35**	(2,18):12
37	56		42	(3,13,2):10	**37**	(10,4,3):20	**36**	(2,19,1):12
38	57	(19,2)	43	(3,13,1):10	**38**	(10,4,2):20	**37**	(2,19):12
39	59		44	(3,13):10	**39**	(10,4,1):20	**38**	(2,20,1):12
40	60	(20,2)	46	(2,20):10	**40**	(10,4):20	**39**	(2,20):12
41	65	(6,7,1)	48	(3,14,1):10	**41**	(11,4,3):20	**40**	(2,21,1):12

n	$T_0(n)$	compound (v_G,v_H,i)	$T_1(n)$	compound (v_G,v_H,i):t_Gt_H	$T_2(n)$	compound (v_G,v_H,i):t_Gt_H	$T_3(n)$	compound (v_G,v_H,i):t_Gt_H
42	66	(6,7)	49	(3,14):10	**42**	(11,4,2):20	**41**	(2,21):12
43	70	(43,1)	52	(2,22,1):10	**43**	(11,4,1):20	**42**	(2,22,1):12
44	72	(11,4)	53	(2,22):10	**44**	(11,4):20	**43**	(2,22):12
45	78	(23,2,1)	54	(3,15):10	48	(2,23,1):11	**44**	
46	79	(23,2)	57	(2,23):10	49	(2,23):11	**45**	
47	83	(24,2,1)	59	(2,24,1):10	50	(2,24,1):11	**46**	
48	83	(24,2)	60	(2,24):10	51	(2,24):11	**47**	
49	94		64	(2,25,1):10	53	(13,4,3):20	**48**	
50	95	(25,2)	65	(2,25):10	54	(13,4,2):20	**49**	
51	99	(26,2,1)	67	(2,26,1):10	55	(13,4,1):20	**50**	
52	99	(26,2)	68	(2,26):10	56	(13,4):20	**51**	
53	103	(27,2,1)	70	(2,27,1):10	57	(14,4,3):20	**53**	(11,5,2):20
54	103	(27,2)	71	(2,27):10	58	(14,4,2):20	**54**	(11,5,1):20
55	111	(28,2,1)	75	(2,28,1):10	59	(14,4,1):20	**55**	(11,5):20
56	111	(28,2)	76	(2,28):10	60	(14,4):20	**56**	(10,6,4):20
57	121	(29,2,1)	80	(2,29,1):10	63	(3,19):10	**57**	(10,6,3):20
58	**121**		81	(2,29):10	64	(3,20,2):10	**58**	(10,6,2):20
59	**124**		89	(5,12,1):00	65	(3,20,1):10	**59**	(10,6,1):20
60	**130**		90	(5,12):00	66	(3,20):10	**60**	(10,6):20
61	**136**		92	(9,7,2):00	68	(3,21,2):10	**61**	(11,6,5):20
62	**155**		93	(9,7,1):00	69	(3,21,1):10	**62**	(11,6,4):20
63	**162**		94	(9,7):00	70	(3,21):10	**63**	(11,6,3):20
64	**192**	(2,32)	100	(5,13,1):00	73	(3,22,2):10	**64**	(11,6,2):20
65	101	(5,13)	74	(3,22,1):10	**65**	(11,6,1):20	**64**	(2,33,1):12
66	105	(6,11)	75	(3,22):10	**66**	(11,6):20	**65**	(2,33):12
67	107	(17,4,1)	78	(3,23,2):10	68	(10,7,3):20	**66**	
68	108	(17,4)	79	(3,23,1):10	69	(10,7,2):20	**67**	
69	111	(5,14,1)	80	(3,23):10	70	(10,7,1):20	**68**	
70	112	(5,14)	82	(3,24,2):10	71	(10,7):20	**69**	
71	115	(9,8,1)	83	(3,24,1):10	72	(11,7,6):20	**70**	
72	116	(9,8)	84	(3,24):10	73	(11,7,5):20	**71**	
73	121	(5,15,2)	88	(3,25,2):10	74	(11,7,4):20	**72**	
74	122	(5,15,1)	89	(3,25,1):10	75	(11,7,3):20	**73**	
75	123	(5,15)	90	(3,25):10	76	(11,7,2):20	**74**	
76	128	(19,4)	92	(3,26,2):10	77	(11,7,1):20	**75**	
77	131	(6,13,1)	93	(3,26,1):10	78	(11,7):20	**76**	
78	132	(6,13)	94	(3,26):10	82	(10,8,2):20	**77**	
79	135	(20,4,1)	96	(3,27,2):10	83	(10,8,1):20	**78**	
80	136	(20,4)	97	(3,27,1):10	84	(10,8):20	**79**	
81	142	(3,27)	98	(3,27):10	85	(11,8,7):20	**80**	
82	145	(6,14,2)	102	(3,28,2):10	86	(11,8,6):20	**81**	
83	146	(6,14,1)	103	(3,28,1):10	87	(11,8,5):20	**82**	
84	147	(6,14)	104	(3,28):10	88	(11,8,4):20	**83**	
85	157	(6,15,5)	108	(3,29,2):10	89	(11,8,3):20	**85**	(11,8,3):21

n	$T_0(n)$	compound (v_G,v_H,i)	$T_1(n)$	compound (v_G,v_H,i):t_Gt_H	$T_2(n)$	compound (v_G,v_H,i):t_Gt_H	$T_3(n)$	compound (v_G,v_H,i):t_Gt_H
86	158	(6,15,4)	109	(3,29,1):10	90	(11,8,2):20	**86**	(11,8,2):21
87	159	(6,15,3)	110	(3,29):10	91	(11,8,1):20	**87**	(11,8,1):21
88	160	(6,15,2)	116	(2,44):10	92	(11,8):20	**88**	(11,8):21
89	161	(6,15,1)	119	(3,30,1):10	98	(13,7,2):20	**89**	(45,2,1):30
90	162	(6,15)	120	(3,30):10	99	(13,7,1):20	**90**	(45,2):30
91	179	(23,4,1)	124	(2,46,1):10	100	(13,7):20	**91**	(46,2,1):30
92	180	(23,4)	125	(2,46):10	101	(14,7,6):20	**92**	(46,2):30
93	188	(24,4,3)	127	(3,31):10	102	(14,7,5):20	**93**	(47,2,1):30
94	188	(24,4,2)	129	(2,48,2):10	103	(14,7,4):20	**94**	(47,2):30
95	188	(24,4,1)	130	(2,48,1):10	104	(14,7,3):20	**95**	(48,2,1):30
96	188	(24,4)	131	(2,48):10	105	(14,7,2):20	**96**	(48,2):30
97	203	(14,7,1)	139	(7,14,1):10	106	(14,7,1):20	**97**	(49,2,1):30
98	203	(14,7)	140	(7,14):10	107	(14,7):20	**98**	(49,2):30
99	220	(25,4,1)	144	(2,50,1):10	114	(3,33):10	**99**	(50,2,1):30
100	220	(25,4)	145	(2,50):10	115	(3,34,2):10	**100**	(50,2):30
101	228	(13,8,3)	148	(2,52,3):10	116	(3,34,1):10	**101**	(51,2,1):30
102	228	(13,8,2)	149	(2,52,2):10	117	(3,34):10	**102**	(51,2):30
103	228	(13,8,1)	150	(2,52,1):10	119	(3,35,2):10	**103**	(52,2,1):30
104	228	(13,8)	151	(2,52):10	120	(3,35,1):10	**104**	(52,2):30
105	236	(27,4,3)	153	(7,15):10	121	(3,35):10	107	(11,10,5):20
106	236	(27,4,2)	155	(2,54,2):10	122	(3,36,2):10	108	(11,10,4):20
107	236	(27,4,1)	156	(2,54,1):10	123	(3,36,1):10	109	(11,10,3):20
108	236	(27,4)	157	(2,54):10	124	(3,36):10	110	(11,10,2):20
109	252	(14,8,3)	164	(2,56,3):10	125	(14,8,3):20	111	(11,10,1):20
110	252	(14,8,2)	165	(2,56,2):10	126	(14,8,2):20	112	(11,10):20
111	252	(14,8,1)	166	(2,56,1):10	127	(14,8,1):20	113	(11,11,10):20
112	252	(14,8)	167	(2,56):10	128	(14,8):20	114	(11,11,9):20
113	276	(29,4,3)	176	(2,58,3):10	132	(3,38,1):10	115	(11,11,8):20
114	276	(29,4,2)	177	(2,58,2):10	133	(3,38):10	116	(11,11,7):20
115	276	(29,4,1)	178	(2,58,1):10	135	(3,39,2):10	117	(11,11,6):20
116	275	(58,2)	179	(2,58):10	136	(3,39,1):10	118	(11,11,5):20
117	285	(59,2,1)	182	(2,59,1):10	137	(3,39):10	119	(11,11,4):20
118	283	(59,2)	183	(2,59):10	138	(3,40,2):10	120	(11,11,3):20
119	292	(60,2,1)	189	(2,60,1):10	139	(3,40,1):10	121	(11,11,2):20
120	290	(60,2)	190	(2,60):10	140	(3,40):10	122	(11,11,1):20
121	317	(61,2,1)	196	(2,61,1):10	145	(3,41,2):10	123	(11,11):20
122	315	(61,2)	197	(2,61):10	146	(3,41,1):10	125	(61,2):30
123	346	(62,2,1)	200	(9,14,3):00	147	(3,41):10	126	(62,2,1):30
124	341	(62,2)	201	(9,14,2):00	148	(3,42,2):10	127	(62,2):30
125	379	(2,63,1)	202	(9,14,1):00	149	(3,42,1):10	128	(63,2,1):30
126	**378**		203	(9,14):00	150	(3,42):10	129	(63,2):30
127	417	(2,64,1)	213	(64,2,1):10	155	(32,4,1):20	130	(64,2,1):30
128	**448**	(2,64)	214	(64,2):10	156	(32,4):20	131	(64,2):30