Foundations of Physics, Vol. 29, No. 3, 1999

Embedding Quantum Universes in Classical Ones

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Received September 7, 1997; revised March 25, 1998

Do the partial order and ortholattice operations of a quantum logic correspond to the logical implication and connectives of classical logic? Rephrased, How far might a classical understanding of quantum mechanics be, in principle, possible? A celebrated result of Kochen and Specker answers the above question in the negative. However, this answer is just one among various possible ones, not all negative. It is our aim to discuss the above question in terms of mappings of quantum worlds into classical ones, more specifically, in terms of embeddings of quantum logics into classical logics; depending upon the type of restrictions imposed on embeddings, the question may get negative or positive answers.

1. INTRODUCTION

Quantum mechanics is a very successful theory which appears to predict novel "counterintuitive" phenomena (see Refs. 12 and 50) even almost a century after its development (cf. Refs. 19, 20, and 42). Yet it can be safely stated that quantum theory is not understood.[10] Indeed, it appears that progress is fostered by abandoning long-held beliefs and concepts rather than by attempts to derive it from some classical basis (Refs. 4, 13, and 18).

But just how far might a classical understanding of quantum mechanics be, in principle, possible? We shall attempt an answer to this question in terms of mappings of quantum worlds into classical ones, more specifically, in terms of embeddings of quantum logics into classical logics.

One physical motivation for this approach is a result proven for the first time by Kochen and Specker[26] [cf. also Refs. 2, 43, and 52; see

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0015-9018/99/0300-0349$16.00/0

reviews in Refs. 32 and 48 and a forthcoming monograph[46]] stating the impossibility to "complete" quantum physics by introducing noncontextual hidden parameter models. Such a possible "completion" had been suggested, though in not very concrete terms, by Einstein, Podolsky, and Rosen (EPR).[9] These authors speculated that "elements of physical reality" exist irrespective of whether they are actually observed. Moreover, EPR conjectured, the quantum formalism can be "completed" or "embedded" into a larger theoretical framework which would reproduce the quantum theoretical results but would otherwise be classical and deterministic from an algebraic and logical point of view.

A proper formalization of the term "element of physical reality" suggested by EPR can be given in terms of two-valued states or valuations, which can take on only one of the two values 0 and 1, and which are interpretable as the classical logical truth assignments *false* and *true*, respectively. Kochen and Specker's results[26] state that for quantum systems representable by Hilbert spaces of dimension higher than two, there does not exist any such valuation $s: L \to \{0, 1\}$ defined on the set of closed linear subspaces of the space L (these subspaces are interpretable as quantum mechanical propositions) preserving the lattice operations and the orthocomplement, even if one restricts the attention to lattice operations carried out among commuting (orthogonal) elements. As a consequence, the set of truth assignments on quantum logics is not separating and not unital. That is, there exist different quantum propositions which cannot be distinguished by any classical truth assignment.

The Kochen and Specker result, as it is commonly argued (e.g., in Refs. 35 and 32), is directed against the noncontextual hidden parameter program envisaged by EPR. Indeed, if one takes into account the entire Hilbert logic (of dimension higher than two) and if one considers all states thereon, any truth value assignment to quantum propositions prior to the actual measurement yields a contradiction. This can be proven by finitistic means, that is, with a finite number of one-dimensional closed linear subspaces [generating an infinite set whose intersection with the unit sphere is dense (cf. Ref. 17)] But, the Kochen–Specker argument continues, it is always possible to prove the existence of separable valuations or truth assignments for classical prepositional systems identifiable with Boolean algebras. Hence, there does not exist any injective morphism from a quantum logic into some Boolean algebra.

Since the previous reviews of the Kochen–Specker theorem by Peres,[34, 35] Redhead,[38] Clifton,[6] Mermin,[32] and Svozil and Tkadlec[48] concentrated on the nonexistence of classical noncontextual elements of physical reality, we discuss here some options and aspects of embeddings in greater detail. Particular emphasis is given to embeddings of quantum universes in classical ones which do not necessarily preserve (binary lattice) operations identifiable with the logical *or* and *and* operations. Stated pointedly, if one is willing to abandon the preservation of quite commonly used logical functions, then it is possible to give a classical meaning to quantum physical statements, thus giving raise to an "understanding" of quantum mechanics.

Quantum logic, according to Refs. 5, 21, 23, 28, and 37, identifies logical entities with Hilbert space entities. In particular, elementary propositions p, q,... are associated with closed linear subspaces of a Hilbert space through the origin (zero vector); the implication relation $\leqslant$ is associated with the set theoretical subset relation $\subseteq$, and the logical *or* $\vee$, *and* $\wedge$, and *not* $'$ operations are associated with the set theoretic intersection $\cap$, with the linear span $\oplus$ of subspaces, and with the orthogonal subspace $\perp$, respectively. The trivial logical statement 1, which is always true, is identified with the entire Hilbert space H, and its complement $\varnothing$ with the zero-dimensional subspace (zero vector). Two propositions p and q are orthogonal if and only if $p \leqslant q'$. Two propositions p, q are comeasurable (commuting) if and only if there exist mutually orthogonal propositions a, b, c such that $p = a \vee b$ and $q = a \vee c$. Clearly, orthogonality implies comeasurability, since if p and q are orthogonal, we may identify a, b, c with 0, p, q, respectively. The negation of $p \leqslant q$ is denoted $p \nleqslant q$.

2. VARIETIES OF EMBEDDINGS

One of the questions already raised in Specker's almost-forgotten first article[43] concerned an embedding of a quantum logical structure L of propositions in classical universe represented by a Boolean algebra B. Thereby, it is taken as a matter of principle that such an embedding should preserve as much logicoalgebraic structure as possible. An embedding of this kind can be formalized as a mapping $\varphi: L \to B$ with the following properties.[4] Let $p, q \in L$.

(i) *Injectivity*: Two different quantum logical propositions are mapped into two different propositions of the Boolean algebra, i.e., if $p \neq q$, then $\varphi(p) \neq \varphi(q)$.

(ii) *Preservation of the order relation*: If $p \leqslant q$, then $\varphi(p) \leqslant \varphi(q)$.

(iii) *Preservation of ortholattice operations*, i.e., preservation of the (ortho-)complement: $\varphi(p') = \varphi(p)'$;
or operation: $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$; and
and operation: $\varphi(p \wedge q) = \varphi(p) \wedge \varphi(q)$.

[4] Specker had a modified notion of embedding in mind; see below.

As it turns out, we cannot have an embedding from the quantum universe to the classical universe satisfying all three requirements (i)–(iii). In particular, a head-on approach requiring (iii) is doomed to failure, since the nonpreservation of ortholattice operations among noncomeasurable propositions is quite evident, given the nondistributive structure of quantum logics.

2.1. Injective Lattice Morphisms

Here we review the rather evident fact that there does not exist an injective lattice morphism from any nondistributive lattice into a Boolean algebra. We illustrate this obvious fact with an example that we need to refer to later on in this paper; the propositional structure encountered in the quantum mechanics of spin-state measurements of a spin one-half particle along two directions (mod π), that is, the modular, orthocomplemented lattice MO_2 drawn in Fig. 1 (where $p_- = (p_+)'$ and $q_- = (q_+)'$).

Clearly, MO_2 is a nondistributive lattice, since for instance,

$$p_- \wedge (q_- \vee q_+) = p_- \wedge 1 = p_-$$

whereas

$$(p_- \wedge q_-) \vee (p_- \wedge q_+) = 0 \vee 0 = 0$$

Hence,

$$p_- \wedge (q_- \vee q_+) \neq (p_- \wedge q_-) \vee (p_- \wedge q_+)$$

In fact, MO_2 is the smallest orthocomplemented nondistributive lattice.

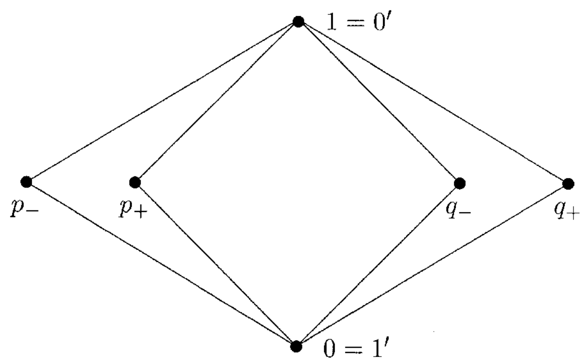

Fig. 1. Hasse diagram of the "Chinese lantern" form of MO_2.

The requirement (iii) that the embedding φ preserves all ortholattice operations (even for noncomeasurable and nonorthogonal propositions) would mean that $\varphi(p_-) \wedge (\varphi(q_-) \vee \varphi(q_+)) \neq (\varphi(p_-) \wedge \varphi(q_-)) \vee (\varphi(p_-) \wedge \varphi(q_+))$. That is, the argument implies that the distributive law is not satisfied in the range of φ. But since the range of φ is a subset of a Boolean algebra and for any Boolean algebra the distributive law is satisfied, this yields a contradiction.

Could we still hope for a reasonable kind of embedding of a quantum universe in a classical one by weakening our requirements, most notably (iii)? In the next three sections we give different answers to this question. In the first section we restrict the set of propositions among which we wish to preserve the three operations *complement* $'$, *or* $\vee$, and *and* $\wedge$. We will see that the Kochen–Specker result gives a very strong negative answer even when the restriction is considerable. In the second section we analyze what happens if we try to preserve not all operations but just the complement. Here we obtain a positive answer. In the third section we discuss a different embedding which preserves the order relation but no ortholattice operation.

2.2. Injective Order Morphisms Preserving Ortholattice Operations Among Orthogonal Propositions

Let us follow Zierler and Schlessinger[52] and Kochen and Specker[26] and weaken (iii) by requiring that the ortholattice operations need only be preserved *among orthogonal* propositions. As shown by Kochen and Specker,[26] this is equivalent to the requirement of separability by the set of valuations or two-valued probability measures or truth assignments on L. As a matter of fact, Kochen and Specker[26] proved nonseparability, but also much more—the *nonexistence* of valuations on Hilbert lattices associated with Hilbert spaces of dimension at least three. For related arguments and conjectures, based upon a theorem by Gleason,[11] see Refs. 2 and 52.

Rather than rephrasing the Kochen and Specker argument[26] concerning the nonexistence of valuations in three-dimensional Hilbert logics in its original form or in terms of fewer subspaces (cf. Refs. 32 and 35), or of Greechie diagrams, which represent orthogonality very nicely (cf. Refs. 46 and 48), we give two geometric arguments which are derived from proof methods for Gleason's theorem (see Refs. 7, 24 and 36).

Let L be the lattice of closed linear subspaces of the three-dimensional real Hilbert space $\mathbb{R}^3$. A *two-valued probability measure* or *valuation* on L is a map v: $L \to \{0, 1\}$ which maps the zero-dimensional subspace containing only the origin $(0, 0, 0)$ to 0, the full space $\mathbb{R}^3$ to 1, and which is additive on orthogonal subspaces. This means that for two orthogonal subspaces $s_1, s_2 \in L$, the sum of the values $v(s_1)$ and $v(s_2)$ is equal to the value of the linear span of s_1 and s_2. Hence, if $s_1, s_2, s_3 \in L$ are a tripod of pairwise orthogonal one-dimensional subspaces, then

$$v(s_1) + v(s_2) + v(s_3) = v(\mathbb{R}^3) = 1$$

The valuation v must map one of these subspaces to 1 and the other two to 0. We show that there is *no* such map. In fact, we show that there is no map v which is defined on all one-dimensional subspaces of $\mathbb{R}^3$ and maps *exactly one subspace out of each tripod of pairwise orthogonal one-dimensional subspaces to 1 and the other two to 0.*

In the following two geometric proofs we often identify a given one-dimensional subspace of $\mathbb{R}^3$ with one of its two intersection points with the unit sphere

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

In the statements "A point (on the unit sphere) has value 0 (or value 1)" and "Two points (on the unit sphere) are orthogonal," we always mean the corresponding one-dimensional subspaces. Note also that the intersection of a two-dimensional subspace with the unit sphere is a great circle.

To start the first proof, let us assume that a function v satisfying the above condition exists. Let us consider an arbitrary tripod of orthogonal points and let us fix the point with value 1. By a rotation we can assume that it is the north pole with the coordinates $(0, 0, 1)$. Then, by the condition above, all points on the equator $\{(x, y, z) \in S^2 \mid z = 0\}$ must have value 0 since they are orthogonal to the north pole.

Let $q = (q_x, q_y, q_z)$ be a point in the northern hemisphere, but not equal to the north pole, that is, $0 < q_z < 1$. Let $C(q)$ be the unique great circle which contains q and the points $\pm (q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator, which are orthogonal to q. Obviously, q is the northernmost point on $C(q)$. To see this, rotate the sphere around the z-axis so that q comes to lie in the $\{y = 0\}$-plane (see Fig. 2). Then the two points in the equator orthogonal to q are just the points $\pm (0, 1, 0)$, and $C(q)$ is the intersection of the plane through q and $(0, 1, 0)$ with the unit sphere, hence

$$C(q) = \{p \in \mathbb{R}^3 \mid (\exists \alpha, \beta \in \mathbb{R})\ \alpha^2 + \beta^2 = 1 \text{ and } p = \alpha q + \beta(0, 1, 0)\}$$

This shows that q has the largest z-coordinate among all points in $C(q)$.

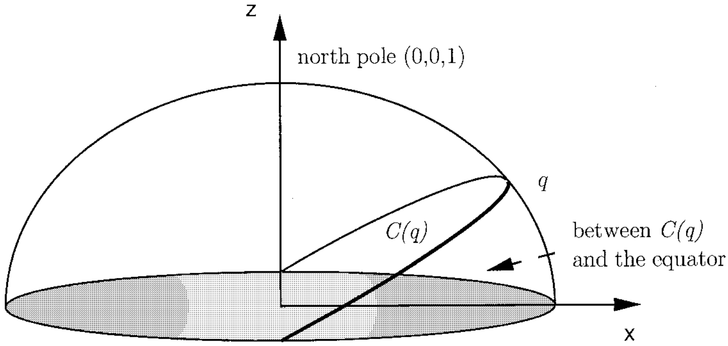

Fig. 2. The great circle $C(q)$.

Assume that q has value 0. We claim that then all points on $C(q)$ must have value 0. Indeed, since q has value 0 and the orthogonal point $(q_y, -q_x, 0)/\sqrt{q_x^2+q_y^2}$ on the equator also has value 0, the one-dimensional subspace orthogonal to both of them must have value 1. But this subspace is orthogonal to all points on $C(q)$. Hence all points on $C(q)$ must have value 0.

Now we can apply the same argument to any point $\tilde{q}$ on $C(q)$ (by the last consideration $\tilde{q}$ must have value 0) and derive that all points on $C(\tilde{q})$ have value 0. The great circle $C(q)$ divides the northern hemisphere into two regions, one containing the north pole and the other consisting of the points below $C(q)$ or "lying between $C(q)$ and the equator" (see Fig. 2). The circles $C(\tilde{q})$ with $\tilde{q} \in C(q)$ certainly cover the region between $C(q)$ and the equator.[5] Hence any point in this region must have value 0.

But the circles $C(\tilde{q})$ cover also a part of the other region. In fact, we can iterate this process. We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. Our analysis above shows that if q has value 0 and p can be reached from q, then also p has value 0.

The following geometric lemma due to Piron[36] (see also Ref. 7 or 24) is a consequence of the fact that the curve $C(q)$ is tangent to the horizontal plane through the point q:

> *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

[5] This is shown formally in the proof of the geometric lemma below.

This result is proved in Appendix A. We conclude that, if a point q in the northern hemisphere has value 0, then every point p in the northern hemisphere with $p_z < q_z$ must have value 0 as well.

Consider the tripod $(1, 0, 0)$, $(0, 1/\sqrt{2}, 1/\sqrt{2})$ $(0, -1/\sqrt{2}, 1/\sqrt{2})$. Since $(1, 0, 0)$ (on the equator) has value 0, one of the two other points has value 0 and one has value 1. By the geometric lemma and our above considerations, this implies that all points p in the northern hemisphere with $p_z < 1/\sqrt{2}$ must have value 0 and all points p with $p_z > 1/\sqrt{2}$ must have value 1. But now we can choose any point p' with $1/\sqrt{2} < p'_z < 1$ as our new north pole and deduce that the valuation must have the same form with respect to this pole. This is clearly impossible. Hence, we have proved our assertion that there is no mapping on the set of all one-dimensional subspaces of $\mathbb{R}^3$ which maps one space out of each tripod of pairwise orthogonal one-dimensional subspaces to 1 and the other two to 0.

In the following we give a second topological and geometric proof for this fact. In this proof we do not use the geometric lemma above.

Fix an arbitrary point on the unit sphere with value 0. The great circle consisting of points orthogonal to this point splits into two disjoint sets, the set of points with value 1, and the set of points orthogonal to these points. They have value 0. If one of these two sets were open, then the other had to be open as well. But this is impossible since the circle is connected and cannot be the union of two disjoint open sets. Hence the circle must contain a point p with value 1 and a sequence of points $q(n)$, $n = 1, 2,...$ with value 0 converging to p. By a rotation we can assume that p is the north pole and the circle lies in the $\{y = 0\}$-plane. Furthermore, we can assume that all points q_n have the same sign in the x-coordinate. Otherwise, choose an infinite subsequence of the sequence $q(n)$ with this property. In fact, by a rotation we can assume that all points $q(n)$ have positive x-coordinate [i.e., all points $q(n)$, $n = 1, 2,...$ lie as the point q in Fig. 2 and approach the north pole as n tends to infinity]. All points on the equator have value 0. By the first step in the proof of the geometric lemma in the appendix, all points in the northern hemisphere which lie between $C(q(n))$ [the great circle through $q(n)$ and $\pm(0, 1, 0)$] and the equator can be reached from $q(n)$. Hence, as we have seen in the first proof, $v(q(n)) = 0$ implies that all these points must have value 0. Since $q(n)$ approaches the north pole, the union of the regions between $C(q(n))$ and the equator is equal to the open right half $\{q \in S^2 \mid q_z > 0, q_x > 0\}$ of the northern hemisphere. Hence all points in this set have value 0. Let q be a point in the left half $\{q \in S^2 \mid q_z > 0, q_x < 0\}$ of the northern hemisphere. It forms a tripod together with the point $(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator and the point $(-q_x, -q_y, (q_x^2 + q_y^2)/q_z)/\|(-q_x, -q_y, (q_x^2 + q_y^2)/q_z)\|$ in the right half. Since these two points have value 0, the point q must have value 1. Hence all points in the left half of the northern hemisphere must have value 1. But this leads to a contradiction because there are tripods with two points in the left half, for example the tripod $(-\frac{1}{2}, 1/\sqrt{2}, \frac{1}{2})$, $(-\frac{1}{2}, -1/\sqrt{2}, \frac{1}{2})$, $(1/\sqrt{2}, 0, 1/\sqrt{2})$. This ends the second proof for the fact that there is no two-valued probability measure on the lattice of subspaces of the three-dimensional Euclidean space which preserves the ortholattice operations at least for orthogonal elements.

2.3. Injective Morphisms Preserving Order as Well as *or* and *and* operations

We have seen that we cannot hope to preserve the ortholattice operations, not even when we restrict ourselves to operations among orthogonal propositions.

An even stronger weakening of condition (iii) would be to require preservation of ortholattice operations merely among the center C, i.e., among those propositions which are comeasurable (commuting) with all other propositions. It is not difficult to prove that in the case of complete Hilbert lattices (and not mere subalgebras thereof), the center consists of just the least lower and the greatest upper bound $C = \{0, 1\}$ and thus is isomorphic to the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$. As it turns out, the requirement is trivially fulfilled and its implications are quite trivial as well.

Another weakening of (iii) is to restrict oneself to particular physical states and study the embeddability of quantum logics under these constraints (see Ref. 1).

In the following sections we analyze a completely different option: Is it possible to embed quantum logic in a Boolean algebra when one does not demand preservation of all ortholattice operations?

One method of embedding an arbitrary partially ordered set in a concrete orthomodular lattice which in turn can be embedded into a Boolean algebra has been used by Kalmbach[22] and extended by Hardin[16] and Mayet and Navara.[31]. In these *Kalmbach embeddings*, as they may be called, the meets and joins are preserved but, not the complement.

The Kalmbach embedding of some bounded lattice L into a concrete orthomodular lattice $K(L)$ may be thought of as the pasting of Boolean algebras corresponding to all maximal chains of L.[15]

First, let us consider linear chains $0 = a_0 \to a_1 \to a_2 \to \cdots \to 1 = a_m$. Such chains generate Boolean algebras $\mathbf{2}^{m-1}$ in the following way: from the first nonzero element a_1 on to the greatest element 1, form $A_n = a_n \wedge (a_{n-1})'$, where $(a_{n-1})'$ is the complement of a_{n-1} relative to 1; i.e., $(a_{n-1})' = 1 - a_{n-1}$. A_n is then an atom of the Boolean algebra generated by the bounded chain $0 = a_0 \to a_1 \to a_2 \to \cdots \to 1$.

Take, for example, a three-element chain $0 = a_0 \rightarrow \{a\} \equiv a_1 \rightarrow \{a, b\} \equiv 1 = a_2$ as depicted in Fig. 3a. In this case,

$$A_1 = a_1 \wedge (a_0)' = a_1 \wedge 1 \equiv \{a\} \wedge \{a, b\} = \{a\}$$

$$A_2 = a_2 \wedge (a_1)' = 1 \wedge (a_1)' \equiv \{a, b\} \wedge \{b\} = \{b\}$$

This construction results in a four-element Boolean Kalmbach lattice $K(L) = \mathbf{2}^2$ with the two atoms $\{a\}$ and $\{b\}$ given in Fig. 3b.

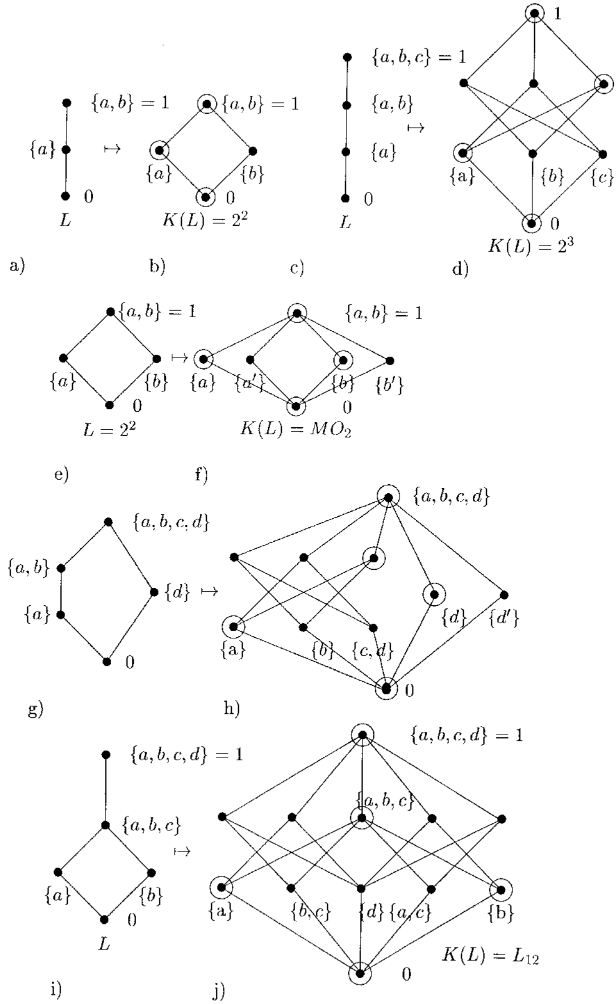

Fig. 3. Examples of Kalmbach embeddings.

Take, as a second example, a four-element chain $0 = a_0 \to \{a\} \equiv a_1 \to \{a, b\} \to \{a, b, c\} \equiv 1 = a_3$ as depicted in Fig. 3c. In this case,

$$A_1 = a_1 \wedge (a_0)' = a_1 \wedge 1 \equiv \{a\} \wedge \{a, b, c\} = \{a\}$$

$$A_2 = a_2 \wedge (a_1)' \equiv \{a, b\} \wedge \{b, c\} = \{b\}$$

$$A_3 = a_3 \wedge (a_2)' = 1 \wedge (a_2)' \equiv \{a, b, c\} \wedge \{c\} = \{c\}$$

This construction results in an eight-element Boolean Kalmbach lattice $K(L) = \mathbf{2}^3$ with the three atoms $\{a\}$, $\{b\}$, and $\{c\}$ depicted in Fig. 3d.

To apply Kalmbach's construction to any bounded lattice, all Boolean algebras generated by the maximal chains of the lattice are pasted together. An element common to two or more maximal chains must be common to the blocks they generate.

Take, as a third example, the Boolean lattice $\mathbf{2}^2$ drawn in Fig. 3e. $\mathbf{2}^2$ contains two linear chains of length three which are pasted together horizontally at their smallest and biggest elements. The resulting Kalmbach lattice $K(\mathbf{2}^2) = MO_2$ is of the "Chinese lantern" type (see Fig. 3f).

Take, as a fourth example, the pentagon drawn in Fig. 3g. It contains two linear chains: one is of length three, the other is of length 4. The resulting Boolean algebras $\mathbf{2}^2$ and $\mathbf{2}^3$ are again horizontally pasted together at their extremities 0, 1. The resulting Kalmbach lattice is given in Fig. 3h.

In the fifth example, drawn in Fig. 3i, the lattice has two maximal chains which share a common element. This element is common to the two Boolean algebras, hence central in $K(L)$. The construction of the five atoms proceeds as follows:

$$A_1 = \{a\} \wedge \{a, b, c, d\} = \{a\}$$

$$A_2 = \{a, b, c\} \wedge \{b, c, d\} = \{b, c\}$$

$$A_3 = B_3 = \{a, b, c, d\} \wedge \{d\} = \{d\}$$

$$B_1 = \{b\} \wedge \{a, b, c, d\} = \{b\}$$

$$B_2 = \{a, b, c\} \wedge \{a, c, d\} = \{a, c\}$$

where the two sets of atoms $\{A_1, A_2, A_3 = B_3\}$ and $\{B_1, B_2, B_3 = A_3\}$ span two Boolean algebras $\mathbf{2}^3$ pasted together at the extremities and at $A_3 = B_3$ and $A'_3 = B'_3$. The resulting lattice is $\mathbf{2} \times MO_2 = L_{12}$, depicted in Fig. 3j.

2.4. Injective Morphisms Preserving Order and Complementation

In the following, we show that *any orthoposet can be embedded in a Boolean algebra*, where, in this case, by an *embedding* we understand an *injective mapping preserving the order relation and the orthocomplementation*.

A slightly stronger version of this fact using more topological notions has already been shown by Katrnoška.[25] Zierler and Schlessinger constructed embeddings with more properties for orthomodular orthoposets (Ref. 52, Theorem 2.1) and mentioned another slightly stronger version of the result above without explicit proof (Ref. 52, Sec. 2, Remark 2).

For completeness' sake we give the precise definition of an orthoposet. An *orthoposet* (or *orthocomplemented poset*) $(L, \leqslant, 0, 1, ')$ is a set L which is endowed with a partial ordering $\leqslant$ [i.e., a subset $\leqslant$ of $L \times L$ satisfying (1) $p \leqslant p$, (2) if $p \leqslant q$ and $q \leqslant r$, then $p \leqslant r$, and (3) if $p \leqslant q$ and $q \leqslant p$, then $p = q$, for all $p, q, r \in L$]. Furthermore, L contains distinguished elements 0 and 1 satisfying $0 \leqslant p$ and $p \leqslant 1$, for all $p \in L$. Finally, L is endowed with a function $'$ (orthocomplementation) from L to L satisfying the conditions (1) $p'' = p$, (2) if $p \leqslant q$, then $q' \leqslant p'$, and (3) the least upper bound of p and p' exists and is 1, for all $p, q \in L$. Note that these conditions imply $0' = 1$, $1' = 0$, and that the greatest lower bound of p and p' exists and is 0, for all $p \in L$.

For example, an arbitrary sublattice of the lattice of all closed linear subspaces of a Hilbert space is an orthoposet, if it contains the subspace $\{0\}$ and the full Hilbert space and is closed under the orthogonal complement operation. Namely, the subspace $\{0\}$ is the 0 in the orthoposet, the full Hilbert space is the 1, the set-theoretic inclusion is the ordering $\leqslant$, and the orthogonal complement operation is the orthocomplementation $'$.

In the rest of this section we always assume that L is an arbitrary orthoposet. We construct a Boolean algebra B and an injective mapping $\varphi: L \rightarrow B$ which preserves the order relation and the orthocomplementation. The construction goes essentially along the same lines as the construction of Zierler and Schlessinger[52] and Katrnoška[25] and is similar to the proof of the Stone representation theorem for Boolean algebras (Ref. 45). It is interesting to note that for a finite orthoposet the constructed Boolean algebra will be finite as well.

We call a nonempty subset K of L an *ideal* if for all $p, q \in L$:

1. if $p \in K$, then $p' \notin K$, and
2. if $p \leqslant q$ and $q \in K$, then $p \in K$.

Clearly, if K is an ideal, then $0 \in K$. An ideal I is *maximal* provided that if K is an ideal and $I \subseteq K$, then $K = I$.

Let $\mathcal{I}$ be the set of all maximal ideals in L, and let B be the power set of $\mathcal{I}$ considered as a Boolean algebra, i.e., B is the Boolean algebra which consists of all subsets of $\mathcal{I}$. The order relation in B is the set-theoretic inclusion, the ortholattice operations *complement*, *or*, and *and* are given by the set-theoretic complement, union, and intersection, and the elements 0 and 1 of the Boolean algebra are just the empty set and the full set $\mathscr{I}$. Consider the map

$$\varphi: L \to B$$

which maps each element $p \in L$ to the set

$$\varphi(p) = \{I \in \mathscr{I} \mid p \notin I\}$$

of all maximal ideals which do not contain p. We claim that the map φ

(i) is injective,

(ii) preserves the order relation, and

(iii) preserves complementation.

This provides an embedding of quantum logic in classical logic which preserves the implication relation and the negation.[6]

The rest of this section consists of the proof of the three claims above. Let us start with claim (ii). Assume that $p, q \in L$ satisfy $p \leqslant q$. We have to show the inclusion

$$\varphi(p) \subseteq \varphi(q)$$

Take a maximal ideal $I \in \varphi(p)$. Then $p \notin I$. If q were contained in I, then by condition 2 in the definition of an ideal also p had to be contained in I. Hence $q \notin I$, thus proving that $I \in \varphi(q)$.

Before we come to claims (iii) and (i), we give another characterization of maximal ideals. We start with the following assertion, which will also be needed later:

$$\begin{array}{l}\text{If } I \text{ is an ideal and } r \in L \text{ with } r \notin I \text{ and } r' \notin I,\\ \text{then also the set } J = I \cup \{s \in L \mid s \leqslant r\} \text{ is an ideal}\end{array} \tag{1}$$

Here is the proof: It is clear that J satisfies condition 2 in the definition of an ideal. To show that it satisfies condition 1 assume to the contrary that there exists $s \in J$ and $s' \in J$, for some $s \in L$. Then one of the following conditions must be true: (I) $s, s' \in J$, (II) $s \leqslant r$ and $s' \leqslant r$, (III) $s \in I$ and $s' \leqslant r$, or (IV) $s \leqslant r$, $s' \in I$. The first case is impossible since I is an ideal. The second case is ruled out by the fact that $r \neq 1$ (namely, $r = 1$ would imply $r' = 0$,

[6] Note that for a finite orthoposet L the Boolean algebra B is finite as well. Indeed, if L is finite, then it has only finitely many subsets, especially only finitely many maximal ideals. Hence $\mathscr{I}$ is finite, and thus also its power set B is finite.

which would contradict our assumption $r' \notin I$). The third case is impossible since $s' \leqslant r$ implies $r' \leqslant s$, which, combined with $s \in I$, would imply $r' \in I$, contrary to our assumption. Finally the fourth case is nothing but a reformulation of the third case with s and s' interchanged. Thus we have proved that J is an ideal and have proved assertion (1).

Next, we prove the following new characterization of maximal ideals:

$$\text{An ideal } I \text{ is a maximal ideal iff } r \notin I \text{ implies } r' \in I \tag{2}$$

To prove this first assume that for all $r \in L$, if $r \notin I$, then $r' \in I$, and suppose that I is a *proper* subset of an ideal K. Then there exists $p \in K$ such that $p \notin I$. By our hypothesis (for all $r \in L$, $r \notin I$ implies $r' \in I$), we have $p' \in I$. Thus both $p \in K$ and $p' \in K$. This contradicts the fact that K is an ideal.

Conversely, suppose that I is a maximal ideal in L and suppose, to the contrary, that for some $r \in L$,

$$r \notin I \qquad \text{and} \qquad r' \notin I \tag{3}$$

Of course $r \neq 1$, since $1' = 0$ and $0 \in I$. Let

$$J = I \cup (r) \tag{4}$$

where $(r) = \{s \in L \mid s \leqslant r\}$ is the principal ideal of r [note that (r) is indeed an ideal]. Then, under assumption (3), using (1) above, we have that J is an ideal which properly contains I. This contradicts the maximality of I and ends the proof of the assertion (2).

For claim (iii) we have to show the relation

$$\varphi(p') = \mathscr{I} \setminus \varphi(p)$$

for all $p \in L$. This can be restated as

$$I \in \varphi(p') \qquad \text{iff} \quad I \notin \varphi(p)$$

for all $I \in \mathscr{I}$. But this means $p' \notin I$ iff $p \in I$, which follows directly from condition 1 in the definition of an ideal and from assertion (2).

We proceed to claim (i), which states that φ is injective, i.e., if $p \neq q$, then $\varphi(p) \neq \varphi(q)$. But $p \neq q$ is equivalent to $p \nleqslant q$ *or* $q \nleqslant p$. Furthermore, if we can show that there is a maximal ideal I such that $q \in I$ and $p \notin I$, then it follows easily that $\varphi(p) \neq \varphi(q)$. Indeed, $p \notin I$ means $I \in \varphi(p)$ and $q \in I$ means $I \notin \varphi(q)$. It is therefore enough to prove that

If $p \nleqslant q$, then there exists a maximal ideal I such that $q \in I$ and $p \notin I$

To prove this we note that since $p \not\leqslant q$, we have $p \neq 0$. Let

$$\mathscr{I}_{pq} = \{K \subseteq L \mid K \text{ is an ideal and } p \notin K \text{ and } q \in K\}$$

We have to show that among the elements of $\mathscr{I}_{p,q}$ there is a maximal ideal. Therefore we use Zorn's lemma. In order to apply it to $\mathscr{I}_{p,q}$, we have to show that $\mathscr{I}_{p,q}$ is not empty and that every chain in $\mathscr{I}_{p,q}$ has an upper bound.

The set $\mathscr{I}_{p,q}$ is not empty since $(q) \in \mathscr{I}_{p,q}$. Now we are going to show that every chain in $\mathscr{I}_{p,q}$ has an upper bound. This means that, given a subset (*chain*) $\mathscr{C}$ of $\mathscr{I}_{p,q}$ with the property

$$\text{for all } J, K \in \mathscr{C} \text{ one has } J \subseteq K \text{ or } K \subseteq J$$

we have to show that there is an element (*upper bound*) $U \in \mathscr{I}_{p,q}$ with $K \subseteq U$ for all $K \in \mathscr{C}$. The union

$$U_{\mathscr{C}} = \bigcup_{K \in \mathscr{C}} K$$

of all ideals $K \in \mathscr{C}$ is the required upper bound! It is clear that all $K \in \mathscr{C}$ are subsets of $U_{\mathscr{C}}$. We have to show that $U_{\mathscr{C}}$ is an element of $\mathscr{I}_{p,q}$. Since $p \notin K$ for all $K \in \mathscr{C}$, we also have $p \notin U_{\mathscr{C}}$. Similarly, since $q \in K$ for some (even all) $K \in \mathscr{C}$, we have $q \in U_{\mathscr{C}}$. We still have to show that $U_{\mathscr{C}}$ is an ideal. Given two propositions r, s with $r \leqslant s$ and $s \in U_{\mathscr{C}}$, we conclude that s must be contained in one of the ideals $K \in \mathscr{C}$. Hence also $r \in K \subseteq U_{\mathscr{C}}$. Now assume $r \in U_{\mathscr{C}}$. Is it possible that the complement r' belongs to $U_{\mathscr{C}}$? The answer is negative, since otherwise $r \in J$ and $r' \in K$, for some ideals $J, K \in \mathscr{C}$. But since $\mathscr{C}$ is a chain, we have $J \subseteq K$ or $K \subseteq J$, hence $r, r' \in K$ in the first case and $r, r' \in J$ in the second case. Both cases contradict the fact that J and K are ideals. Hence, $U_{\mathscr{C}}$ is an ideal and thus an element of $\mathscr{I}_{p,q}$. We have proved that $\mathscr{I}_{p,q}$ is not empty and that each chain in $\mathscr{I}_{p,q}$ has an upper bound in $\mathscr{I}_{p,q}$.

Consequently, we can apply Zorn's lemma to $\mathscr{I}_{p,q}$ and obtain a maximal element I in the ordered set $\mathscr{I}_{p,q}$. Thus

$$p \notin I \qquad \text{and} \qquad q \in I \tag{5}$$

It remains to show that I is a maximal ideal in L. Thus suppose, to the contrary, that I is *not* a maximal ideal in L.

By (2) there exists $r \in L$ such that both $r \notin I$ and $r' \notin I$. Furthermore, since $p \neq 0$, then either $p \not\leqslant r$ or $p \not\leqslant r'$. Without loss of generality, suppose

$$p \not\leqslant r \tag{6}$$

It follows, by (1), and since $r \notin I$ and $r' \notin I$, that $I \cup (r)$ is an ideal properly containing I. But since, by Conditions (5) and (6), $q \in I$ and $p \not\leqslant r$, we have

$$p \notin I \cup (r) \qquad \text{and} \qquad q \in I \cup (r)$$

Thus $I \cup (r) \in \mathscr{I}_{pq}$, and since $r \notin I$, we deduce that $I \cup (r)$ properly contains I, contradicting the fact that I is a *maximal element* in $\mathscr{I}_{pq}$. This ends the proof of claim (i), the claim that the map φ is injective.

We have shown:

Any orthoposet can be embedded into a Boolean algebra where the embedding preserves the order relation and the complementation

2.5. Injective Order-Preserving Morphisms

In this section we analyze a different embedding suggested by Malhas.[29, 30]

We consider an orthocomplemented lattice $(L, \leqslant, 0, 1, ')$, i.e., a lattice $(L, \leqslant, 0, 1)$ with $0 \leqslant x \leqslant 1$ for all $x \in L$, with orthocomplementation, that is, with a mapping $': L \to L$ satisfying the following three properties: (a) $x'' = x$, (b) if $x \leqslant y$, then $y' \leqslant x'$ and (c) $x \cdot x' = 0$ and $y \vee y' = 1$. Here $x \cdot y = \mathrm{glb}(x, y)$ and $x \vee y = \mathrm{lub}(x, y)$.

Furthermore, we assume that L is atomic[7] and satisfies the following additional property:

$$\text{for all } x, y \in L, x \leqslant y \text{ iff for every atom } a \in L, a \leqslant x \text{ implies } a \leqslant y \qquad (7)$$

Every atomic Boolean algebra and the lattice of closed subspaces of a separable Hilbert space satisfy the above conditions.

Consider next a set U[8] and let $W(U)$ be the smallest set of words over the alphabet $U \cup \{', \to\}$ which contains U and is closed under negation [if $A \in W(U)$, then $A' \in W(U)$] and implication [if $A, B \in W(U)$, then $A \to B \in W(U)$][9] The elements of U are called *simple propositions* and the elements of $W(U)$ are called (*compound*) *propositions*.

A *valuation* is a mapping

$$t\colon W(U) \to \mathbf{2}$$

[7] For every $x \in L \setminus \{0\}$, there is an atom $a \in L$ such that $a \leqslant x$. An atom is an element $a \in L$ with the property that if $0 \leqslant y \leqslant a$, then $y = 0$ or $y = a$.

[8] Not containing the logical symbols $\cup$, $'$, $\to$.

[9] Define in a natural way $A \cup B = A' \to B$, $A \cap B = (A \to B')'$, $A \leftrightarrow B = (A \to B) \cap (B \to A)$.

such that $t(A) \neq t(A')$ and $t(A \to B) = 0$ iff $t(A) = 1$ and $t(B) = 0$. Clearly, every assignment $s\colon U \to \mathbf{2}$ can be extended to a unique valuation t_s.

A *tautology* is a proposition A which is true under every possible valuation, i.e., $t(A) = 1$, for every valuation t. A set $\mathscr{K} \subseteq W(U)$ is consistent if there is a valuation making true every proposition in $\mathscr{K}$. Let $A \in W(U)$ and $\mathscr{K} \subseteq W(U)$. We say that A *derives* from $\mathscr{K}$, and write $\mathscr{K} \models A$, in case $t(A) = 1$ for each valuation t which makes true every proposition in $\mathscr{K}$ [that is, $t(B) = 1$, for all $B \in \mathscr{K}$] We define the set of consequences of $\mathscr{K}$ by

$$Con(\mathscr{K}) = \{A \in W(U) \mid K \models A\}$$

Finally, a set $\mathscr{K}$ is a *theory* if $\mathscr{K}$ is a fixed-point of the operator *Con*:

$$Con(\mathscr{K}) = \mathscr{K}$$

It is easy to see that *Con* is in fact a finitary closure operator, i.e., it satisfies the following four properties:

- $\mathscr{K} \subseteq Con(\mathscr{K})$,
- if $\mathscr{K} \subseteq \tilde{\mathscr{K}}$, then $Con(\mathscr{K}) \subseteq Con(\tilde{\mathscr{K}})$,
- $Con(Con(\mathscr{K})) = Con(\mathscr{K})$, and
- $Con(\mathscr{K}) = \bigcup_{\{X \subseteq \mathscr{K},\, X \text{ finite}\}} Con(X)$.

The first three properties can be proved easily. A topological proof for the fourth property is given in Appendix B.

The main example of a theory can be obtained by taking a set X of valuations and constructing the set of all propositions true under all valuations in X:

$$Th(X) = \{A \in W(U) \mid t(A) = 1, \text{ for all } t \in X\}$$

In fact, every theory is of the above form, that is, *for every theory* $\mathscr{K}$ *there exists a set of valuations* X *(depending upon* $\mathscr{K}$*) such that* $\mathscr{K} = Th(X)$. Indeed, take

$$X_{\mathscr{K}} = \{t\colon W(U) \to \mathbf{2} \mid t \text{ valuation with } t(A) = 1, \text{ for all } A \in \mathscr{K}\}$$

and note that

$$\begin{aligned} Th(X_{\mathscr{K}}) &= \{B \in W(U) \mid t(B) = 1, \text{ for all } t \in X_{\mathscr{K}}\} \\ &= \{B \in W(U) \mid t(B) = 1, \text{ for every valuation with } t(A) = 1, \\ &\qquad \text{for all } A \in \mathscr{K}\} \\ &= Con(\mathscr{K}) = \mathscr{K} \end{aligned}$$

In other words, *theories are those sets of propositions which are true under a certain set of valuations (interpretations).*

Let now $\mathcal{T}$ be a theory. Two elements $p, q \in U$ are $\mathcal{T}$-equivalent, written $p \equiv_{\mathcal{T}} q$, in case $p \leftrightarrow q \in \mathcal{T}$. The relation, $\equiv_{\mathcal{T}}$ is an equivalence relation. The equivalence class of p is $[p]_{\mathcal{T}} = \{q \in U \mid p \equiv_{\mathcal{T}} q\}$ and the factor set is denoted by $U_{\equiv_{\mathcal{T}}}$; for brevity, we sometimes write $[p]$ instead of $[p]_{\mathcal{T}}$. The factor set comes with a natural partial order:

$$[p] \leqslant [q] \qquad \text{if} \quad p \rightarrow q \in \mathcal{T}$$

Note that in general, $(U_{\equiv_{\mathcal{T}}}, \leqslant)$ is not a Boolean algebra.[10]

In a similar way we can define the $\equiv_{\mathcal{T}}$-equivalence of two propositions:

$$A \equiv_{\mathcal{T}} B \qquad \text{if} \quad A \leftrightarrow B \in \mathcal{T}$$

Denote by $[[A]]_{\mathcal{T}}$ (shortly, $[[A]]$) the equivalence class of A and note that for every $p \in U$,

$$[p] = [[p]] \cap U$$

The resulting Boolean algebra $W(U)_{\equiv_{\mathcal{T}}}$ is the Lindenbaum algebra of $\mathcal{T}$.

Fix now an atomic orthocomplemented lattice $(L, \leqslant, 0, 1, ')$ satisfying (7). Let U be a set of cardinality greater or equal to L and fix a surjective mapping $f\colon U \rightarrow L$. For every atom $a \in L$, let $s_a\colon U \rightarrow \mathbf{2}$ be the assignment defined by $s_a(p) = 1$ iff $a \leqslant f(p)$. Take

$$X = \{t_{s_a} \mid a \text{ is an atom of } L\}^{11} \qquad \text{and} \qquad Th(X)$$

Malhas[29, 30] has proven that the *lattice* $(U_{\equiv_{\mathcal{T}}}, \leqslant)$ *is orthocomplemented*, and, in fact, *isomorphic to* L. Here is the argument. First, note that there exist two elements 0, 1 in U such that $f(0) = 0$, $f(1) = 1$. Clearly, $0 \notin \mathcal{T}$, but $1 \in \mathcal{T}$. Indeed, for every atom a, $a \leqslant f(1) = 1$, so $s_a(1) = 1$, a.s.o.

Second, for every $p, q \in U$,

$$p \rightarrow q \in \mathcal{T} \qquad \text{iff} \quad f(p) \leqslant f(q)$$

If $p \rightarrow q \notin \mathcal{T}$, then there exists an atom $a \in L$ such that $t_{s_a}(p \rightarrow q) = 0$, so $s_a(p) = t_{s_a}(p) = 1$, $s_a(q) = t_{s_a}(q) = 0$, which according to the definition of s_a, mean $a \leqslant f(p)$, but $a \nleqslant f(q)$. If $f(p) \leqslant f(q)$, then $a \leqslant f(q)$, a contradiction. Conversely, if $f(p) \nleqslant (q)$, then by (7) there exists an atom a such

[10] For instance, in case $\mathcal{T} = Con(\{p\})$, for some $p \in U$. If U has at least three elements, then $(U_{\equiv_{\mathcal{T}}}, \leqslant)$ does not have a minimum.

[11] Recall that t_s is the unique valuation extending s.

that $a \leqslant f(p)$ and $a \nleqslant f(q)$. So $s_a(p) = t_{s_a}(p) = 1$, $s_a(q) = t_{s_a}(q) = 0$, i.e., $(p \to q) \notin \mathcal{T}$.

As immediate consequences we deduce the validity of the following three relations: for all $p, q \in U$,

- $f(p) \leqslant f(q)$ iff $[p] \leqslant [q]$,
- $f(p) = f(q)$ iff $[p] = [q]$, and
- $[0] \leqslant [p] \leqslant [1]$.

Two simple propositions $p, q \in U$ are *conjugate* in case $f(p)' = f(q)$.[12] Define now the operation $*$: $U_{\mathcal{T}} \to U_{\mathcal{T}}$ as follows: $[p]^* = [q]$ in case q is a conjugate of p. It is not difficult to see that the operation $*$, is well defined and actually is an orthocomplementation. It follows that $(U_{\mathcal{T}}, \leqslant_{\mathcal{T}}, *)$ is an orthocomplemented lattice.

To finish the argument we show that this lattice is *isomorphic* with L. The isomorphism is given by the mapping ψ: $U_{\mathcal{T}} \to L$ defined by the formula $\psi([p]) = f(p)$. This is a well-defined function (because $f(p) = f(q)$ iff $[p] = [q]$), which is bijective ($\psi([p]) = \psi([q])$ implies $f(p) = f(q)$, and surjective because f is onto). If $[p] < [q]$, then $f(p) \leqslant f(q)$, i.e., $\psi([p]) \leqslant \psi([q])$. Finally, if q is a conjugate of p, then

$$\psi([p]^*) = \psi([q]) = f(q) = f(p)' = \psi([p])'$$

In particular, *there exists a theory whose induced orthoposet is isomorphic to the lattice of all closed subspaces of a separable Hilbert space.* How does this relate to the Kochen–Specker theorem? *The natural embedding*

$$\Gamma: U_{\equiv_{\mathcal{T}}} \to W(U)_{\equiv_{\mathcal{T}}}, \qquad \Gamma([p]) = [[p]]$$

is order preserving and one-to-one, but in general it does not preserve orthocomplementation, i.e., in general, $\Gamma([p]^*) \neq \Gamma([p])'$. We always have $\Gamma([p]^*) \leqslant \Gamma([p])'$, but sometimes $\Gamma([p])' \nleqslant \Gamma([p]^*)$. The reason is that for every pair of conjugate simple propositions p, q one has $(p \to q') \in \mathcal{T}$, but the converse is not true.

By combining the inverse ψ^{-1} of the isomorphism ψ with Γ, we obtain an embedding φ of L into the Boolean Lindenbaum algebra $W(U)_{\equiv_{\mathcal{T}}}$. Thus, the above construction of Malhas gives us another method *to embed any quantum logic in a Boolean logic in case we require that only the order is preserved.*[13]

[12] Of course, this relation is symmetrical.

[13] In Section 2.4 we saw that it is possible to embed quantum logic into a Boolean logic preserving the order and the complement.

Next we give a simple example of a Malhas type embedding $\varphi\colon MO_2 \to \mathbf{2}^4$. Consider again the finite quantum logic MO_2 represented in Fig. 1. Let us choose

$$U = \{A, B, C, D, E, F, G, H\}$$

Since U contains more elements than MO_2, we can map U surjectively onto MO_2; e.g.,

$$f(A) = 0$$
$$f(B) = p_-$$
$$f(C) = p_-$$
$$f(D) = p_+$$
$$f(E) = q_-$$
$$f(F) = q_+$$
$$f(G) = 1$$
$$f(H) = 1$$

For every atom $a \in MO_2$, let us introduce the truth assignment $s_a\colon U \to \mathbf{2} = \{0, 1\}$ as defined above [i.e., $s_a(r) = 1$ iff $a \leqslant f(r)$] and thus a valuation on $W(U)$ separating it from the rest of the atoms of MO_2. That is, for instance, associate with $p_- \in MO_2$ the function s_{p_-} as follows:

$$s_{p_-}(A) = s_{p_-}(D) = s_{p_-}(E) = s_{p_-}(F) = 0$$
$$s_{p_-}(B) = s_{p_-}(C) = s_{p_-}(G) = s_{p_-}(H) = 1$$

The truth assignments associated with all the atoms are listed in Table I.

Table I. Truth Assignments on U Corresponding to Atoms $p_-, p_+, q_-, q_+ \in MO_2$

	A	B	C	D	E	F	G	H
s_{p_-}	0	1	1	0	0	0	1	1
s_{p_+}	0	0	0	1	0	0	1	1
s_{q_-}	0	0	0	0	1	0	1	1
s_{q_+}	0	0	0	0	0	1	1	1

The theory $\mathscr{T}$ we are thus dealing with is determined by the union of all the truth assignments; i.e.,

$$X = \{t_{s_{p-}}, t_{s_{p+}}, t_{s_{q-}}, t_{s_{q+}}\} \qquad \text{and} \qquad \mathscr{T} = Th(X)$$

The way it was constructed, U splits into six equivalence classes with respect to the theory $\mathscr{T}$; i.e.,

$$U_{\equiv_{\mathscr{T}}} = \{[A], [B], [D], [E], [F], [G]\}$$

Since $[p] \to [q]$ if and only if $(p \to q) \in \mathscr{T}$, we obtain a partial order on $U_{\equiv_{\mathscr{T}}}$ induced by T which isomorphically reflects the original quantum logic MO_2. The Boolean Lindenbaum algebra $W(U)_{\equiv_{\mathscr{T}}} = \mathbf{2}^4$ is obtained by forming all the compound propositions of U and imposing a partial order with respect to $\mathscr{T}$. It is represented in Fig. 4. The embedding is given by

$$\varphi(0) = [[A]]$$
$$\varphi(p_-) = [[B]]$$
$$\varphi(p_+) = [[D]]$$
$$\varphi(q_-) = [[E]]$$
$$\varphi(q_+) = [[F]]$$
$$\varphi(1) = [[G]]$$

It is order preserving but does not preserve operations such as the complement. Although, in this particular example, $f(B) = (f(D))'$ implies $(B \to D') \in \mathscr{T}$, the converse is not true in general. For example, there is no $s \in X$ for which $s(B) = s(E) = 1$. Thus, $(B \to E') \in T$, but $f(B) \neq (f(E))'$.

One needs not be afraid of order-preserving embeddings which are no lattice morphisms, after all. Even automaton logics [see Refs. 8, 39–41, and 47 (Chap. 11)] can be embedded in this way. Take, again, the lattice MO_2 depicted in Fig. 1. A partition (automaton) logic realization is, for instance,

$$\{\{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}\}$$

with

$$\{1\} \equiv p_-$$
$$\{2, 3\} \equiv p_+$$
$$\{2\} \equiv q_-$$
$$\{1, 3\} \equiv q_+$$

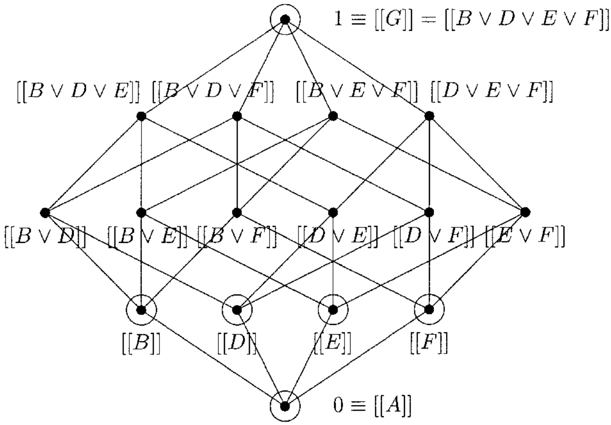

Fig. 4. Hasse diagram of an embedding of the quantum logic MO_2 represented by Fig. 1. Concentric circles indicate the embedding.

respectively. If we take $\{1\}$, $\{2\}$, and $\{3\}$ as atoms, then the Boolean algebra $\mathbf{2}^3$ generated by all subsets of $\{1, 2, 3\}$ with the set theoretic inclusion as order relation suggests itself as a candidate for an embedding. The embedding is quite trivially given by

$$\varphi(p) = p \in \mathbf{2}^3$$

The particular example considered above is represented in Fig. 5. It is not difficult to check that the embedding satisfies requirements (i) and (ii), that is, it is injective and order preserving.

It is important to realize at this point that, although different automaton partition logical structures may be isomorphic from a logical point of view (one-to-one translatable elements, order relations, and operations),

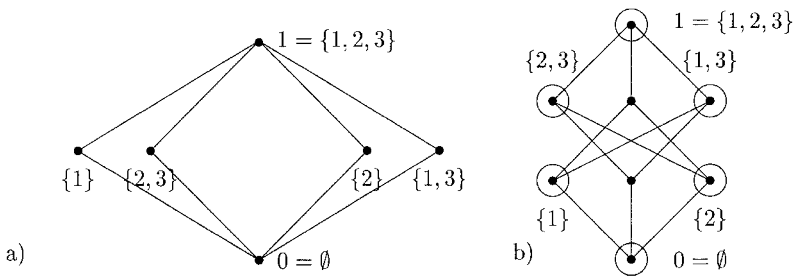

Fig. 5. Hasse diagram of an embedding of MO_2 (a) into $\mathbf{2}^3$ (b). Concentric circles indicate points of $\mathbf{2}^3$ included in MO_2.

Table II. The Four Valuations s_1, s_2, s_3, and s_4 on MO_2 Take on the Values Listed in the Rows

	p_-	p_+	q_-	q_+
s_1	1	0	1	0
s_2	1	0	0	1
s_3	0	1	1	0
s_4	0	1	0	1

they may be very different with respect to their embeddability. Indeed, any two distinct partition logics correspond to two distinct embeddings.

It should also be pointed out that, in the case of an automaton partition logic and for all finite subalgebras of the Hilbert lattice of two-dimensional Hilbert space, it is always possible to find an embedding corresponding to a logically equivalent partition logic which is a lattice morphism for comeasurable elements [modified requirement (iii)]. This is due to the fact that partition logics and MO_n have a separating set of valuations. In the MO_2 case, this is, for instance

$$\{\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}\}$$

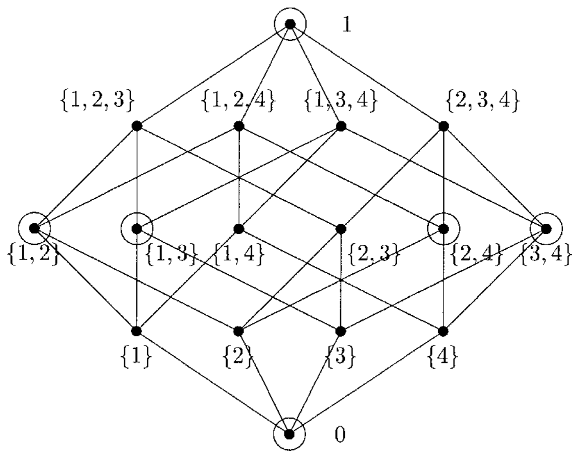

Fig. 6. Hasse diagram of an embedding of the partition logic $\{\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}\}$ into $\mathbf{2}^4$ preserving ortholattice operations among comeasurable propositions. Concentric circles indicate the embedding.

with

$$\{1,2\} \equiv p_-$$
$$\{3,4\} \equiv p_+$$
$$\{1,3\} \equiv q_-$$
$$\{2,4\} \equiv q_+$$

respectively. This embedding is based upon the set of all valuations listed in Table II These are exactly the mappings from MO_2 to **2** preserving the order relation and the complementation. They correspond to the maximal ideals considered in Sec. 2.3. In this special case the embedding is just the embedding obtained by applying the construction of Sec. 2.3, which had been suggested by Zierler and Schlessinger (Ref. 52, Theorem 2.1). The embedding is drawn in Fig. 6.

3. SURJECTIVE EXTENSIONS?

The original proposal put forward by EPR[9] in the last paragraph of their paper was some form of completion of quantum mechanics. Clearly, the first type of candidate for such a completion is the sort of embedding reviewed above. The physical intuition behind an embedding is that the "actual physics" is a classical one, but because of some yet unknown reason, some parts of this "hidden arena" becomes observable while others remain hidden.

Nevertheless, there exists at least one other alternative to complete quantum mechanics. This is best described by a *surjective map* $\phi: B \to L$ of a classical Boolean algebra onto a quantum logic, such that $|B| \geqslant |L|$.

Plato's cage metaphor applies to both approaches, in that observations are mere shadows of some more fundamental entities.

4. SUMMARY

We have reviewed several options for a classical "understanding" of quantum mechanics. Particular emphasis has been given to techniques for embedding quantum universes into classical ones. The term "embedding" is formalized here as usual. That is, an embedding is a mapping of the entire set of quantum observables into a (bigger) set of classical observables such that different quantum observables correspond to different classical ones (infectivity).

The term "observables" here is used for quantum propositions, some of which (the complementary ones) might not be comeasurable (see Ref. 14). It might therefore be more appropriate to conceive these "observables" as "potential observables." After a particular measurement has been chosen, some of these observables are actually determined and others (the complementary ones) become "counterfactuals" by quantum mechanical means [cf. Schrödinger's catalogue of expectation values (Ref. 42, p. 823)]. For classical observables, there is no distinction between "observables" and "counterfactuals," because everything can be measured precisely, at least in principle.

We should mention also a *caveat*. The relationship between the states of a quantum universe and the states of a classical universe into which the former one is embedded is beyond the scope of this paper.

As might have been suspected, it turns out that, in order to be able to perform the mapping from the quantum universe into the classical one consistently, important structural elements of the quantum universe have to be sacrificed.

- Since *per definition*, the quantum propositional calculus is nondistributive (nonboolean), a straightforward embedding which preserves all the logical operations among observables, irrespective of whether or not they are comeasurable, is impossible. This is due to the quantum mechanical feature of *complementarity* .

- One may restrict the preservation of the logical operations to be valid only among mutually orthogonal propositions. In this case it turns out that, again, a consistent embedding is impossible, since no consistent meaning can be given to the classical existence of "counterfactuals." This is due to the quantum mechanical feature of *contextuality*. That is, quantum observables may appear different, depending on the way in which they were measured (and inferred).

- In a further step, one may abandon preservation of lattice operations such as *not* and the binary *and* and *or* operations altogether. One may merely require the preservation of the implicational structure (order relation). It turns out that, with these provisos, it is indeed possible to map quantum universes into classical ones. Stated differently, definite values can be associated with elements of physical reality, irrespective of whether they have been measured or not. In this sense, that is, in terms of more "comprehensive" classical universes (the hidden parameter models), quantum mechanics can be "understood."

At the moment we can neither say if the nonpreservation of the binary lattice operations (interpreted as *and* and *or*) is too high a price for value definiteness nor speculate whether or not the entire program of embedding quantum universes in classical theories is a progressive or a degenerative case (compare Ref. 27).

APPENDIX A: PROOF OF THE GEOMETRIC LEMMA

In this appendix we prove the geometric lemma due to Piron[36] which was formulated in Sec. 2.2. First let us restate it. Consider a point q in the northern hemisphere of the unit sphere $S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$. By $C(q)$ we denote the unique great circle which contains q and the points $\pm (q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$, in the equator, which are orthogonal to q, (cf. Fig. 2). We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. The lemma states:

> *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

For the proof we follow Cooke *et al.*[7] and Kalmbach.[24] We consider the tangent plane $H = \{p \in \mathbb{R}^3 \mid p_z = 1\}$ of the unit sphere in the north pole and the projection h from the northern hemisphere onto this plane which maps each point $q2$ in the northern hemisphere to the intersection $h(q)$ of the line through the origin and q with the plane H. This map h is a bijection. The north pole $(0, 0, 1)$ is mapped to itself. For each q in the northern hemisphere (not equal to the north pole), the image $h(C(q))$ of the great circle $C(q)$ is the line in H which goes through $h(q)$ and is orthogonal to the line through the north pole and through $h(q)$. Note that $C(q)$ is the intersection of a plane with S^2, and $h(C(q))$ is the intersection of the same plane with H (see Fig. 7). The line $h(C(q))$ divides H into two half-planes. The half-plane not containing the north pole is the image of the region in the northern hemisphere between $C(q)$ and the equator. Furthermore, note that $q_z > p_z$ for two points in the northern hemisphere if and only if $h(p)$ is further away from the north pole than $h(q)$. We proceed in two steps.

Step 1. First, we show that, if p and q are points in the northern hemisphere and p lies in the region between $C(q)$ and the equator, then p can be reached from q. In fact, we show that there is a point $\tilde{q}$ on $C(q)$ such that p lies on $C(\tilde{q})$. Therefore we consider the images of q and p in the plane H (see Fig. 8). The point $h(p)$ lies in the half-plane bounded by $h(C(q))$ not containing the north pole. Among all points $h(q')$ on the line $h(C(q))$, we set $\tilde{q}$ to be one of the two points such that the line trough the,

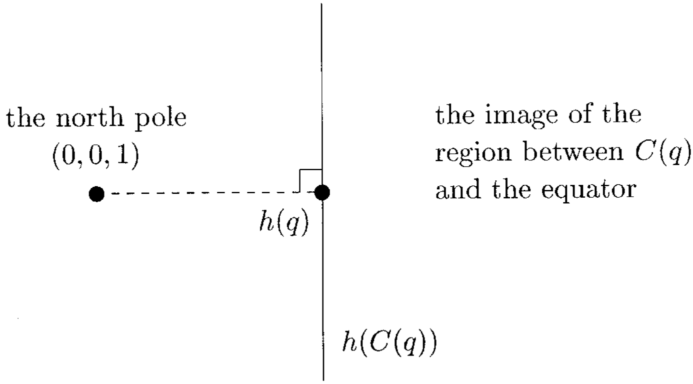

Fig. 7. The plane H viewed from above.

north pole and $h(q')$ and the line through $h(q')$ and $h(p)$ are orthogonal. Then this last line is the image of $C(\tilde{q})$, and $C(\tilde{q})$ contains the point p. Hence p can be reached from q. Our first claim is proved.

Step 2. Fix a point q in the northern hemisphere. Starting from q we can wander around the northern hemisphere along great circles of the form $C(p)$ for points p in the following way: for $n \geqslant 5$ we define a sequence $q_0, q_1, \ldots, q_n$ by setting $q_0 = q$ and by choosing q_{i+1} to be that point on the great circle $C(q_i)$ such that the angle between $h(q_{i+1})$ and $h(q_i)$ is $2\pi/n$. The image in H of this configuration is a shell where $h(q_n)$ is the point farthest away from the north pole (see Fig. 9). First, we claim that any point p on the unit sphere with $p_z < q_{n_z}$ can be reached from q. Indeed, such a point

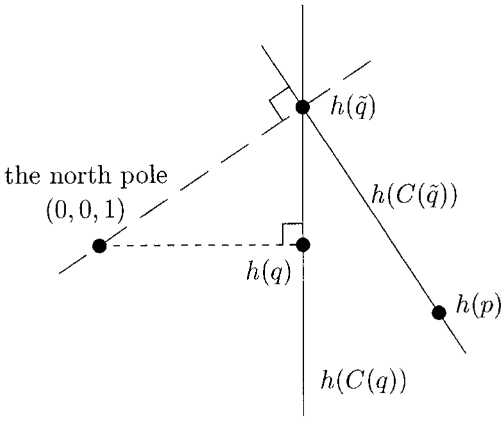

Fig. 8. The point p can be reached from q.

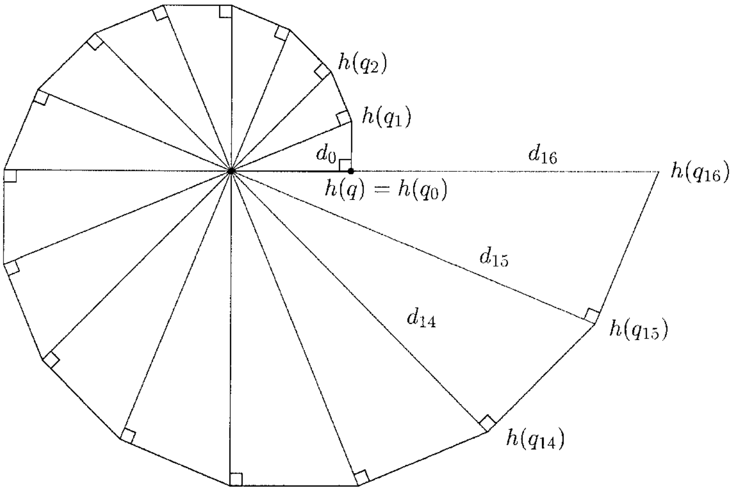

Fig. 9. The shell in the plane H for $n = 16$.

corresponds to a point $h(p)$ which is farther away from the north pole than $h(q_n)$. There is an index i that $h(p)$ lies in the half-plane bounded by $h(C(q_i))$ and not, containing the north pole, hence such that p lies in the region between $C(q_i)$ and the equator. Then, as we have already seen, p can be reached from q_i and hence also from q. Second, we claim that q_n approaches q as n tends to infinity. This is equivalent to showing that the distance of $h(q_n)$ from $(0, 0, 1)$ approaches the distance of $h(q)$ from $(0, 0, 1)$. Let d_i denote the distance of $h(q_i)$ from $(0, 0, 1)$ for $i = 0, \ldots, n$. Then $d_i/d_{i+1} = \cos(2\pi/n)$ (see Fig. 9). Hence $d_n = d_0 \cdot (\cos(2\pi/n))^{-n}$. That d_n approaches d_0 as n tends to infinity follows immediately from the fact that $(\cos(2\pi/n))^n$ approaches 1 as n tends to infinity. For completeness' sake[14] we prove it by proving the equivalent statement that $\log((\cos(2\pi/n))^n)$ tends to 0 as n tends to infinity. Namely, for small x we know the formulae $\cos(x) = 1 - x^2/2 + \mathcal{O}(x^4)$ and $\log(1+x) = x + \mathcal{O}(x^2)$. Hence, for large n,

$$\log((\cos(2\pi/n))^n) = n \cdot \log\left(1 - 2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})\right)$$

$$= n \cdot \left(-2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})\right) = -\frac{2\pi^2}{n} + \mathcal{O}(n^{-3})$$

This ends the proof of the geometric lemma.

[14] Actually, this is an exercise in elementary analysis.

APPENDIX B: PROOF OF A PROPERTY OF THE SET OF CONSEQUENCES OF A THEORY

In Section 2.5 we introduced the set $Con(\mathscr{K})$ of consequences of a set $\mathscr{K}$ of propositions over a set U of *simple propositions* and the logical connectives negation $'$ and implication $\rightarrow$. We mentioned four properties of the operator *Con*. In this appendix we prove the fourth property:

$$Con(\mathscr{K}) = \bigcup_{\{X \subseteq \mathscr{K},\ X \text{ finite}\}} Con(X)$$

The inclusion $Con(\mathscr{K}) \supseteq \bigcup_{\{X \subseteq \mathscr{K},\ X \text{ finite}\}} Con(X)$ follows directly from the second property of *Con*, i.e., from the monotonicity: if $X \subseteq \mathscr{K}$, then $Con(X) \subseteq Con(\mathscr{K})$. For the other inclusion we assume that a proposition $A \in Con(\mathscr{K})$ is given. We have to show that there exists a finite subset $X \subseteq \mathscr{K}$ such that $A \in Con(X)$.

In order to do this we consider the set $\mathscr{V}(W(U))$ of all valuations. This set can be, identified with the power set of U and viewed as a topological space with the product topology of $|U|$ copies of the discrete topological space $\{0, 1\}$. By Tychonoff's theorem (see Ref. [33]) $\mathscr{V}(W(U))$ is a compact topological space. For an arbitrary proposition B and valuation t, the set $\{t \in \mathscr{V}(W(U)) \mid t(B) = 0\}$ of valuations t with $t(B) = 0$ is a compact and open subset of valuations because the value $t(B)$ depends only on the finitely many simple propositions occurring in B.

Note that our assumption $A \in Con(\mathscr{K})$ is equivalent to the inclusion

$$\{t \in \mathscr{V}(W(U)) \mid t(A) = 0\} \subseteq \bigcup_{B \in \mathscr{K}} \{t \in \mathscr{V}(W(U)) \mid t(B) = 0\}$$

Since the set on the left-hand side is compact, there exists a finite subcover of the open cover on the right-hand side, i.e., there exists a finite set $X \subseteq \mathscr{K}$ with

$$\{t \in \mathscr{V}(W(U)) \mid t(A) = 0\} \subseteq \bigcup_{B \in X} \{t \in \mathscr{V}(W(U)) \mid t(B) = 0\}$$

This is equivalent to $A \in Con(X)$.

ACKNOWLEDGMENTS

This paper was completed during the visits of the first author at the University of Technology Vienna (1997) and of the third author at the

University of Auckland (1997). The first author was partially supported by AURC A18/XXXXX/62090/F3414056, 1996. The second author was supported by DFG Research Grant HE 2489/2-1. The authors thank the anonymous referees for their extremely helpful suggestions and comments, leading to a better form of the paper.

REFERENCES

1. J. L. Bell and R. K. Clifton, *Int. J. Theor. Phys.* **43**, 2409–2421 (1995).
2. J. S. Bell, *Rev. Modern Phys.* **38**, 447–452 (1966). Reprinted in Ref. 3, pp. 1–13.
3. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).
4. C. H. Bennett, *Nature* **371**, 479–480 (1994).
5. G. Birkhoff and J. von Neumann, *Ann. Math.* **37**(4), 823–843 (1936).
6. R. Clifton, *Am. J. Phys.* **61**, 443–447 (1993).
7. R. Cooke, M. Keane, and W. Moran, *Math. Proc. Cambr. Soc.* **98**, 117–128 (1985).
8. A. Dvurečenskij, S. Pulmannová, and K. Svozil, *Helv. Phys. Acta* **68**, 407–428 (1995).
9. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777–780 (1935). Reprinted in Ref. 51, pp. 138–141.
10. R. P. Feynman, *The Character of Physical Law* (MIT Press, Cambridge, MA, 1965).
11. A. M. Gleason, *J. Math. Mech.* **6**, 885–893 (1957).
12. D. B. Greenberger, M. Horne, and A. Zeilinger, *Phys. Today* **46**, 22–29 (1993).
13. D. B. Greenberger and A. YaSin, *Found. Phys.* **19**(6), 649–704 (1989).
14. S. P. Gudder, *J. Math. Phys.* **11**, 431–436 (1970).
15. H. Harding, personal communication to K. Svozil, March (1998).
16. J. Harding, *Order* **8**, 93–103 (1991).
17. H. Havlicek and K. Svozil, *Math. Phys.* **37**(11), 5337–5341 (1996).
18. T. J. Herzog, P. G. Kwiat, H. Weinfurter, and A. Zeilinger, *Phys. Rev. Lett.* **75**, 3034–3037 (1995).
19. M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw–Hill, New York, 1966).
20. M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New-York, 1974).
21. J. M. Jauch, *Foundations of Quantum Mechanics*, (Addison–Wesley, Reading, MA, 1968).
22. G. Kalmbach, *Arch. Math.* **28**, 7–8 (1977).
23. G. Kalmbach, *Orthomodular Lattices* (Academic, New York, 1983).
24. G. Kalmbach, *Measures and Hilbert Lattices* (World Scientific, Singapore, 1985).
25. F. Katrnoška, *Comment. Math. Univ. Carolinae* **23**, 489–498 (1982).
26. S. Kochen and E. P. Specker, *J. Math. Mech.* **17**(1), 59–87 (1967). Reprinted in Ref. 44, pp. 235–263.
27. I. Lakatos, *Philosophical Papers. 1. The Methodology of Scientific Research Programmes* (Cambridge University Press, Cambridge, 1978).
28. G. W. Mackey, *Am. Math. Month Suppl.* **64**, 45–57 (1957).
29. O. Q. Malhas, *J. Symbol. Logic* **52**(3), 837–841 (1987).
30. O. Q. Malhas, *Int. J. Theor. Phys.* **31**(9), 1699–1712 (1992).
31. R. Mayet and M. Navara, in *Current Issues in Quantum Logic*, G. Pilz, ed. (Teubner, Stuttgart, Wien, 1995), pp. 241–248.
32. N. D. Mermin, *Rev. Mod. Phys.* **65**, 803–815 (1993).

33. J. R. Munkres, *Topology. A First Course* (Prentice–Hall, Englewood Cliffs, NJ, 1975).
34. A. Peres, *J. Phys. A* **24**, L175–L178 (1991). Cf. Ref. 35, pp. 186–200.
35. A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).
36. C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, MA, 1976).
37. P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics* (Kluwer Academic, Dordrecht, 1991).
38. M. Redhead, *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics* (Clarendon Press, Oxford, 1990).
39. M. Schaller and K. Svozil, *Nuovo Cimento B* **109**, 168–176 (1994).
40. M. Schaller and K. Svozil, *Int. J. Theor. Phys.* **34**(8), 1741–1750 (1995).
41. M. Schaller and K. Svozil, *Int. J. Theor; Phys.* **35**(5), 911–940 (1996).
42. E. Schrödinger, *Naturwissenschaften* **23**, 807–812, 823–828, 844–849 (1935). English translation in Ref. 49 and Ref. 51, pp. 152–167.
43. E. Specker, *Dialectica* **14**, 175–182 (1960). (in German). Reprinted in Ref. 44, pp. 175–182.
44. E. Specker, *Selecta* (Birkhäuser, Basel, 1990).
45. M. H. Stone, *Trans. Am. Math. Soc.* **40**, 37–111 (1936).
46. K. Svozil, *Quantum Logic* (Springer, Singapore, 1998).
47. K. Svozil, *Randomness & Undecidability in Physics* (World Scientific, Singapore, 1993).
48. K. Svozil and J. Tkadlec, *J. Math. Phys.* **37**(11), 5380–5401 (1996).
49. D. J. Trimmer, *Proc. Am. Phil. Soc.* **124**, 323–338 (1980). Reprinted in Ref. 51, pp. 152–167.
50. J. A. Wheeler, in *Quantum Theory and Measurement*, J. A. Wheeler and W. H Zurek, eds. (Princeton University Press, Princeton, NJ, 1983), pp. 182–213. See Ref. 51.
51. J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, NJ, 1983).
52. N. Zierler and M. Schlessinger, *Duke Math. J.* **32**, 251–262 (1965).