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A computational attack on the conjectures of Graffiti: New counterexamples and proofs

Tony L. Brewster, Michael J. Dinneen*, Vance Faber

Computer Research and Applications, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

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Abstract

Graffiti is a computer program that checks for relationships among certain graph invariants. It uses a database of graphs and has generated well over 700 conjectures. Having obtained a readily available computer tape of all the nonisomorphic graphs with 10 or fewer vertices, we have tested approximately 200 of the Graffiti conjectures and have found counterexamples for over 40 of them. For each conjecture that failed we display a counterexample. We also provide results that came from analyzing those conjectures which had a small number of counterexamples. Finally, we prove some results about four of the conjectures.

1. Introduction

One of the diffculties of obtaining a new result in graph theory has been finding the statement of the result. The situation may soon change as computers are now capable of generating interesting mathematical conjectures.

A conjecture generating program, called Graffiti, was developed by Siemion Fajtlowicz in 1986 (see "Written on the Wall" [5]). It uses a database of graphs and heuristically checks for relationships among certain graph invariants. The main task of the program is to decide which of these relationships should be accepted as conjectures. Out of all the conjectures generated by Graffiti, more than 700 of them have been included in [5], and many of these have created considerable mathematical interest [6–9, 11].

Having obtained a readily available computer tape of all the nonisomorphic graphs with 10 or fewer vertices (see [3]), we have tested approximately 200 of the Graffiti conjectures and have found counterexamples for over 40 of them. For each of the failed conjectures we present a counterexample, the values of the invariants, and the number of counterexamples. The curious reader should consult the discussion in

* Corresponding author.

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the latest version of "Written on the Wall" for conjectures that have passed our brute-force test and still remain open.

We begin in the next section by focusing on those conjectures with less than 10 counterexamples. In certain groups of these counterexamples a pattern is apparent. Consequently, in Section 3 we construct an infinite class of counterexamples for four of the conjectures: 107, 293, 551, and 581. In Section 4 we give one counterexample for each of the remaining failed conjectures. Next in Section 5, we provide various results about four of Graffiti's conjectures.

For the reader's convenience, regarding some of the Graffiti's nonstandard graph theory terms, we provide an extensive glossary near the end of this paper.

2. Conjectures with less than 10 counterexamples

Those conjectures in which only a small number of counterexamples were found have been emphasized since it is natural to question if they are true for all but a finite number of graphs. There were a total of 18 conjectures for which we found less than 10 counterexamples within the over 12 million nonisomorphic graphs with 10 or fewer vertices. (The cutoff of 10 counterexamples was chosen quite arbitrarily.)

Eleven of Graffiti's conjectures failed for only one graph. Surprisingly, six of these eleven conjectures failed for the same graph: K_2, the complete graph on two vertices. By serendipity of Graffiti's database not included K_2, our partial verification gives strong evidence that those conjectures that failed only for K_2 can be made true by simply adding the condition 'of order at least 3'.

Another popular counterexample was P_6, the path on six vertices. It appeared as a counterexample to four conjectures with less than 10 counterexamples, which include two conjectures in which it was the only counterexample found.

We first list the six conjectures that failed only for K_2, the two conjectures that failed only for P_6, the other three conjectures for which we found only one counterexample, and then each of the remaining conjectures with fewer than 10 counter examples.

To help the reader verify the counterexamples, Table 1 contains the value of the terms associated with the *first* counterexample given for each of the conjectures in this section. For example, a reference to Conjecture 107 of Section 2.4 and to Table 1 reveals that the first graph listed as a counterexample has mode of distance matrix of 3, radius of 2, and is even regular[1] of degree 5.

2.1. Conjectures with K_2 as the only counterexample found

Conjecture 179. The inverse transmission of gravity matrix $\leqslant$ the average transmission of gravity matrix (connected graphs).

[1] The definition of the term *even regular* maybe misleading — see glossary.

Table 1
The value of the terms (invariants) associated with the first counterexample given for each of the conjectures with less than 10 counterexamples

Conjecture	Number of counterexamples	1st term	2nd term	3rd term	4th term
107	3	3	2	5	
165	6	3	6	2.0667	
166	3	5.2915	5		
179	1	2	1		
293	8	1.1691	8	7	∞
306	1	4.0405	4	9	10
353	1	3.0202	3	8	8
551	5	6	5.9286		
581	6	10	4	5	
596	1	3	2		
606	1	1	1	0	
610	1	2	1.4142		
612	1	1.4142	2	1	
613	1	1.4142	1		
615	1	2	1	1.4142	
665	3	2.0667	2	18	18
670	1	3	2	18	18
725	1	6	2	0.6180	4

Conjecture 606. Size/independence $\leqslant$ the secnd largest eigenvalue of Laplacian (connected triangle-free graphs).

Conjecture 610. The maximum eigenvalue of Laplacian $\leqslant$ the length of dual-degree sequence (connected triangle-free graphs).

Conjecture 612. The length of 1-residue vector $\leqslant$ order $-$ independence.

Conjecture 613. The length of 1-residue vector $\leqslant$ harmonic.

Conjecture 615. Order/average distance $\leqslant$ length of degree sequence (connected triangle-free graphs).

2.2. *Conjectures with P_6 as the only counterexample found*

Conjecture 596. The radius $\leqslant$ the maximal frequency of mid-degree sequence (triangle-free graphs).

Conjecture 670. The radius $\leqslant$ the frequency of the mode of mid-degree sequence (graphs with sum of odd vector $\leqslant$ sum of even vector).

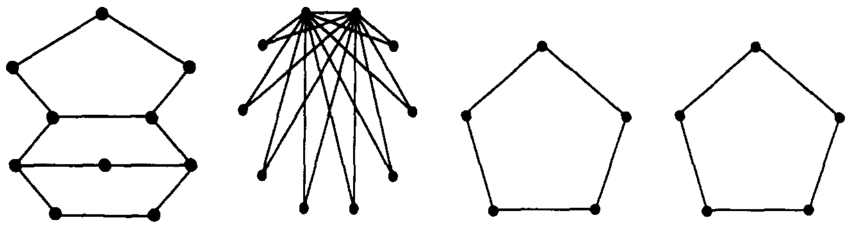

Fig. 1. Conjectures 306, 353 and 725.

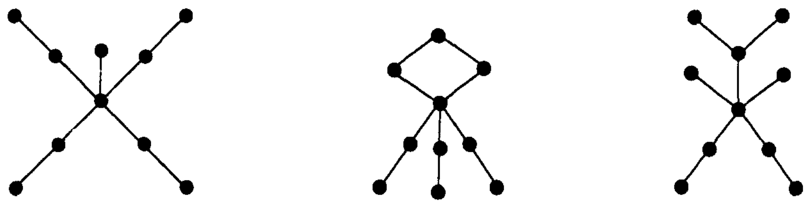

Fig. 2. Conjecture 107.

2.3. Other conjectures where only one counterexample was found (see Fig. 1)

Conjecture 306. The inverse of dual-degree sequence $\leqslant$ the number of nonpositive eigenvalues (graphs with distance rank < rank).

Conjecture 353. The harmonic $\leqslant$ rank (heliotropic plants).

Conjecture 725. The number of nonnegative eigenvalues − largest eigenvalue + smallest nonnegative eigenvalue $\leqslant$ independence.

2.4. Other conjectures with less than 10 counterexamples

Conjecture 107. The mode of distance matrix $\leqslant$ radius (even regular graphs); see Fig. 2.

Conjecture 165. The mode of the eigenvalues of Laplacian $\leqslant$ size/average distance; see Fig. 3.

Conjecture 166. The square root of size $\leqslant$ the number of nonpositive eigenvalues of distance matrix; see Fig. 4.[2]

[2] The plus symbol denotes the join of two graphs, where the join of graphs G and H is formed by adding an edge between each vertex of G and each vertex of H.

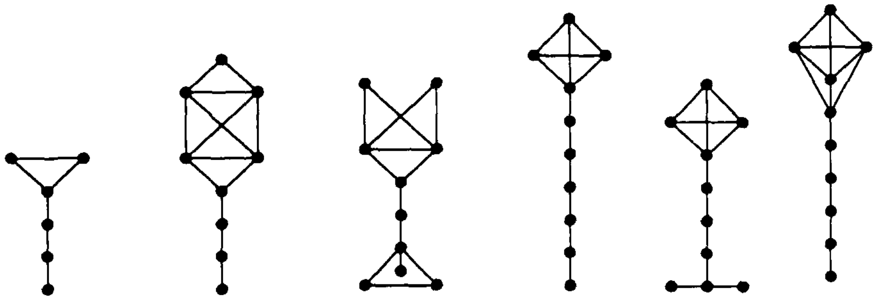

Fig. 3. Conjecture 165.

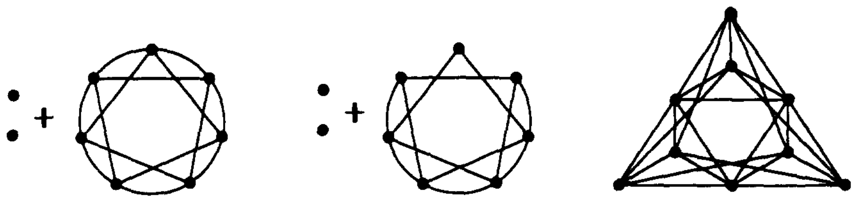

Fig. 4. Conjecture 166.

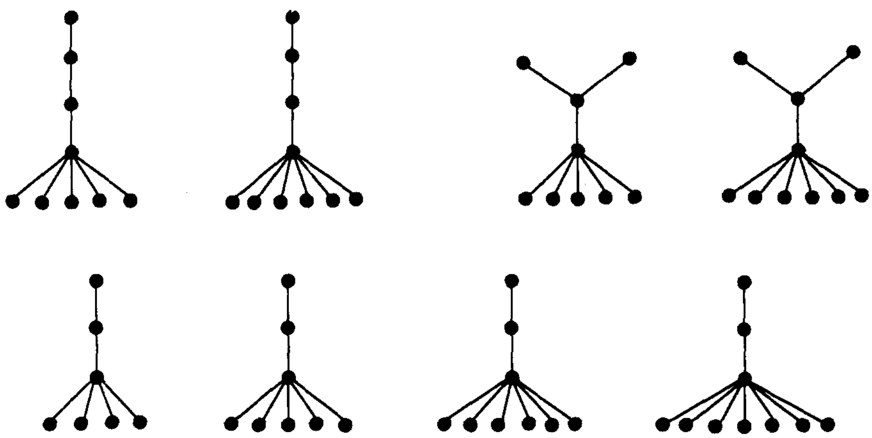

Fig. 5. Conjecture 293.

Conjecture 293. The minimum of the derivative of positive eigenvalues $\leqslant$ size/independence (graphs with girth $\geqslant 5$; see discussion in Section 3.2); see Fig. 5.

Conjecture 551. The maximum of mid-degree sequence $\leqslant$ the mean of dual-degree sequence; see Fig. 6.

Conjecture 581. The order $-$ the frequency of the mode of mid-degree sequence $\leqslant$ independence (trees); see Fig. 7.

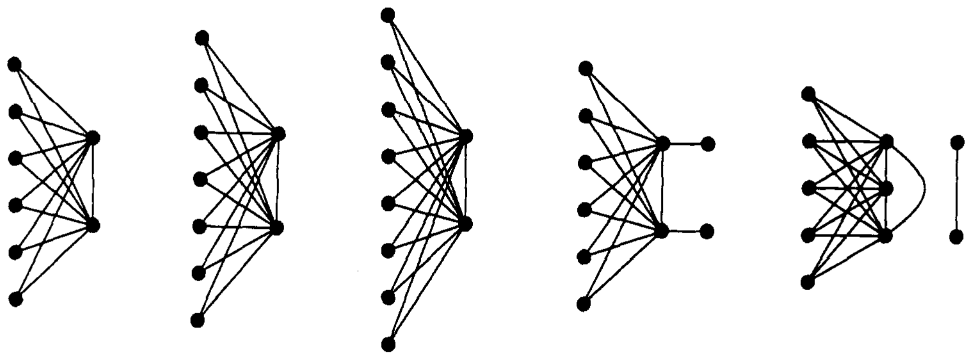

Fig. 6. Conjecture 551.

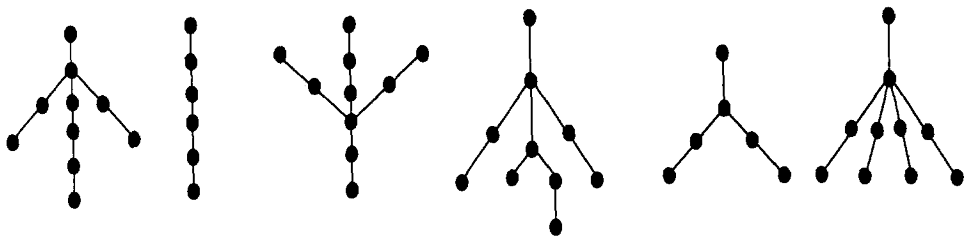

Fig. 7. Conjecture 581.

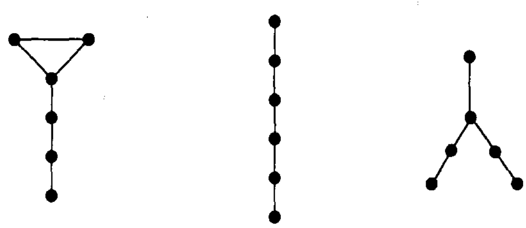

Fig. 8. Conjecture 665.

Conjecture 665. The average distance $\leqslant$ the frequency of the mode of mid-degree sequence (graphs with sum of even vector $\leqslant$ sum of odd vector); see Fig. 8.

3. Constructing infinite classes of counterexamples

As noted above, those conjectures in which only a small number of counterexamples were found have been emphasized since it is natural to question if they are true for all but a finite number of graphs. Surprisingly, isolating those particular conjectures and examining their associated counterexamples revealed patterns (in most cases) that allowed us to conclude just the opposite: that an infinite class of counterexamples exist. In this section we demonstrate the process by generalizing four of the conjectures taken from Section 2.4.

3.1. Conjecture 107

Conjecture 107. Mode of distance $\leqslant$ radius (even regular graphs).

Motivated by the first given counterexample to this conjecture in Section 2.4, we now show that the graph G_R in Fig. 9 is a counterexample for each even positive integer R.

Lemma. *The graph G_R is even regular for each even positive integer R.*

Proof. Locally on the labeled path $R, R-1, \ldots, 1, a, b$ each vertex begins $R/2+1$ even-length subpaths. Furthermore, there are $3 \cdot R/2$ even-length minimal paths from each labeled vertex to the set of nonlabeled vertices. Therefore these graphs are even regular by symmetry. $\square$

Lemma. *The graph G_R has $R+1$ as the mode of its distance matrix for each even positive integer R.*

Proof. The frequency of $R+1$, denoted by $freq(R+1)$, in the distance matrix is $4+4(3(R-1)+4)=12R+8$ where there are four paths from vertex b, $3(R-1)$ paths from vertices $1, 2, \ldots, R-1$, and four more paths from vertex R.

The frequency of $k > R+1$ is $2\binom{4}{2}(R-(k-(R+1))) = 24R-12k+12$ where we counted the paths between two of the four long branches. One now sees that $24R-12k+12 < 24R-12(R+1)+12 = 12R < freq(R+1)$ for all $k > R+1$.

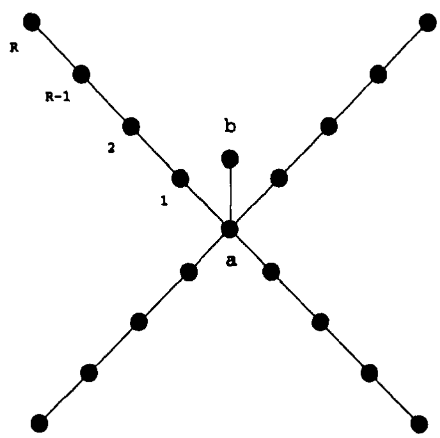

Fig. 9. The graph G_R.

The frequency of $2 \leqslant k \leqslant R$ is $4(2(R+2-k)) + 2\binom{4}{2}(k-1) = 8R + 4k + 4$ where the first term represents the paths local to the labeled vertices and the second term counts paths through vertex a to a nonlabeled vertex. We also have $8R + 4k + 4 < 8R + 4R + 4 = 12R + 4 < \mathit{freq}(R+1)$ for all $2 \leqslant k \leqslant R$.

The case for $k = 1$ is trivial $(4R + 1 < \mathit{freq}(R+1))$. Hence $R + 1$ is the mode of the distance matrix for the graph G_R. $\square$

Combining the previous two lemmas with the fact that the graph G_R has radius R completes the proof of the claim that G_R is a counterexample to Conjecture 107 for each even positive integer R.

3.2. Conjecture 293

The separator is defined to be the difference between the largest and the second largest eigenvalue of the adjacency matrix. The derivative of the positive eigenvalues is defined to be a vector whose ith component is $p'(i) = p(i+1) - p(i)$, where p is the vector of positive eigenvalues sorted in increasing order.

The next conjecture led to the discovery of a class of graphs in which the minimum derivative of the positive eigenvalues equals the separator and can be made arbitrarily large while keeping the ratio of size to independence bounded by a constant.

Our initial counterexamples for the next conjecture follow Bondy and Murty [1], who consider the girth of acyclic graphs infinite. Graffiti, on the other hand, considers the girth of acyclic graphs undefined and hence excludes trees from consideration. At the end of this section, we point out how most graphs with cycles can be extended to counterexamples which Graffiti does accept.

Conjecture 293. The minimum of the derivative of the positive eigenvalues $\leqslant$ size/independence (graphs with girth $\geqslant 5$).

Motivated by the first two trees given in Section 2.4 as counterexamples to this conjecture, we now show that the tree in Fig. 10 is a counterexample for each integer $n \geqslant 10$.

Proof. Label the vertices so that the first $n - 2$ of them is a maximal (in fact, the maximum) independent set, the motivation being that the adjacency matrix A will be almost triangular.

After a few row operations, $A - \lambda I$ can be triangulated and the determinant obtained by simply taking the product of the diagonal elements. The characteristic equation is then given by

$$(-\lambda)^{n-2}\left[\frac{2-\lambda^2}{\lambda}\right]\left[\frac{\lambda^4 + (1-n)\lambda^2 + (2n-7)}{\lambda(2-\lambda^2)}\right] = 0, \quad \text{or}$$

$$\lambda^{n-4}[\lambda^4 + (1-n)\lambda^2 + (2n-7)] = 0.$$

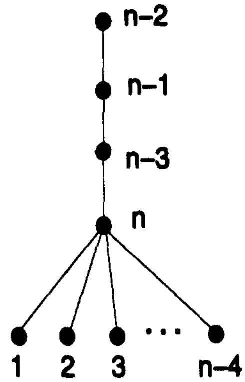

Fig. 10.

Noting that the fourth-degree term is quadratic in λ^2, we find that $\lambda = 0$ is a root of multiplicity $n-4$ and that the other four roots are the roots of

$$\lambda^2 = \frac{(n-1) \pm \sqrt{n^2 - 10n + 29}}{2}.$$

The only two positive eigenvalues are

$$\lambda_1 = \sqrt{\frac{(n-1) + \sqrt{n^2 - 10n + 29}}{2}} \quad \text{and} \quad \lambda_2 = \sqrt{\frac{(n-1) - \sqrt{n^2 - 10n \rightarrow 29}}{2}}.$$

Thus the minimum derivative of the positive eigenvalues is the separator, the difference between the two largest eigenvalues.

Since

$$\lambda_1 > \sqrt{\frac{(n-1) + \sqrt{(n-5)^2}}{2}} = \sqrt{\frac{2n-6}{2}} = \sqrt{n-3} \qquad \text{and}$$

$$\lambda_2 < \sqrt{\frac{(n-1) - \sqrt{(n - \sqrt{29})^2}}{2}} = \sqrt{\frac{\sqrt{29} - 1}{2}},$$

$$\lambda_1 - \lambda_2 > \sqrt{n-3} - \sqrt{\frac{\sqrt{29} - 1}{2}}.$$

Thus the minimum derivative of the positive eigenvalue can be made arbitrarily large.

The claim that this can be done while keeping the ratio of size to independence bounded by a constant is now elementary since the graphs in question are trees with independence $= n-2$. So size/independence $= (n-1)/(n-2) = 1 + 1/(n-2) \leqslant \frac{9}{8}$ for $n \geqslant 10$. After substituting $n = 10$ in the lower bound given for the separator, we see that for all $n \geqslant 10$ the minimum derivative of the positive eigenvalues $= \lambda_1 - \lambda_2 > \sqrt{10-3} - \sqrt{(\sqrt{29}-1)/2} > \frac{9}{8} \geqslant$ size/independence. $\square$

Adding isolated vertices to a graph only increases the number of zero eigenvalues: i.e. the derivative of the positive eigenvalues remains unchanged. Since the size of a graph is defined to be the number of edges of the graph, it too remains unchanged when isolated vertices are added. It follows that any graph of girth $\geqslant 5$ with at least two positive eigenvalues (so that the derivative of the positive eigenvalues is well-defined) and with no two positive eigenvalues the same (so that the minimum of the derivative is nonzero, and hence positive since the eigenvalues are sorted in increasing order) can be extended to be a counterexample of Conjecture 293 by adding isolated vertices. For example a 5-cycle can be extended to be a counterexample by adding 12 isolated vertices.

3.3. Conjecture 551

Conjecture 551. The maximum of the mid-degree sequence $\leqslant$ the mean of the dual-degree sequence.

Referring to the first three counterexamples given for this conjecture in Section 2.4, a pattern is evident. Generalizing these graphs, we now show that the graph G_n in Fig. 11 is a counterexample for each integer $n \geqslant 8$:

Proof. Note that the vertices of G_n have been labeled so that its degree sequence $\{n-1, n-1, \overbrace{2, \ldots, 2}^{n-2}\}$ is in decreasing order as is required when computing the mid-degree sequence. After two steps the derived sequence has all components equal 0, and hence the depth equals 2. It follows that the mid-degree sequence is given by $\{n-2, \overbrace{1, \ldots, 1}^{n-2}\}$ and that the maximum of the mid-degree sequence is $n-2$.

Since the dual-degree sequence is the vector whose ith component is the mean of the degrees of the neighbors of vertex i, the mean of the dual-degree sequence of G_n is given by

$$\frac{2\left[\dfrac{2(n-2)+(n-1)}{n-1}\right]+(n-2)(n-1)}{n}, \quad \text{or} \quad \frac{n^3-4n^2+11n-12}{n(n-1)}.$$

So the maximum of the mid-degree sequence is greater than the mean of dual-degree sequence when

$$n-2 > \frac{n^3-4n^2+11n-12}{n(n-1)},$$

or when $n^2-9n+12>0$. Therefore, G_n is a counterexample for each integer $n \geqslant 8$. $\square$

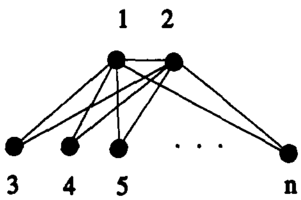

Fig. 11. The graph G_n.

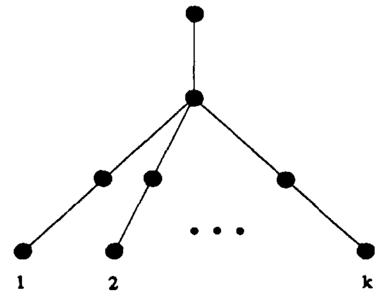

Fig. 12. The graph T_k.

3.4. Conjecture 581

Conjecture 581. The order – frequency of mode of mid-degree sequence $\leqslant$ independence (trees).

While there are only six trees with 10 or fewer vertices that fail this conjecture, an infinite class of counterexamples can be constructed as shown in Fig. 12, provided that k is an even positive integer for the tree T_k.

Proof. Note that the order of T_k is $n = 2 \cdot k + 2$. These trees have independence equal to $k + 1$ since the $k + 1$ vertices of degree one form an independent set and are each incident to an edge of a perfect matching. The sorted degree sequence $\{k + 1, \overbrace{2, \ldots, 2}^{k}, \overbrace{1, \ldots, 1}^{k+1}\}$ of T_k has depth $k + 1$ and yields $\{\overbrace{1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{k/2+1}\}$ as the mid-degree sequence. (By definition the mid-degree sequence is found after $\lceil \text{depth}/2 \rceil$ derived operations.) The frequency of the mode of the mid-degree sequence is k. Hence $(2 \cdot k + 2) - k \nleqslant k + 1$ for the tree T_k when k is an even positive integer. $\square$

4. Conjectures with 10 or more counterexamples

We now list those conjectures for which we found a relatively large number of counterexamples during our exhaustive search of the over 12 million nonisomorphic graphs with 10 or fewer vertices. Although in each instance we display only one counterexample (displayed in Fig. 13), Table 2 contains the total number of counterexamples found as well as the value of the terms for the displayed graphs. Recall that the terms are defined in the glossary.

Conjecture 187. The mode of eigenvalues of Laplacian $\leqslant n$ – independence (graphs with sum of odd vector $\leqslant$ sum of even vector).

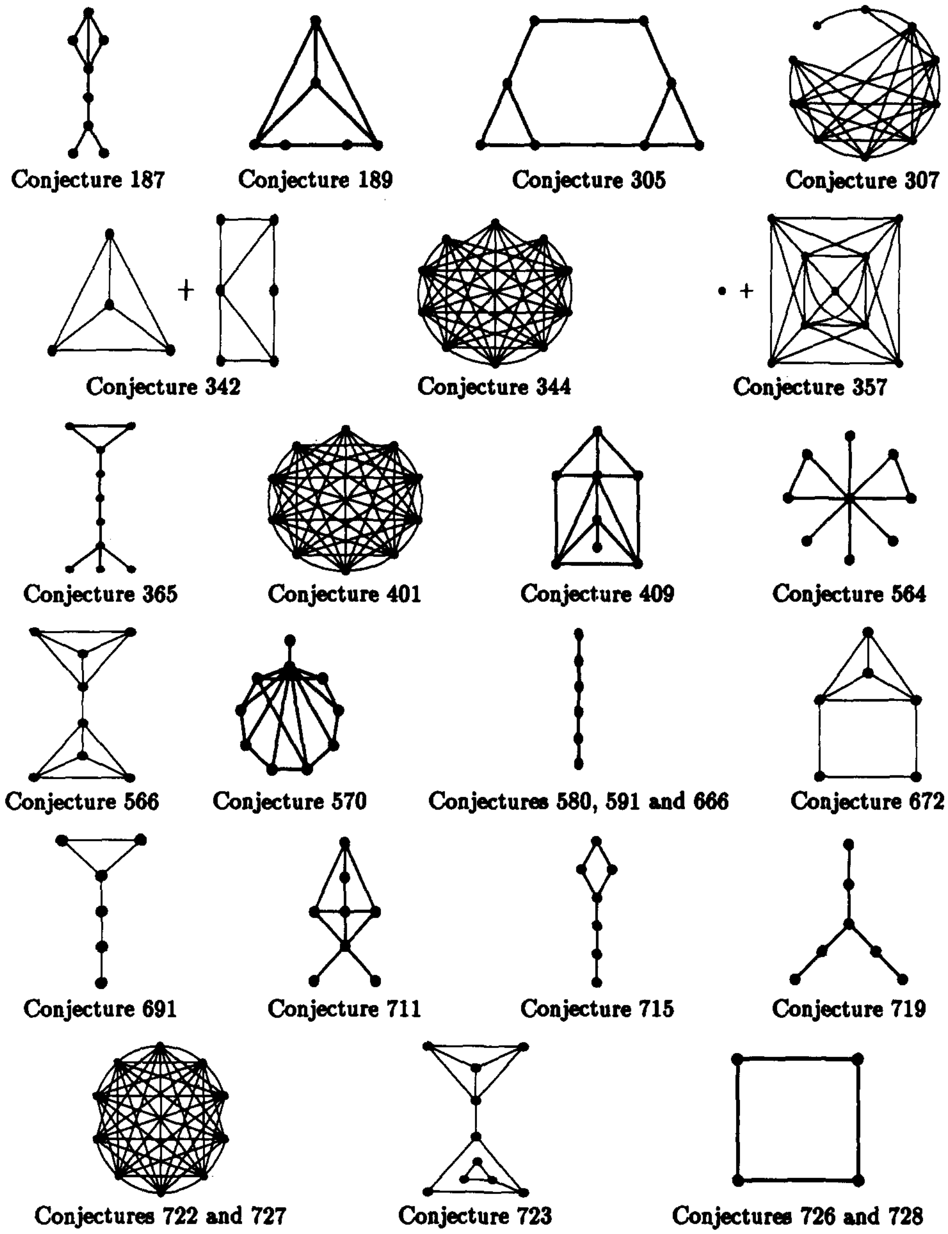

Fig. 13. Sample counterexamples for conjectures with 10 or more counterexamples.

Conjecture 189. The mode of the eigenvalues of Laplacian $\leqslant$ the number of nonpositive eigenvalues (graphs with sum of odd vector $\leqslant$ sum of even vector).

Conjecture 305. The inverse of dual-degree sequence $\leqslant$ the number of nonnegative eigenvalues (graphs with distance rank $<$ rank).

Table 2
The value of the terms associated with the counterexample (of Fig. 13) given for each of the conjectures with 10 or more counterexamples

Conjecture	Number of counterexamples[a]	1st term	2nd term	3rd term	4th term	5th term
187	3154	4	7	4	24	25
189	5116	4	3	16	20	
305	16	3.0738	3	6	8	
307	13	1.5333	10	6.6323	9	10
342	268	9	8	3	3	
344	469	8.4772	8	3	3	
357	135	31	10	14	5	5
365	10	10	0.1875	44	6	6
401	1081	8.2448	8.0741	2		
409	3860	5	4	4	4	
564	415	5	4			
566	20	2.7913	2.5			
570	12	6	4	1		
580	18	6	2			
591	17	2.3333	2			
666	345	2.3333	2	18	18	
672	206802	6	2	2	18	18
691	63	2.0667	2	18	18	
711	2794	6	5			
715	37	2.1889	2.1667			
719	2098	2	1.7143	0		
722	97291	9	−6.1160	2		
723	15	7	3.9881	3		
726	1718371	3	0.6667	2		
727	47661	3	0.1160	2		
728	58259	3	0.5	2		

[a] Due to numerical roundoff, the individual counts may be slightly off. However, all counterexamples displayed in Fig. 13 have been verified by hand.

Conjecture 307. The average distance $\leqslant$ order/largest eigenvalue (graphs with distance rank < rank).

Conjecture 342. The maximum degree $\leqslant$ the rank of distance matrix (plants).

Conjecture 344. The maximum eigenvalue $\leqslant$ the number of the negative eigenvalues of distance matrix (plants).

Conjecture 357. The size − the order $\leqslant$ the number of nonedges (geotropic plants).

Conjecture 365. Size/separator $\leqslant$ sum of odd vector (geotropic plants).

Conjecture 401. The minimum derivative of the positive eigenvalues $\leqslant$ mean of gravity matrix (graphs with independence $\leqslant$ 2).

Conjecture 409. The scope of degree sequence $\leqslant$ maximum of even vector (graphs with the number of negative eigenvalues $\leqslant$ number of positive eigenvalues).

Conjecture 564. The frequency of the mode of the eigenvalues of Laplacian $\leqslant$ the frequency of the mode of degree sequence (connected graphs).

Conjecture 566. The second largest eigenvalue $\leqslant$ the mean of even vector (connected graphs).

Conjecture 570. The number of positive eigenvalues $-$ the number of negative eigenvalues $\leqslant$ the minimum of even vector (connected graphs).

Conjecture 580. (Order $-$ 1)/2 $\leqslant$ the frequency of the mode of 1-residue vector (trees).

Conjecture 591. The average distance $\leqslant$ the frequency of the maximum of 1-residue vector (trees).

Conjecture 666. The average distance $\leqslant$ the frequency of the maximum of 1-residue vector (graphs with sum of even vector $\leqslant$ sum of odd vector).

Conjecture 672. Order/independence $\leqslant$ the frequency of the mode of even vector (graphs with sum of odd vector $\leqslant$ sum of even vector).

Conjecture 691. The average distance $\leqslant$ the mode of even vector (graphs with sum of odd vector $\leqslant$ sum of even vector).

Conjecture 711. The range of deficiency vector $\leqslant$ the range of eigenvalues.

Conjecture 715. The scope of nonpositive eigenvalues $\leqslant$ the mean of degrees greater than or equal to the average degree.

Conjecture 719. The mean of dual-degree sequence $-$ mean degree sequence $\leqslant$ scope of dual-degree sequence.

Conjecture 722. The number of nonpositive eigenvalues $+$ the sum of the reciprocal of negative eigenvalues $\leqslant$ independence.

Conjecture 723. The number of nonpositive eigenvalues $-$ the sum of the temperature vector $\leqslant$ independence.

Conjecture 726. The number of nonnegative eigenvalues $-$ the mean of nonnegative eigenvalues $\leqslant$ independence.

Conjecture 727. The number of nonnegative eigenvalues − the sum of the reciprocal of positive eigenvalues ⩽ independence.

Conjecture 728. The number of nonnegative eigenvalues + the sum of the reciprocal of negative eigenvalues ⩽ independence.

5. Theorems

This section contains some results about four of Graffiti's conjectures, namely Conjectures 128, 158, 539, and 717.

5.1. Conjecture 128

Conjecture 128. The second smallest eigenvalue of Laplacian ⩽ n/average distance.

For a graph of order n let $0 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of the Laplacian and let avg_D denote the average distance.

Lemma. *If* $\frac{1}{2}\lambda_n \ln(2n-2) \leqslant n - \lambda_2$ *and* $\lambda_2 \neq \lambda_n$ *then* $\lambda_2 \leqslant n/avg_D$.

Proof. By the main result in [4], the diameter D of any graph satisfies the following:

$$D \leqslant \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}\left(\dfrac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right)} \right\rfloor + 1.$$

From this bound and the identity $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$, $x \geqslant 1$, we have

$$D \leqslant \frac{\ln(n-1+\sqrt{(n-1)^2-1})}{\ln\left(\dfrac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n} + \sqrt{\left(\dfrac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right)^2 - 1}\right)} + 1 \leqslant \frac{\ln(2n-2)}{\ln\left(\dfrac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right)} + 1.$$

Using the Taylor's series expansion $\ln((1+x)/(1-x)) = 2(x + x^3/3 + x^5/5 + \cdots)$, we find that

$$D \leqslant \frac{\ln(2n-2)}{2(\lambda_2/\lambda_n)} + 1 = \frac{1}{2}\lambda_n \frac{\ln(2n-2)}{\lambda_2} + 1.$$

Hence if $\frac{1}{2}\lambda_n \ln(2n-2) \leqslant n - \lambda_2$, then $D \leqslant (n-\lambda_2)/\lambda_2 + 1 = n/\lambda_2$. Finally, after rearranging terms, $\lambda_2 \leqslant n/D \leqslant n/avg_D$. □

Theorem. *For a fixed maximum degree, there exist at most a finite number of counter-examples to Conjecture* 128.

Proof. Assume we have a connected counterexample of order n, minimum degree δ, and maximum degree Δ.

If $\lambda_2 = \lambda_n$ then the graph must be a complete graph K_n (exercise for reader). We know from [10] that $\lambda_2 \leqslant (n/(n-1))\delta$, which means for $\delta = n-1$, $\lambda_2 = n \leqslant n/avg_D$. So Conjecture 128 holds when $\lambda_2 = \lambda_n$.

If $\lambda_2 \neq \lambda_n$ then using the restriction $n - \lambda_n < n - \lambda_2 < \frac{1}{2}\lambda_n \ln(2n-2)$ from the above lemma, along with fact that $\lambda_n \leqslant 2\Delta$ (see [10]), the following holds:

$$n \leqslant \tfrac{1}{2}\lambda_n \ln(2n-2) + \lambda_n \leqslant \Delta(\ln(2n-2)+2), \quad \text{or} \quad \frac{n}{\ln(2n-2)+2} \leqslant \Delta.$$

Thus n is bounded when Δ is fixed, and it follows that there can be at most a finite number of counterexamples within the graphs with maximum degree Δ. □

Remark. Again using the fact that $\lambda_2 \leqslant (n/(n-1))\delta$ while comparing Conjectures 62 and 128,

$$avg_D \leqslant \frac{n}{\Delta} \quad \text{and} \quad avg_D \leqslant \frac{n}{\lambda_2},$$

we see that Conjecture 62 implies Conjecture 128 for all irregular graphs.

5.2. Conjectures 158 and 539

We give characterizations for when equalities hold for two of the conjectures proved by Fajtlowicz.

Conjecture 158. For each graph G the minimum degree $\leqslant n -$ independence.

It is straightforward to prove the inequality. In general, equality holds if and only if the complement of G has a component consisting of a clique and all other components (if any) have maximum degree less than or equal to that of the degree of the clique. This claim is easily seen from the following elementary proof of the inequality of Conjecture 158.

Proof. For any graph H, clique $\leqslant$ maximum degree $+ 1$. Hence $n -$ maximum degree of $H - 1 \leqslant n -$ clique of H. But this says that the minimum degree of H complement $\leqslant n -$ independence of H complement. Finally take G to be H complement. □

Now consider the following conjecture.

Conjecture 539. For every graph G the minimum degree $\leqslant$ size/independence.

Let δ denote the minimum degree. For this conjecture, equality holds if and only if G has a maximal independent set that is regular of degree δ and that edge covers G.

Proof. Say independence equals k, and $I = \{v_1, v_2, \ldots, v_k\}$ is a maximal independent set. Then since none of the v_i's are adjacent to each other and since the degree of each v_i is at least δ, the number of edges of $G \geqslant k \cdot \delta$. That is, *minimum degree* $= \delta \leqslant |E(G)|/k =$ size/independence. Clearly equality holds if and only if the edges incident to the v_i's are the only edges of G and each v_i has degree δ. $\square$

Two of our observations regarding Conjecture 539 are listed below. The simple proofs may be found in our technical report [2].

Observation 1. If a graph has minimum degree = size/independence, then it is bipartite.

Observation 2. A regular graph has minimum degree = size/independence if and only in independence $= n/2$.

5.3. Conjecture 717

We believe that we were the first (see [5]) to prove the following conjecture. A similar proof was published by Favaron et al. [9]. The proof depends upon the fact that $x + 1/x \geqslant 2$ for any positive x.

Conjecture 717. The mean degree $\leqslant$ mean of the dual-degree sequence.

Proof. For a vertex u let $N(u)$ denote the set of vertices adjacent to u. Then the mean dual-degree sequence is given by

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \text{dual-degree } v_i &= \frac{1}{n} \sum_{\substack{u \in v(G) \\ |N(u)| \geqslant 1}} \frac{1}{|N(u)|} \sum_{v \in N(u)} \text{degree } v \\
&= \frac{1}{n} \sum_{\substack{u \in v(G) \\ |N(u)| \geqslant 1}} \sum_{v \in N(u)} \frac{\text{degree } v}{\text{degree } u} \\
&= \frac{1}{n} \sum_{\substack{e \in E(G) \\ e = \{u, v\}}} \left(\frac{\text{degree } v}{\text{degree } u} + \frac{\text{degree } u}{\text{degree } v} \right) \\
&\geqslant \frac{1}{n} \sum_{e \in E(G)} 2 \\
&= \frac{1}{n} (2|E(G)|) \\
&= \frac{1}{n} \sum_{i=1}^{n} \text{degree } v_i \\
&= \text{mean degree.} \qquad \square
\end{aligned}$$

Dedication

This paper is dedicated to the memory of Tony Brewster who died suddenly on 8 September 1993. His death was a great tragedy and we will all miss his energy, dedication and love for mathematics.

Glossary

Adjacency matrix. The usual $n \times n$ matrix for a graph of order n where the i, jth entry is 1 if the graph contains the edge (i, j) and is 0 otherwise.

Average of a matrix. The average of the entries above the diagonal.

Average of a vector. The average of the components.

Deficiency vector. The vector whose ith component is the number of nonedges in the graph induced by the neighbors of vertex i.

Degree sequence. The vector whose ith component is the degree of vertex i.

Depth. The number of derived operations required on the degree sequence to obtain a vector of zeros.

Derivative of a vector V. The vector V' obtained from V by first sorting it in increasing order and then setting $V'(i) = V(i+1) - V(i)$.

Derived operation. The following operation on a vector sorted in decreasing order: if p is the largest component then delete this first component and subtract 1 from the p following components.

Distance matrix. The usual distance matrix whose i, jth component is the number of edges in the shortest path from vertex i to vertex j.

Dual-degree sequence. The vector whose ith component is the mean of the degrees of the neighbors of vertex i.

Eigenvalue(s). Unless otherwise specified, the eigenvalue(s) of the adjacency matrix of the graph.

Even vector. The vector whose ith component is the number of vertices an even distance (including zero) from vertex i.

Even regular. A graph in which each component of the even vector is the same.

Girth. The length of the smallest cycle of a graph; in the case of no cycles the girth is considered infinite.

Gravity matrix. The matrix (indexed by vertices of the graph) whose u, vth entry is 0 if $u = v$ or if there is no path joining u to v; otherwise it is $(1/(n-1))(deg(u) \cdot deg(v))/d(u,v)$, where $deg(v)$ denotes the degree of v and $d(u,v)$ is the distance from u to v.

Geotropic plant. A graph in which independence equals the number of nonpositive eigenvalues.

Harmonic. The sum of the weights over all the edges, where the weight of an edge incident with vertices of degree p and degree q is taken to be $2/(p+q)$.

Heliotropic plant. A graph in which independence equals the number of nonnegative eigenvalues.

Independence. The size of the largest independent set, where an independent set is a set of vertices with no edges between any two vertices in the set.

Inverse (of a vector). The sum of the reciprocals of the nonzero components.

Laplacian. The matrix (indexed by the vertices of the graph) having the degree of vertex v on the corresponding entry of the diagonal, a negative one if the corresponding vertices are adjacent and a zero otherwise.

Length (of a vector). The square root of the sum of the squares of the components.

Mean of a vector. See average of a vector.

Mean of a matrix. See average of a matrix.

Mid-degree sequence. The vector obtained from the degree sequence by repeating the derived operation depth/2 (rounded up) many times.

Mode (of a vector). The component which occurs most often; in case of ties, take the smallest component.

n. See order.

Nonedge. A two-element independent set.

Odd vector. The vector whose ith component is the number of vertices an odd distance from vertex i.

Order. The number of vertices of a graph.

Plant. A graph in which independence equals the minimum of the number of non-negative and the number of nonpositive eigenvalues.

Radius. The minimum eccentricity of the vertices, where the eccentricity $e(v)$ of vertex v in a connected graph G is $\max d(u,v)$ for all u in G. As usual $d(u,v)$ is the distance from vertex u to vertex v.

Range (of a vector). The number of distinct components.

Rank. Unless otherwise specified, the rank of the adjacency matrix.

Regular. A graph in which each vertex has the same degree.

1-residue vector. The vector obtained by repeating the derived operation until all components are less than or equal to 1.

Scope (of a vector). The difference between the largest and the smallest components.

Separator. The difference between the largest and the second largest eigenvalue of the adjacency matrix.

Size. The number of edges of a graph.

Sum of a matrix. The sum of the entries above the diagonal.

Sum of a vector. The sum of the components.

Temperature vector. The vector whose ith component is $d/(n-d)$, where d is the degree of vertex i and n is the number of vertices of the graph.

Transmission (of a matrix M). The vector whose components are the sums of the rows of M.

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