Undecidability in quantum information theory

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Computability theory

- a subfield of mathematical logic
- founded by Gödel, Turing, Church, Kleene in the 1930s
- studies (relative) computability in principle, without considering resource bounds.

Fundamental notions and facts of computability theory

- Computable set of natural numbers (formalized via Turing machines)
- computable function on the natural numbers
- recursively enumerable (r.e.) set
- preorderings such as $\leq_m$ to compare the complexity of sets
- The halting problem K (goes a given TM halt?) is undecidable.
- In fact it is $\leq_m$-complete for the class of r.e. sets.

The spectral gap problem undecidable

Cubitt et al. 2015 (Nature); Bausch et al. 2020

It is undecidable whether there is a spectral gap for spin square lattices.

- Announced in 2015 by Cubitt, Perez Garcia and Wolf (Undecidability of the Spectral Gap, Nature 528, 2015, 6 pages)
- The full paper has only appeared in 2022 in the top journal Forum Mathematics Pi, 102 pages.

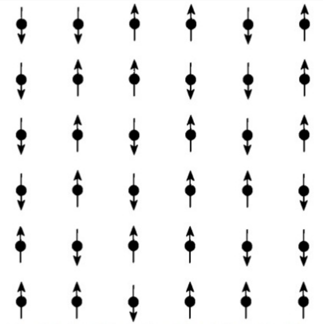
It is undecidable whether there is a spectral gap for abstract spin chains. Proved by Bausch, Cubitt, Lucia and Perez Garcia (Phys. Review X.10, 2020, 20 pages), building on many of the results in the long paper posted in 2015.

Spin chains and spin lattices

Spin chains were introduced to understand magnetism. A classical spin chain consists of N dipoles arranged linearly:

$$\underbrace{\uparrow\downarrow\downarrow\uparrow\ \dots\ \uparrow}_{N}$$

Higher-dimensional arrangements of dipoles have also been studied, in particular square lattices.



Ising model in 1D: Hamiltonian

The 1D Ising model is due to Lenz (1920), and was "solved" by his student Ising in his thesis (1925).

The positions $i = 1, \ldots, N$ in a spin chain are called sites.

The energy of a state of the system is given by a Hamiltonian.

For the 1D Ising model with N sites, the Hamiltonian is

$$H_N = -\frac{1}{2} \sum_{k=1}^{N} [s\sigma_k + J\sigma_k\sigma_{k+1}]$$

- $\sigma_i = 1$ for spin $\uparrow$ at site i, and $\sigma_i = -1$ for spin $\downarrow$ at site i
- periodic boundary conditions: $\sigma_{N+1} = \sigma_1$
- s is the strength of the external magnetic field
- J is the interaction strength between neighbours.

Quantum setting: Heisenberg (1928) model

- n-chain, each site contains a spin $1/2$ particle (e.g., electron).
- state is unit vector in $(\mathbb{C}^2)^{\otimes n}$
- Spins in x, y, z directions, corresponding to observables given by the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$.
- Let $\mathbb{I}$ denote the identity 2×2 matrix. For $\alpha = x, y, z$ let

$$\sigma_k^\alpha = \mathbb{I}^{\otimes(k-1)} \otimes \sigma^\alpha \otimes \mathbb{I}^{\otimes(n-k)} \in L((\mathbb{C}^2)^{\otimes n}).$$

The Hamiltonian is now a Hermitian operator on $(\mathbb{C}^2)^{\otimes n}$:

$$H = -\frac{1}{2} \sum_{k=1}^{n} [s\sigma_k^z + \sum_{\alpha=x,y,z} J_\alpha \sigma_k^\alpha \sigma_{k+1}^\alpha].$$

The $J_\alpha \in \mathbb{R}$ are called coupling constants.

Abstract spin chains

For $d \geq 2$ (sometimes suppressed), a qudit is a unit vector in d-dimensional Hilbert space $\mathbb{C}^d$.

- An abstract spin chain is a system of n qudits, arranged linearly. The positions are referred to as sites.
- The state of such a system is given by a vector in the d^n-dimensional Hilbert space $(\mathbb{C}^d)^{\otimes n}$.

One also considers higher dimensional arrangements of qudits, e.g. square lattices.

Local Hamiltonians

Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices.

As in the case of the Ising and Heisenberg chains, the behaviour of an abstract spin chain is described by local Hamiltonians.

Let $h^{(1)} \in M_d(\mathbb{C})$ and $h^{(2)} \in M_{d^2}(\mathbb{C})$ be Hermitian matrices, where

- $h^{(1)}$ describes the one-site "interactions", and
- $h^{(2)}$ describes the nearest-neighbour interactions.

The global Hamiltonian of a spin chain of n qudits is given by shifting and adding up these interactions as the indices vary:

$$H_n = \sum_{k=1}^{n} h_k^{(1)} + \sum_{k=1}^{n-1} h_{k,k+1}^{(2)}.$$

Asymptotic spectral gap

The spectral gap of a nontrivial Hamiltonian H acting on a finite-dimensional Hilbert space is $\Delta(H) = \gamma_1(H) - \gamma_0(H)$, the difference between its least two eigenvalues.

When writing a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ we will assume that H_n is a Hamiltonian on the d^n-dimensional Hilbert space.

The asymptotic spectral gap of such a sequence can be defined as

$$\Delta\langle H_n \rangle = \liminf_n \Delta(H_n).$$

(The ground energy $\gamma_0(H_n)$ might increase with n.)

Gapped and gapless sequences of Hamiltonians

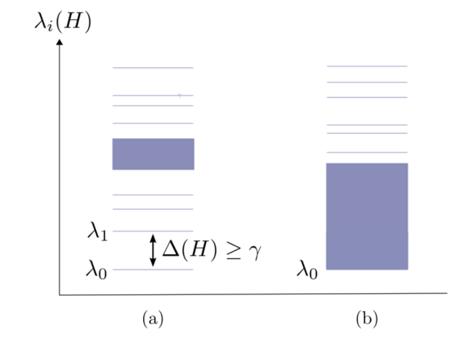
Intuitively, the system is gapped if $\Delta\langle H_n\rangle$ is positive, and gapless otherwise. Cubitt et al. (2015) and then Bausch et al. (2020) use definitions making both the gapped and the gapless case more restricted, so that some sequences have neither property.

$\langle H_n\rangle$ is gapped if $\Delta\langle H_n\rangle = \liminf_n \Delta(H_n)$ is positive; moreover, for sufficiently large n, the least eigenvalue $\lambda_0(H_n)$ is non-degenerate. Physically the additional condition means that there is a unique ground state of the system (up to phase).

$\langle H_n\rangle$ is gapless if there is some $c > 0$ such that for each $\varepsilon > 0$, for all sufficiently large n, each point in the interval $[\lambda_0(H_n), \lambda_0(H_n) + c]$ is ε-close to some eigenvalue of H_n.

(a) $\langle H_n \rangle$ is gapped if $\Delta\langle H_n \rangle = \liminf_n \Delta(H_n)$ is positive and for sufficiently large n, the least eigenvalue $\lambda_0(H_n)$ is non-degenerate.

(b) $\langle H_n \rangle$ is gapless if there is some $c > 0$ such that for each $\varepsilon > 0$, for all sufficiently large n, each point in the interval $[\lambda_0(H_n), \lambda_0(H_n) + c]$ is ε-close to some eigenvalue of H_n.



From Cubitt et al., Nature 2015

1D case due to Bausch et al.

Even in the 1-dimensional case, it has now been shown to be undecidable whether there is a spectral gap by Bausch, Cubitt, Lucia and Perez Garcia 2020. Given a Turing machine M, they determine a (fairly large) dimension d. Then, given an input $\eta \in \mathbb{N}$ to M they compute local Hamiltonians $h^{(1)} \in M_d(\mathbb{C})$ and $h^{(2)} \in M_{d^2}(\mathbb{C})$ as above such that

- if $M(\eta)$ halts then the sequence $\langle H_n(\eta) \rangle$ (defined as above by shifting the local interactions) is gapless,
- otherwise the sequence $\langle H_n(\eta) \rangle$ is gapped.

They fully rely on the methods of Cubitt et al. (2015) who showed that the spectral gap problem is undecidable in the 2D case, using square lattices of qudits. The definitions there are similar, except that there are two types of nearest-neighbour interactions, for rows and for columns.

Remarks

1. In the 2D case, the relationship between machines and Hamiltonians is the other way round: if $M(\eta)$ halts then the sequence is gapped, else gapless.

2. The entries of the Hamiltonians are easy "complex" numbers:
   - Let R be the subring of $\mathbb{C}$ generated by
   $$\mathbb{Q} \cup \{\sqrt{2}\} \cup \{\exp(2\pi i\theta) \colon \theta \in \mathbb{Q}\}.$$
   - The entries of the local Hamiltonians, and hence of the $H_n(\eta)$, are all in R.
   - So the undecidability of the spectral gap is not an artefact of the well-known fact that equality of two computable reals is undecidable.

Elements of the proofs in 2D and 1D

In 2 dimensions:

- quantum Turing machines (Bernstein and Vazirani)
- history state Hamiltonian $T^{-1/2}\sum_{t=0}^{T-1}|t\rangle|\psi_t\rangle$ due to Feynman/Kitaev
- Gottesman and Irani (FOCS 2013): The ground state encodes the whole computation of a quantum TM up to stage T; here the quantum TM is not related to M; rather, it is related to the phase estimation algorithm (e.g. Nielsen/Chuang)
- Quasi-periodic Wang tiling due to Robinson 1971.

In 1 dimension:

the Wang tiling, for which the second spatial dimension in the lattice was needed, is replaced by a "marker Hamiltonian".

Infinite spin chains, and

the quantum SMB theorem

Von Neumann entropy of a density matrix

- M_n is the algebra of $2^n \times 2^n$ complex matrices. M_∞ is the CAR algebra, a C^* algebra that is some kind of limit of the M_n's.
- The von Neumann entropy of a density matrix $S \in M_n$ is $H(S) = -\mathsf{tr}(S \log_2 S)$.
- This is the usual entropy of the distribution that S induces on its eigenvectors.
- Its maximum value is n, when the distribution is uniform.

Effective SMB theorem

- A 1950s theorem due to Shannon, McMillan and Breiman says that the (global) entropy of an ergodic measure μ on $\{0,1\}^{\mathbb{N}}$ equals the empirical entropy $-\lim_n \frac{1}{n}\log_2 \mu[Z \upharpoonright n]$ for μ-almost every bit sequence Z.
- If μ is computable, and Z is ML-random w.r.t. μ, then its empirical entropy determines the entropy of μ (Hochman, 2009).
- Bjelakovic, Krueger, Siegmund and Szkola (2004) proved a version of the Shannon-McMillan theorem for ergodic quantum lattice systems. Since there is no notion of null set of states, they use an ad hoc, finitary definition of "for almost".
- We present a conjecture that attempts to remedy this, using that quantum Martin-Loef-tests are a quantum analog of effective null sets of bit sequences.

Effective quantum SMB theorem?

- A state μ on M_∞ is called ergodic if it is an extreme point on the convex set of shift invariant states.
- $h(\mu) = \lim_n \frac{1}{n} H(\mu_n)$ is the von Neumann entropy of μ.

Conjecture (with Marco Tomamichel, Nies Logic Blogs 2017, 2020)

Suppose that for some $D > 0$, for each n, the diagonal entries of μ_n are bounded below by 2^{-nD}. Let ρ be a state that is quantum Martin-Loef random with respect to μ. Then

$$h(\mu) = -\lim \frac{1}{n} \mathsf{tr}(\rho_n \log_2 \mu_n).$$

This is known when μ is a diagonal state. See Logic Blog 2020 Prop 9.3 (arxiv.org/abs/2101.09508).

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