

Computably t.d.l.c. groups

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The main questions

- (A) How can one define a computable presentation of a t.d.l.c. group? Which t.d.l.c. groups have such a presentation?
- (B) Given a computable presentation of a t.d.l.c. group, are objects such as the rational valued Haar measures, the modular function, or the scale function computable?
- (C) Do constructions that lead from t.d.l.c. groups to new t.d.l.c. groups have algorithmic versions?
- (D) When is a computable presentation of a t.d.l.c. group unique up to computable isomorphism?

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Talk 1 mainly addresses Question A

- We will introduce two notions of computable presentation of a t.d.l.c. group G , and show their equivalence.
- The first notion relies on standard notions of computability in the uncountable setting.
- The second notion restricts computation to a countable structure of approximations of the elements, the “meet groupoid” of compact open cosets.
- Based on these, we obtain various examples of computably t.d.l.c. groups.
- The first talk also outlines some computability theoretic notions needed.

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Talk 2 mainly addresses Question B-D

- We show that given a computable presentation of a t.d.l.c. group G , the modular function and the Cayley-Abels graphs (in the compactly generated case) are computable.
- We discuss algorithmic properties of the scale function on G , but leave open whether it can be non-computable.
- We explain why the class of computably t.d.l.c. groups is closed under most of the constructions studied by Wesolek (PLMS, 2015).
- We will give a criterion based on meet groupoids when the computable presentation is unique up to computable isomorphism.

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Examples of computably t.d.l.c. groups

- All computable profinite groups and all computable discrete groups
- $(\mathbb{Q}_p, +)$, the additive group of p -adic numbers for a prime p
- The semidirect product $\mathbb{Z} \ltimes \mathbb{Q}_p$ where $g \in \mathbb{Z}$ acts as $x \mapsto xp$ on \mathbb{Q}_p
- The groups $SL_n(\mathbb{Q}_p)$ for $n \geq 2$
- $Aut(T_d)$, the automorphisms of a homogeneous undirected tree of degree d .

By convention, all t.d.l.c. groups will be separable and infinite.

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Computable structures: the countable case

A structure in the model theoretic sense consists of a nonempty set D , called the domain, with relations and functions defined on it.

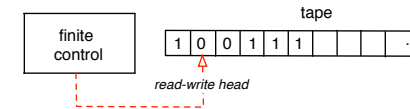
Definition (Mal'cev and Rabin independently, 1960s)

- A **computable structure** is a structure such that the domain is a computable set $D \subseteq \mathbb{N}$, and the functions and relations of the structure are computable.
- A countable structure S is called **computably presentable** if some computable structure W is isomorphic to it. In this context we call W a **computable copy** of S .

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Computable functions

Picture of a Turing machine with one tape:



Definition

Given a set $S \subseteq \mathbb{N}^k$, where $k \geq 1$, a function $f: S \rightarrow \mathbb{N}$ is called **computable** if there is a Turing machine that on inputs n_1, \dots, n_k decides whether the tuple of inputs (n_1, \dots, n_k) is in S , and if so outputs $f(n_1, \dots, n_k)$.

One version of the Church-Turing thesis states that computability in this sense is the same as being computable by some algorithm.

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Example

For each $k \geq 1$, the group $GL_k(\mathbb{Q})$ is computably presentable.

To obtain a computable copy:

- fix an algorithmic encoding of the rational $k \times k$ matrices by natural numbers
- let the domain D be the computable set of numbers that encode a matrix with nonzero determinant
- the matrix operations are computable.

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Computable structures: the uncountable case

- computable analysis represents all the elements by “names”
- Let \mathbb{N}^* denote the tree of strings with natural number entries. Names usually are elements of the set $[T]$ of paths on some computable subtree T of \mathbb{N}^* .
- For instance, as a name for a real number r , take a path coding a sequence of rationals $\langle q_n \rangle_{n \in \mathbb{N}}$ such that
$$\forall n |q_n - q_{n+1}| \leq 2^{-n} \text{ and } \lim_n q_n = r.$$
- Such names are directly accessible to computation of Turing machines with tapes that hold the infinite inputs.
- One can now define computability of functions and relations on $[T]$: one requires that they are computable on the names.
- For example, $\exp: \mathbb{R} \rightarrow \mathbb{R}$ is computable.

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Ad hoc ways to define computability for some uncountable structures

The following sometimes works for particular classes of uncountable structures: impose algorithmic constraints on the definition of the class.

An example is the definition of when a profinite group G is computable, due to Smith and la Roche (two papers from 1981):

$$G = \varprojlim_i (A_i, \psi_i)$$

for a **computable** diagram $(A_i, \psi_i)_{i \in \mathbb{N}}$ of finite groups and epimorphisms $\psi_i: A_i \rightarrow A_{i-1}$ ($i > 0$).

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Computable presentations of t.d.l.c. groups

- We aim at a robust definition of the class of t.d.l.c. groups with a computable presentation.
- We ask that our definition extend the existing definitions for discrete, and for profinite groups.
- We want this class to have good algorithmic closure properties.

We provide two types of computable presentations, which will turn out to be equivalent in the sense that from a presentation of one type one can construct a presentation of the other type.

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1. Computable Baire presentations

Each totally disconnected Polish space is homeomorphic to the set of paths $[T]$ for some subtree T of \mathbb{N}^* (the tree of strings with natural number entries). So there is no need to distinguish between names and objects in our setting.

For a computable Baire presentation,

- the domain of G equals $[T]$ for what we call an effectively locally compact subtree of \mathbb{N}^*
- the operations are computable in the sense of Turing machines holding the infinite inputs on tapes of which the content keeps unchanged during the computation.

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2. Computable presentations via a meet groupoid

- We introduce an algebraic structure $\mathcal{W}(G)$ on the countable set of compact open cosets in G , together with \emptyset .
- This structure is a partially ordered groupoid, with the usual set inclusion, and multiplication of a left coset of a subgroup U with a right coset of U (which is a coset).
- The intersection of two compact open cosets is such a coset itself, unless it is empty, so we have a meet semilattice.

A computable presentation of G via meet groupoids is a computable copy of the meet groupoid of G such that the index function on compact open subgroups, namely $U, V \mapsto |U: U \cap V|$, is also computable.

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Notation for strings of natural numbers

- Let \mathbb{N}^* denote the set of strings with natural numbers as entries. We use letters σ, τ, ρ etc. for elements of \mathbb{N}^* .
- The set \mathbb{N}^* can be seen as a directed tree:
 - the empty string is the root, and
 - the successor relation is given by appending a number at the end of a string.
- $\sigma \preceq \tau$ denotes that σ is an initial segment of τ .
- can identify finite strings of length n with partial functions on \mathbb{N} having domain $\{0, \dots, n-1\}$.

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Section 2:

Computability on paths of rooted trees

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Some notation for rooted trees

- By a (directed) tree we mean a nonempty subset T of \mathbb{N}^* such that $\sigma \in T$ and $\rho \prec \sigma$ implies $\rho \in T$.
- By $[T]$ one denotes the set of paths of a tree T .
- If T has no leaves, $[T]$ is a closed set in Baire space $\mathbb{N}^{\mathbb{N}}$ equipped with the usual product topology.
- For $\sigma \in T$ let

$$[\sigma]_T = \{X \in [T] : \sigma \prec X\}.$$

That is, $[\sigma]_T$ is the cone of paths on T that extend σ .

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Computable functions on $[T]$

Let $T \subseteq \mathbb{N}^*$ be a computable rooted tree without leaves.

Definition

- A function $\Phi : [T] \times \mathbb{N} \rightarrow \mathbb{N}$ is computable if there is a Turing machine that when it has the list of the values $f(0), f(1), f(2), \dots$ written on the read-only tape and w on some other tape, it can determine the value $\Phi(f, w)$.
- A function $\Phi : [T] \rightarrow [\mathbb{N}^*]$ is computable if the function $\tilde{\Phi} : [T] \times \mathbb{N} \rightarrow \mathbb{N}$ given by $\tilde{\Phi}(g, n) = \Phi(g)(n)$ is computable.
- In a similar way, define that $\Phi : [T] \times [S] \rightarrow [\mathbb{N}^*]$ is computable, using a TM with two read-only tapes.

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Discussion of the definition of computability for functions $\Phi : [T] \rightarrow [T]$

Recall $T \subseteq \mathbb{N}^*$ is a rooted tree, and we consider functions Φ on the path space $[T] = \{f : \forall n f \upharpoonright \{0, \dots, n-1\} \in T\}$.

Specify an arbitrary Φ by conditions of the type $\Phi(f)(w) = k$, $w, k \in \mathbb{N}$.

Specify a continuous Φ by a sufficiently large, consistent list of conditions of the type $C(\sigma, w, k)$, where $\sigma \in T$, saying that for each $f \in [T]$ extending σ , we have $\Phi(f)(w) = k$.

Specify a computable Φ by a sufficiently large, consistent **effective** list of conditions $C(\sigma, w, k)$. (This is what the oracle Turing machine provides.)

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Example

Let $T = \mathbb{N}^*$. The function $\Phi : [T] \times \mathbb{N} \rightarrow \mathbb{N}$ with $\Phi(f, w) = \sum_{i=0}^w f(i)$ is computable.

Proof.

The oracle TM with f written on the read-only tape queries the values of $f(i)$ for $i = 0, \dots, w$ one by one and adds them up. \square

Fact

Φ is computable $\Rightarrow \Phi$ is continuous.

So e.g. the function $f \mapsto \min(\text{range} f)$ is not computable.

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(2.3) Computably locally compact (c.l.c.) trees

Definition (computably locally compact trees)

- Let T be a computable subtree of \mathbb{N}^* without leaves such that only the root can have infinitely many successors.
- We say that T is **computably locally compact**, or c.l.c. for short, if there is a computable binary function H such that, if $\rho \in T$ is a nonempty string, then

$$\rho(i) \leq H(\rho(0), i) \text{ for each } i < |\rho|.$$

We note that in the paper, the definition is slightly more general: there is a $k \in \mathbb{N}$ such that only strings of length $< k$ can have infinitely many successors.

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Section 3:

Defining computably t.d.l.c. groups
via Baire presentations

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Computable Baire presentation

Definition (3.1)

A **computable Baire presentation** is a t.d.l.c. group of the form $H = ([T], \text{Mult}, \text{Inv})$ where

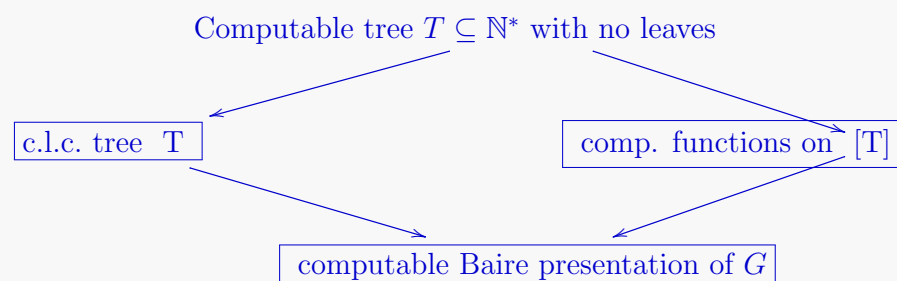
- 1 T is computably locally compact as defined in 2.3;
- 2 $\text{Mult}: [T] \times [T] \rightarrow [T]$ and $\text{Inv}: [T] \rightarrow [T]$ are computable.

We say that a t.d.l.c. group G is **computably t.d.l.c.** (via a Baire presentation) if $G \cong H$ for such a group H .

Such a definition works for t.d.l.c. algebraic structures in general.

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Dependency of definitions leading to computable Baire presentations



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The ring \mathbb{Q}_p has a computable Baire presentation

Let Q be the tree of strings $\sigma \in \mathbb{N}^*$ such that

- all entries, except possibly the first, are among $\{0, \dots, p-1\}$,
- $r0 \not\prec \sigma$ for each $r > 0$.

A string $r\sigma \in Q$ denotes the rational $p^{-r}n_\sigma \in \mathbb{Z}[1/p]$, where n_σ is the number which has σ as a p -ary expansion, written in reverse order:

$$n_\sigma = \sum_{i < |\sigma|} p^i \sigma(i).$$

For instance, let $p = 3$; then

$$(3, 1, 0, 2) \text{ denotes the rational } 3^{-3} \cdot (1 + 2 \cdot 9) = 19/27.$$

We allow the case that σ ends in 0. The condition that $r0 \not\prec \sigma$ for each $r > 0$ says that p does not divide n_σ .

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Addition and multiplication on \mathbb{Q}_p are computable

- For the addition operation, consider an oracle Turing machine with two oracle tapes starting with notations $r\sigma$ and $s\tau$ of numbers $p^{-r}m$ and $p^{-s}n$, where $|\sigma| + r - s = |\tau|$.
- Then $p^{-r}m + p^{-s}n = p^{-s}(p^{s-r}m + n)$.
- The machine can output a string denoting $p^{-r}m + p^{-s}n$.
- A similar argument works for multiplication.

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An auxiliary fact on c.l.c. trees

Given a computable subtree S of a c.l.c. tree T , when can we get rid of the leaves but keep the same paths?

Proposition (3.2)

- Let T be a c.l.c. tree. Let S be a computable subtree of T .
- Suppose that there is a uniformly computable dense sequence $(f_i)_{i \in \mathbb{N}}$ in $[S]$.

Then the tree $\tilde{S} = \{\sigma : [\sigma]_S \neq \emptyset\}$ is decidable. It follows that \tilde{S} is c.l.c. Of course, $[\tilde{S}] = [S]$.

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Computable Baire presentation of $SL_n(\mathbb{Q}_p)$

The presentation is based on the computable tree

$$T = \{\sigma : \forall i < n^2 [\sigma^i \in \mathbb{Q}]\},$$

where σ^i is the string of entries of σ in positions of the form $kn^2 + i$ for some $k \in \mathbb{N}$.

- The determinant function $[T] \rightarrow \mathbb{Q}_p$ is computable.
- It is easy to show that there is a computable subtree S of T such that paths on S denote matrices with determinant 1.
- $SL_n(\mathbb{Z}[1/p])$ is dense in $SL_n(\mathbb{Q}_p)$ (strong approximation for algebraic groups).
- The paths on S corresponding to matrices in $SL_n(\mathbb{Z}[1/p])$ are precisely the ones that are 0 from some point on. Clearly there is a computable listing $(f_i)_{i \in \mathbb{N}}$ of these paths.

So we can use Prop 3.2 to get rid of the leaves, and so to turn S into a computably locally compact tree \tilde{S} with $[\tilde{S}] = [S]$.

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Section 4:

Defining computably t.d.l.c. groups

via meet groupoids

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Axioms for groupoids

Intuitively, the notion of a groupoid generalizes the notion of a group by allowing that the binary operation is partial. A groupoid is given by a domain \mathcal{W} on which a unary operation $(\cdot)^{-1}$ and a partial binary operation, denoted by “ \cdot ”, are defined. These operations satisfy the following conditions:

- (a) associativity in the sense that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, with either both sides or no side defined (and so the parentheses can be omitted);
- (b) $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are always defined;
- (c) if $A \cdot B$ is defined then $A \cdot B \cdot B^{-1} = A$ and $A^{-1} \cdot A \cdot B = B$.

Category view: a groupoid is a small category in which each morphism has an inverse.

$A: U \rightarrow V$ means that U, V are idempotent, and $A = UA = AV$.

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Axioms for meet groupoids

Definition

A **meet groupoid** is a groupoid $(\mathcal{W}, \cdot, (\cdot)^{-1})$ that is also a meet semilattice $(\mathcal{W}, \cap, \emptyset)$ of which \emptyset is the least element.

Writing $A \subseteq B \iff A \cap B = A$, it satisfies the conditions

- (d) $\emptyset^{-1} = \emptyset = \emptyset \cdot \emptyset$, and $\emptyset \cdot A$ and $A \cdot \emptyset$ are undefined for each $A \neq \emptyset$,
- (e) if U, V are idempotents such that $U, V \neq \emptyset$, then $U \cap V \neq \emptyset$,
- (f) $A \subseteq B \iff A^{-1} \subseteq B^{-1}$,
- (g) if $A_i \cdot B_i$ are defined ($i = 0, 1$) and $A_0 \cap A_1 \neq \emptyset \neq B_0 \cap B_1$, then

$$(A_0 \cap A_1) \cdot (B_0 \cap B_1) = A_0 \cdot B_0 \cap A_1 \cdot B_1.$$

Item (g) implies that if U and V are idempotent, then so is $U \cap V$.

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The meet groupoid of a t.d.l.c. group

Definition

Let G be a t.d.l.c. group. We define a meet groupoid $\mathcal{W}(G)$.

- Its domain consists of the compact open cosets in G , as well as the empty set.
- We define $A \cdot B$ to be the usual product AB in case that $A = B = \emptyset$, or A is a left coset of a subgroup V and B is a right coset of V ;
- otherwise $A \cdot B$ is undefined.

We will use the usual group theoretic terminology for elements of an abstract meet groupoid \mathcal{W} .

- If U is an idempotent of \mathcal{W} we call U a **subgroup**
- if $AU = A$ we call A a **left coset** of U
- if $UB = B$ we call B a **right coset** of U .

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Where the approximation structures come from

The idea to study appropriate Polish groups via an algebraic structure on their open cosets is due to Katrin Tent, and first appeared in Kechris, N. and Tent, 2018. This idea was further elaborated in a 2021 paper by Nies, Schlicht and Tent on the complexity of the isomorphism problem for oligomorphic groups. In both works, approximation structures are used that are given by the ternary relation “ $AB \subseteq C$ ”, where A, B, C are certain open cosets. They are called “coarse groups”.

In the present work, it will be important that we have explicit access to the combination of the groupoid and the meet semilattice structures (which coarse groups don’t provide). Coarse groups are too “coarse” an algebraic structure to analyse algorithmic aspects of t.d.l.c. groups.

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Computationally t.d.l.c. groups via meet groupoids

A meet groupoid \mathcal{W} is called **Haar computable** if

- (a) its domain is a computable subset D of \mathbb{N} ;
- (b) the groupoid and meet operations are computable; in particular, the relation $\{\langle x, y \rangle : x, y \in S \wedge x \cdot y \text{ is defined}\}$ is computable;
- (c) the partial function with domain contained in $D \times D$ sending a pair of subgroups $U, V \in \mathcal{W}$ to $|U : U \cap V|$ is computable.

Definition (Computationally t.d.l.c. groups via meet groupoids)

Let G be a t.d.l.c. group. We say that G is **computationally t.d.l.c.** via a meet groupoid if $\mathcal{W}(G)$ has a Haar computable copy \mathcal{W} .

Condition (c) is equivalent to saying that every Haar measure μ on G that assigns a rational to some (and hence every) compact open coset is computable on \mathcal{W} .

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Computationally t.d.l.c. via meet groupoid

Example (4.9)

For any prime p , the additive group \mathbb{Q}_p and the group $\mathbb{Z} \ltimes \mathbb{Q}_p$ are computationally t.d.l.c. via a meet groupoid.

\mathbb{Q}_p : proper open subgroups are of the form $U_r := p^r \mathbb{Z}_p$ for some $r \in \mathbb{Z}$, all compact. For each r there is a canonical epimorphism $\pi_r : \mathbb{Q}_p \rightarrow C_{p^\infty}$ with kernel U_r . So each compact open coset of U_r can be uniquely written in the form $D_{r,a} = \pi_r^{-1}(a)$ for some $a \in C_{p^\infty}$.

$\mathbb{Z} \ltimes \mathbb{Q}_p$: it has the same compact open subgroups as \mathbb{Q}_p .

We have $D_{r,a} = g^{-z} D_{r-z,a} g^z$ for each $z \in \mathbb{Z}$. So we have

$A_{r,z} : U_{r-z} \rightarrow U_r$ where $A_{r,z} = g^z U_r$. In particular $A_{1,1} : U_0 \rightarrow U_1$.

For both groups, the groupoid operation and index function are computable.

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Section 5:

Equivalence

of the two types of computable presentations

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Equivalence of computable presentations in the sense of Def. 3.1 and Def. 4.6

Theorem

A group G is computationally t.d.l.c. via a Baire presentation \iff
 G is computationally t.d.l.c. via a meet groupoid.

From a presentation of G of one type, one can uniformly obtain a presentation of G of the other type.

First discuss \Leftarrow .

We need a few preliminaries.

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For “ \Leftarrow ”: a computable presentation of the group of permutations of \mathbb{N}

For strings $\sigma_0, \sigma_1 \in \mathbb{N}^*$ of the same length n , let $\sigma_0 \otimes \sigma_1$ denote the string τ of that length such that $\tau(k) = \langle \sigma_0(k), \sigma_1(k) \rangle$ for each $k < n$ (where $\langle \cdot, \cdot \rangle$ is a computable pairing function, such as Cantor’s). Define a computable tree without leaves by $\text{Tree}(S_\infty) = \{\sigma \otimes \tau : \sigma, \tau \in \mathbb{N}^* \wedge$

$$\sigma, \tau \text{ are 1-1} \wedge \sigma(\tau(k)) = k \wedge \tau(\sigma(i)) = i \text{ whenever defined}\}.$$

The paths of $\text{Tree}(S_\infty)$ can be viewed as the permutations of \mathbb{N} , paired with their inverses:

$$[\text{Tree}(S_\infty)] = \{f \otimes f^{-1} : f \text{ is permutation of } \mathbb{N}\}.$$

The group operations on $\text{Tree}(S_\infty)$ are computable.

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1. $\tilde{G} = \mathcal{G}_{\text{comp}}(\mathcal{W})$ is isomorphic to G

Define a group homomorphism $\Phi: G \rightarrow \tilde{G}$ by letting $\Phi(g)$ be the element of S_∞ corresponding to the left action of g , i.e.

$$\Phi(g)(A) = gA$$

where $A \in \mathcal{W}(G)$ (identified with the computable copy \mathcal{W}).

Claim

$$\Phi: G \cong \tilde{G}.$$

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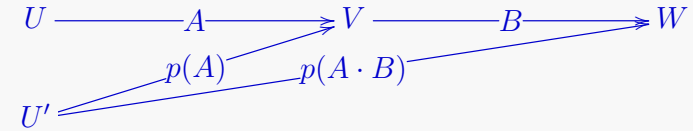
\Leftarrow : From meet groupoid to Baire presentation

Given: a Haar computable meet groupoid \mathcal{W} with domain \mathbb{N} such that $\mathcal{W}(G) \cong \mathcal{W}$. We can use this isomorphism to identify $\mathcal{W}(G)$ and \mathcal{W} .

Definition

Let $\tilde{G} = \mathcal{G}_{\text{comp}}(\mathcal{W})$ be the closed subgroup of S_∞ consisting of elements p that preserve the meet operation of \mathcal{W} , and satisfy $p(A) \cdot B = p(A \cdot B)$ whenever $A \cdot B$ is defined.

The following diagram displays the condition in the definition above in category terms as a commutative diagram.



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2: $\text{Tree}(\tilde{G})$ is c.l.c.

Using the assumption that \mathcal{W} is Haar computable, we show that $\text{Tree}(\tilde{G})$ is c.l.c. To show that $\text{Tree}(\tilde{G})$ is computable we use:

Claim

A finite injection α on \mathbb{N} can be extended to some $p \in \tilde{G}$
 $\iff B \cdot A^{-1}$ is defined whenever $\alpha(A) = B$, and

$$\bigcap \{B \cdot A^{-1} : \alpha(A) = B\} \neq \emptyset.$$

To show that $\text{Tree}(\tilde{G})$ has the computable bound on branching:

Lemma (Effectively finite suborbits)

Suppose that $U \in \mathcal{W}$ is a subgroup and L is a left coset of U . Let $F \in \mathcal{W}$. One can uniformly in U, L and F compute a strong index for the finite set $\mathcal{L} = \{p(F) : p \in [S] \wedge p(U) = L\}$.

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Theorem (Recall)

A group G is computably t.d.l.c. via a Baire presentation \iff
 G is computably t.d.l.c. via a meet groupoid.

From a presentation of G of one type, one can uniformly obtain a presentation of G of the other type.

Now discuss the implication \Rightarrow .

We also need a few preliminaries.

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Computing with compact open sets (1)

Let $h: \mathbb{N}^* \rightarrow \mathbb{N}$ be the canonical encoding given by
 $h(w) = \prod_{i < |w|} p_i^{w_i+1}$, where p_i is the i -th prime number.

Definition (Strong indices for finite sets of strings)

For a finite set $u \subseteq \mathbb{N}^*$ let $n_u = \sum_{\eta \in u} 2^{h(\eta)}$; one says that n_u is the **strong index** for u .

Definition (Code numbers for compact open sets)

Let T be a c.l.c. tree. For a finite set $u \subseteq T - \{\emptyset\}$, let

$$\mathcal{K}_u = \bigcup_{\eta \in u} [\eta]_T,$$

(note that this set is compact). By a **code number** for a compact open set $\mathcal{K} \subseteq [T]$ we mean the strong index for a finite set $u \subseteq \mathbb{N}^*$ such that $\mathcal{K} = \mathcal{K}_u$.

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Computing with compact open sets (2)

Inclusion, Boolean operations are decidable for the \mathcal{K}_u

Lemma

Let T be a c.l.c. tree. Given code numbers u, w ,

- (i) one can compute code numbers for $\mathcal{K}_u \cup \mathcal{K}_w$ and $\mathcal{K}_u \cap \mathcal{K}_w$;
- (ii) one can decide whether $\mathcal{K}_u \subseteq \mathcal{K}_w$. In particular, one can, given a code number $u \in \mathbb{N}$, compute the minimal code number $u^* \in \mathbb{N}$ such that $\mathcal{K}_{u^*} = \mathcal{K}_u$.

Definition

Given a c.l.c. tree T , let E_T denote the set of **minimal** code numbers for compact open subsets of $[T]$. By the foregoing lemma, E_T is decidable.

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Computing with compact open sets (3)

Lemma

Let T, S be c.l.c. trees. Suppose a function $\Phi: [T] \rightarrow [S]$ is computable via a partial computable function P_Φ . Given code numbers u, w , one can decide whether $\Phi(\mathcal{K}_u^T) \subseteq \mathcal{K}_w^S$.

This is used to prove the following.

Lemma

Let G be computably t.d.l.c. via a computable Baire presentation $([T], \text{Mult}, \text{Inv})$.

- (i) There is a computable function $I: E_T \rightarrow E_T$ such that for each $u \in E_T$, one has $\mathcal{K}_{I(u)} = (\mathcal{K}_u)^{-1}$.
- (ii) For $u, v, w \in E_T$ one can decide whether $\mathcal{K}_u \mathcal{K}_v \subseteq \mathcal{K}_w$.

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“ \Rightarrow ”: From Baire presentation to meet groupoid

- The set $\{u \in E_T : \mathcal{K}_u \text{ is a coset}\}$ can be obtained via an existential quantification over a computable binary relation.
- So there is a computable 1-1 function θ defined on $\mathbb{N} - \{0\}$ such that the range of θ equals this set.
- Define a meet groupoid \mathcal{W} with domain \mathbb{N} , thinking of n as coding $A_n = \mathcal{K}_{\theta(n)}$ for $n > 0$, and $A_0 = \emptyset$.
- Show the meet groupoid operations are computable.
- For subgroups $U, V \in \mathcal{W}$ compute $|U : U \cap V|$ by finding in \mathcal{W} the distinct left cosets of $U \cap V$ contained in U with union U .

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The actions on \mathcal{W} are computable

Corollary (5.9, a corollary to “ \Rightarrow ” of the proof above)

- As before, let \mathcal{W} be a Haar computable copy of $\mathcal{W}(G)$ (with domain \mathbb{N}).
- The left and right actions $[T] \times \mathbb{N} \rightarrow \mathbb{N}$, given by
$$(g, A) \mapsto gA \text{ and } (g, A) \mapsto Ag,$$
are computable.

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A condition for computability in the abelian case

Let G be abelian t.d.l.c. There is a compact open $K \leq G$ and $L = G/K$ is discrete. The sequence

$$0 \rightarrow K \rightarrow G \rightarrow L \rightarrow 0 \text{ is exact;}$$

G is a topological extension of L by K .

Theorem (Lupini, Melnikov and N., J Algebra, 2022)

Let G be abelian t.d.l.c. group. The following are equivalent.

- (1) G is computably t.d.l.c.
- (2) There are a computable profinite group K and a computable discrete group L such that G is an extension of L by K via a computable co-cycle $c: L \times L \rightarrow K$.

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Section 6:

Algorithmic properties of objects
associated with a t.d.l.c. group

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The modular function is computable

Throughout, let G be computably t.d.l.c. via a Baire presentation $([T], \text{Mult}, \text{Inv})$, and let \mathcal{W} be the Haar computable copy of $\mathcal{W}(G)$

Recall that $\Delta(g) = \mu(Ug)/\mu(U)$, where U is any compact open subgroup, μ a left Haar measure; may assume μ is rational valued, and hence that μ is computable.

Since the right action of G on \mathcal{W} is computable, we have:

Proposition

The modular function $\Delta: [T] \rightarrow \mathbb{Q}^+$ is computable.

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Algorithmic properties of the scale function

For a compact open subgroup V of G and an element $g \in G$ let $m(g, V) = |V^g: V \cap V^g|$.

Define the scale function $[T] \rightarrow \mathbb{N}$ by

$$s(g) = \min\{m(g, V): V \text{ is a compact open subgroup}\}.$$

E.g., in $\mathbb{Z} \times \mathbb{Q}_p$, where $g \in \mathbb{Z}$ acts as $x \rightarrow xp$, we have $s(g) = 1$, $s(g^{-1}) = p$.

Example

For $d \geq 3$, the scale function on $\text{Aut}(T_d)$ in the computable presentation of Ex. 5.10 is computable.

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Cayley-Abels graphs are computable

If G is compactly generated, there is a compact open subgroup U , and a set $S = \{s_1, \dots, s_k\} \subseteq G$ such that $S = S^{-1}$ and $U \cup S$ algebraically generates G . The **Cayley-Abels graph**

$$\Gamma_{S,U} = (V_{S,U}, E_{S,U})$$

of G is given as follows. The vertex set $V_{S,U}$ is the set $L(U)$ of left cosets of U , and the edge relation is

$$E_{S,U} = \{gU, gsU\}: g \in G, s \in S\}.$$

Theorem

Suppose that G is computably t.d.l.c. and compactly generated.

- (i) Each Cayley-Abels graph $\Gamma_{S,U}$ of G has a computable copy \mathcal{L} .
- (ii) Let $\Gamma_{T,V}$ be another Cayley-Abels graph. Any two computable copies of $\Gamma_{S,U}$ and $\Gamma_{T,V}$ are computably quasi-isometric.

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Basic algorithmic properties of the scale function

The following are immediate from Cor 5.9 (that the left and right actions on the Haar computable meet groupoid \mathcal{W} are computable):

- (i) The function $m: [T] \times \mathbb{N} \rightarrow \mathbb{N}$, $m(g, V) = |V^g: V \cap V^g|$ (defined to be 0 if the second argument is not a subgroup) is computable.
- (ii) The scale is computable if and only if one can algorithmically decide whether a compact open subgroup V is minimizing for g (Fact 6.5).
- (iii) The scale function is computably approximable from above. That is, there is a computable function $\Theta: [T] \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\Theta(f, r) \geq \Theta(f, r+1)$ for each $f \in [T], r \in \mathbb{N}$, and $s(f) = \lim_r \Theta(f, r)$ (Fact 6.6).

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An open question

Question

Given a computable presentation of a t.d.l.c. group G based on a tree T , is the scale function $s: [T] \rightarrow \mathbb{N}$ computable?

- Willis' tidying procedure, and Möller's spectral radius formula don't answer this in the affirmative.
- If the answer is in the negative, one can further ask whether there is a computably presented G such that the scale is non-computable for **each** of its computable presentations.

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The class of computably t.d.l.c. groups is closed under suitable algorithmic versions of many constructions that have been studied in the theory of t.d.l.c. groups: (1), (2), (3) and (6) described in Thm 1.3. of Wesolek, **Elementary t.d.l.c. groups** (2015).

- passing to closed subgroups,
- taking group extensions via continuous actions,
- forming "local" direct products, and
- taking quotients by closed normal subgroups

The first three are reasonably straightforward. All computable presentations will be Baire. By $Tree(G)$ we denote the c.l.c. tree underlying such a presentation of G .

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Section 7:

Closure properties

of the class of computably t.d.l.c. groups

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Computable closed subgroups

Fact

Let G be a computably t.d.l.c. group. Let H be a closed subgroup of G (so that $Tree(H)$ is a subtree of $Tree(G)$). Then H is computably t.d.l.c. via

- the Baire presentation based on $Tree(H)$ (which is c.l.c.)
- the operations of G restricted to H .

Example

Burger and Mozes studied the closed subgroups $U(F)$ of $Aut(T_d)$, where $d \geq 3$ and F is a subgroup of S_d .

Using the computable presentation of $Aut(T_d)$ together with the preceding fact, each group $U(F)$ is computably t.d.l.c.

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Computable semidirect products

Proposition

- Let G, H be computably t.d.l.c. groups.
- Suppose $\Phi: G \times H \rightarrow H$ is a computable function that specifies an action of G on H via topological automorphisms.

Then the topological semidirect product $L = G \rtimes_{\Phi} H$ is computably t.d.l.c.

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Local direct products

Proposition (Prop 11.5 in the paper with Melnikov)

- Let $(G_i)_{i \in \mathbb{N}^+}$ be computably t.d.l.c. groups uniformly in i , and
- for each i let U_i be a compact open subgroup of G_i , uniformly in i .

Then $G = \bigoplus_{i \in \mathbb{N}^+} (G_i, U_i)$ is computably t.d.l.c.

Here the domain of $G = \bigoplus_{i \in \mathbb{N}^+} (G_i, U_i)$ consists of the functions in $f \in \prod_i G_i$ such that $f(i) \in U_i$ for sufficiently large i .

$\prod_i U_i$ is a compact open subgroup. See Wesolek 2015, Def. 2.3.

This result might be useful to show that the scale function can be noncomputable.

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Quotients by computable closed normal subgroups

Theorem (Thm. 11.11 in paper with Melnikov)

Let N be a closed normal subgroup of G such that $Tree(N)$ is a computable subtree of $Tree(G)$. Then G/N is computably t.d.l.c.

We prove this by building a Haar computable copy of the meet groupoid of G/N .

On the way we have to show that “ $\mathcal{K} \subseteq N\mathcal{L}$ ” is decidable, where \mathcal{K}, \mathcal{L} are compact open sets.

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An application of the theorem

Example (7.6)

For each prime p and each $n \geq 2$, the group $\mathrm{PGL}_n(\mathbb{Q}_p)$ is computably t.d.l.c.

Proof.

In Example 7.2 we obtain a computable Baire presentation $(T, \mathrm{Mult}, \mathrm{Inv})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$.

We employ the closed embedding $F: \mathrm{GL}_n(\mathbb{Q}_p) \rightarrow \mathrm{SL}_{n+1}(\mathbb{Q}_p)$ which extends a matrix A to the matrix B where the new row and new column vanish except for the diagonal element (which necessarily equals $(\det A)^{-1}$).

In this presentation, the centre N of $\mathrm{GL}_n(\mathbb{Q}_p)$ is given by the diagonal $(n+1) \times (n+1)$ matrices such that the first n entries of the diagonal agree. So clearly $Tree(N)$ is a computable subtree of the tree in Ex. 7.2.

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