How expressive is the first–order language for groups?

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May 21, 2001

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Logic in group theory

Applying concepts from computability theory: undecidability, computably enumerable sets.

- Novikov (1955), Boone (1955-57) proved the existence of a finitely presented (f.p.) group with unsolvable word problem.
- Many other problems related to f.p. groups are undecidable - for instance the triviality problem.
- Higman (1960) found out which finitely generated (f.g.) groups are subgroups of a f.p. groups:

  precisely the ones with a computably enumerable word problem.

The word problem for a f.g. group G presented by F_n/N is the complexity of N.

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First–order language

- The first–order language of groups consists of *formulas* built up from equations $t(x_1, \ldots, x_k) = 1$ t in $F(x_1, \ldots, x_n)$ using $\neg, \&, \vee, \rightarrow, \exists x, \forall x$.
- A *sentence* is a formula which has only bound variables.
- For a group G, $\mathrm{Th}(G)$ is the set of sentences which hold in G.

Examples: let [x,y] denote $x^{-1}y^{-1}xy$.

- The sentence $\forall x \forall y \; [x, y] = 1$ expresses that the group is abelian.
- The following sentence is in $\mathrm{Th}(F_n)$):

$$\forall x \forall y \forall z (y \neq 1 \;\&\; [x, y] = 1 \;\&\; [y, z] = 1 \\ \rightarrow [x, z] = 1).$$

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G versus $\mathrm{Th}(G)$

$\mathrm{Th}(G)$ “knows” whether G

- is nilpotent
- has torsion.

But, many properties are not formalizable in first–order logic: for instance,

- finitely generated
- simple
- maximum condition (every subgroup is f.g.)

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Why first–order?

- A f.o. property of G is *intrinsic*. One does not have to go beyond G to verify it.
- Toolbox for first–order logic- compactness theorem, etc There are fewer tools for more expressive languages.
- First–order language is based on equations
- $\mathrm{Th}(G)$ is an interesting "invariant".

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F.g. groups

Let G be f.g. infinite. **To what extent is** G **determined by** $\mathrm{Th}(G)$**?**

By basic results of model theory:

- We need some extra information- at least that G is countable. (There is some uncountable model of $\mathrm{Th}(G)$, since G is infinite.)
- There is in fact some other *countable* model.

Is G the only f.g. model of $\mathrm{Th}(G)$?

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Quasi-axiomatizable

Definition. *A f.g. group G is quasi-axiomatizable if, whenever H is a* **f.g.** *group with the same theory as G, $G \cong H$.*

All f.g. abelian groups G are quasi-axiomatizable. For instance, if $G = \mathbb{Z}^n$, G is the only f.g. group such that

- G abelian, torsion free
- $|G/2G| = 2^n$.

These properties can be captured by an infinite axiom system.

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Free groups

Recently, long-open questions of Tarski (1945) were answered.

Let F_n be the rank n free group.

Theorem 1 (Kharlampovich/Myasnikov 1998) *For each $n > 2$, $\mathrm{Th}(F_n) = \mathrm{Th}(F_2)$. Moreover, this theory is decidable.*

In fact, they showed that the natural embedding $F_n \leq F_m$ $(2 \leq n < m)$ is elementary.

Recall: $F_n \ncong F_m$ for $n \neq m$. So, the Theorem implies that **no** F_n, $n \geq 2$, is quasi-axiomatizable.

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Nilpotent groups

Theorem 2 (Hirshon (1977), Oger (1990))

- *Each f.g. torsion-free step-2 nilpotent group is quasi-axiomatizable.*
- *There are f.g. torsion-free step-3 nilpotent groups G, H such that $\mathrm{Th}(G) = \mathrm{Th}(H)$, but $G \not\cong H$.*

Oger's part was: for f.g. nilpotent G, H,

$$\mathrm{Th}(G) = \mathrm{Th}(H) \;\Leftrightarrow\; G \cdot \mathbb{Z} \cong H \cdot \mathbb{Z}.$$

The direction $\Leftarrow$ is actually true for any groups G, H.

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QFA groups

We consider f.g. groups G where one axiom, along with the information that G is f.g., suffices.

Definition 3 *An infinite f.g. group G is quasi-finitely axiomatizable (QFA) if there is a first–order sentence φ such that*

- $G \models \varphi$
- *H is a f.g. group such that $H \models \varphi$, then $G \cong H$.*

Thus the axiom provides a characterization of G among the f.g. groups, using only elementary properties of G.

Using quantifier elimination for the theory of abelian groups, one can prove that no f.g. abelian group G is QFA. Thus one needs an infinite axiom system to describe G.

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Results on QFA

Theorem 4 (N)
- *For each prime p, the restricted wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ is QFA.*
- *$UT_3^3(\mathbb{Z})$, or equivalently the free step-2 nilpotent group of rank 2, is QFA.*

"Separating classes of groups by first–order sentence", available on my home page http://www.math.uchicago.edu/∼nies

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Recall that for groups G, B, E one writes $G = B \rtimes E$ if

- $BE = G$
- $B \lhd G$
- $E \cap B = \{1\}$.

Then $E \cong G/B$.

By the definition, $W = \mathbb{Z}_p \wr \mathbb{Z}$ is a split extension $A \rtimes C$, where

- $A = \bigoplus_{r \in \mathbb{Z}} \mathbb{Z}_p^{(r)}$
- $\mathbb{Z}_p^{(r)}$ is a copy of $\mathbb{Z}_p$
- $C = \langle d \rangle$ with d of infinite order
- d acts on A by shifting, i.e. $(\mathbb{Z}_p^{(r)})^d = \mathbb{Z}_p^{(r+1)}$.

W has the presentation

$$\langle a, d \mid a^p,\ [v_r, v_s]\ (r, s \in \mathbb{Z}, r < s) \rangle$$

where $v_r = a^{d^r}$.

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Classes of groups

Let $\mathcal{E}$ be a class of groups. The theory $\mathrm{Th}(\mathcal{E})$ *is the set of first–order sentences which hold for all members of $\mathcal{E}$.*

Example. The sentence

$$\forall u, v \exists r, s, t\ [u, v] = r^2 s^2 t^2$$

holds for all groups.

(Let $r = u^{-1}v^{-1}$, $s = vuv^{-1}$, $t = v$).

$\mathrm{Th}(\mathcal{E})$ is undecidable for many classes $\mathcal{E}$. For instance when

- $\mathcal{E}$= all groups or
- $\mathcal{E}$ = finite p–groups for a fixed p (Mal'cev 1965).

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Separating classes

If $\mathcal{C} \subset \mathcal{D}$, then $\mathrm{Th}(\mathcal{C}) \supseteq \mathrm{Th}(\mathcal{D})$. One can gauge the expressiveness of first–order logic in group theory by asking:

for which inclusions $\mathcal{C} \subset \mathcal{D}$ of natural classes are the theories different?

An example, where $\mathcal{C}$ = "finite", $\mathcal{D}$= "f.p. with solvable WP".

Proposition 5 (Higman) *There is an existential sentence φ which holds for a f.p. group H with solvable word problem, but fails for all finite groups.*

Let H be $\langle a_0, a_1, a_2, a_3 \mid r_0, r_1, r_2, r_3 \rangle$, where $r_i = a_i^{a_{i+1}} a_i^{-2}$ $(i = 0, 1, 2, 3)$, and $i + 1$ is computed modulo 4.

Let φ be

$\exists x_0 \ldots \exists x_3 (x_0 \neq 1\ \&\ \bigwedge_{i=0,1,2,3} r_i(x_0, \ldots, x_3) = 1).$

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