Approximations to bi-interpretability with $\mathbb{N}$

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- Analize structures from computability theory by proving approximations to bi-interpretability with $\mathbb{N}$.
- This can give information about
  - the degree of the theory
  - definability
  - restrictions on automorphism
  - model theoretic properties
- The structures are:
  - Turing degrees of Δ^0_2-sets and computably enumerable (c.e.) sets
  - C.e. many-one degrees
  - C.e. sets under inclusion
  - Lattices of theories which form Σ^0_n-sets ($n \geq 1$)

What is bi-interpretability with $\mathbb{N}$?

$\boldsymbol{A} = (A; R_1, \ldots, R_n)$ is bi-interpretable with $\mathbb{N}$ if there is a structure with the same domain

$$\boldsymbol{A}_{\mathbb{N}} = (A; \oplus, \otimes)$$

such that

- $(A; \oplus, \otimes) \cong (\mathbb{N}, +, \times)$
- $R_1, \ldots, R_n$ and $\oplus, \otimes$ are interdefinable

For $\boldsymbol{B} = \mathbb{N}$, this coincides with the usual model-theoretic notion of bi-interpretability of structures $\boldsymbol{A}, \boldsymbol{B}$

Example: $\mathbb{Q}$ is b.i. with $\mathbb{N}$

Consequences

If $\boldsymbol{A}$ is b.i. with $\mathbb{N}$, then $\boldsymbol{A}$ has the same model theoretic properties as $\mathbb{N}$.

- $\mathrm{Th}(\boldsymbol{A}) \equiv_m \mathrm{Th}(\mathbb{N})$
- $\boldsymbol{A}$ is a prime model and
  a minimal model of $\mathrm{Th}(\boldsymbol{A})$
- $\boldsymbol{A}$ is rigid

BI with $\mathbb{N}$ in parameters

$\boldsymbol{A}$ is b.i. with $\mathbb{N}$ *in parameters* if

$$(\boldsymbol{A}, \overline{\boldsymbol{a}}) \text{ is b.i. with } \mathbb{N},$$

for some list of constants $\overline{\boldsymbol{a}}$ in A.

- Still $\mathrm{Th}(\boldsymbol{A}) \equiv_m \mathrm{Th}(\mathbb{N})$
- $\boldsymbol{A}$ is still a prime model
  and a minimal model of $\mathrm{Th}(\boldsymbol{A})$
- $\mathrm{Aut}(\boldsymbol{A})$ is countable

$\boldsymbol{D}_T(\leq \emptyset')$

Let $\boldsymbol{D}_T(\leq \emptyset')$ be the structure of Turing degrees of Δ^0_2-sets

- upper semilattice with least element $\deg_T(\emptyset)$, greatest element $\deg_T(K)$
- has minimal elements

Theorem 1 (Slaman,Woodin) *$\boldsymbol{D}_T(\leq \emptyset')$ is b.i. with $\mathbb{N}$ in params.*

Arithmetical structures

In all examples, $\boldsymbol{A}$ will be an *arithmetical structure*:

> there is $\beta : \mathbb{N} \mapsto A$ onto such that the preimages of the relations of $\boldsymbol{A}$ under β are arithmetical.

Example: $\boldsymbol{D}_T(\leq \emptyset')$

Let $\beta(e) =$

- $\deg_T(\{e\}^K)$ if $\{e\}^K$ is total
- $\deg_T(\emptyset)$ otherwise

Then "$\beta(i) \leq \beta(j)$" is Σ^0_4.

We call $R \subseteq A^n$ *arithmetical* if $\beta^{-1}(R)$ is.

Definable sets

- Let $\boldsymbol{A}$ be arithmetical. Then all parameter definable relations are arithmetical.

- BI with params says that being arithmetical is also sufficient:

$$\boldsymbol{A} \text{ is b.i. with } \mathbb{N} \text{ in params}$$

$$\Leftrightarrow$$

$$\forall R \; R \text{ arithmetical} \Rightarrow R \text{ parameter definable.}$$

C.e. Turing degrees

(Joint work with R. Shore and T. Slaman.)

Let $\mathcal{R}_T$ be the structure of Turing degrees of c.e. sets.

- upper semilattice with least element $\deg_T(\emptyset)$, greatest element $\deg_T(K)$
- is dense

Let $\boldsymbol{x}^{(n)}$ be the n-th jump of a Turing degree $\boldsymbol{x}$.

Let

$$Low_n = \{\boldsymbol{x} : \boldsymbol{x}^{(n)} = 0^{(n)}\},$$

$$High_n = \{\boldsymbol{x} : \boldsymbol{x}^{(n)} = 0^{(n+1)}\}$$

BI with $\mathbb{N}$ up to second jump

We write $\boldsymbol{x} \sim_2 \boldsymbol{y}$ if $\boldsymbol{x}^{(2)} = \boldsymbol{y}^{(2)}$.

Theorem 2 (with Shore and Slaman) *The relation $\sim_2$ is definable*

- *There are definable ternary relations $S_\oplus, S_\otimes$ on $\mathcal{R}_T$ which are compatible with $\sim_2$ such that*

$$(\mathcal{R}_T, S_\oplus, S_\otimes)/_{\sim_2} \cong \mathbb{N}.$$

Definability in $\mathcal{R}_T$

Corollary 3 *$\mathbb{N}$ can be interpreted in $\mathcal{R}_T$ without params*

- *if $R \subseteq \mathcal{R}_T$ is invariant under $\sim_2$ and arithmetical, then R is definable.*
- *$Low_n, High_n$ is definable for each $n \geq 2$*

Theorem 4 *$High_1$ is definable in $\mathcal{R}_T$.*

$$\boldsymbol{x} \in High_1 \Leftrightarrow \mathcal{R}_T \models \forall \boldsymbol{y} \exists \boldsymbol{z} \leq \boldsymbol{x} \, [\boldsymbol{z} \sim_2 \boldsymbol{y}].$$

C.e. many-one degrees

For $X, Y \subseteq \mathbb{N}$, let

$$X \leq_m Y \Leftrightarrow X = f^{-1}(Y)$$

for some computable f.

$\mathcal{R}_m$ = structure of m-degrees of c.e. sets.

- *distributive* upper semilattice with least, greatest elements
- has minimal elements

Bi-interpretability with params fails: $\mathcal{R}_m$ has uncountably many automorphisms.

BI of initial segments

Theorem 5 *Let* $\boldsymbol{e} < 1$ *such that* $[0, \boldsymbol{e}]_{\mathcal{R}_m}$ *is infinite. Then there is a uniformly parameter definable structure*

$$([0, \boldsymbol{e}], \oplus, \otimes) \cong \mathbb{N}.$$

Arithmetical $\Rightarrow$ definable

An equivalent formulation:

For each $k \geq 3$, the class of n -ary relation which have a Σ^0_k index set and are included in $[0, \boldsymbol{e}]^n$ for some $\boldsymbol{e} < 1$ is uniformly definable. I.e. there is a fmla $\varphi_k(x_1, \ldots, x_n; \boldsymbol{e}, b)$ such that, as $\boldsymbol{b}$ varies,

$$\{x_1, \ldots, x_n : \mathcal{R}_m \models \varphi_k(x_1, \ldots, x_n; \boldsymbol{e}, \boldsymbol{b})\}$$

includes all such relations (and more).

The complexity of Th($\mathcal{R}_m$)

Theorem 6 *Th*($\mathbb{N}$) *can be interpreted in Th*($\mathcal{R}_m$).

Let $\boldsymbol{M}_{\boldsymbol{e},\overline{\boldsymbol{b}}}$ denote a model for the language $L(+, x)$ uniformly defined with the parameters $\boldsymbol{e}, \overline{\boldsymbol{b}}$.

- We can express as a f.o. property of $\boldsymbol{e}, \overline{\boldsymbol{b}}$ that $\boldsymbol{e} < 1$, $\boldsymbol{M}_{\boldsymbol{e},\overline{\boldsymbol{b}}}$ has $[0, \boldsymbol{e}]$ as a domain and $\boldsymbol{M}_{\boldsymbol{e},\overline{\boldsymbol{b}}}$ satisfies Robinson's Q.
- Then the standard part of $\boldsymbol{M}_{\boldsymbol{e},\overline{\boldsymbol{b}}}$ is an arithmetical subset of $[0, \boldsymbol{e}]$ of a fixed complexity.
- So, we can express that $\boldsymbol{M}_{\boldsymbol{e},\overline{\boldsymbol{b}}}$ is standard by quantifying over such sets in the f.o.-language of $\mathcal{R}_m$.

This gives the desired interpretation of Th($\mathbb{N}$). (Stronger: we can interpret $\mathbb{N}$ without parameters.)

C.e. sets under inclusion

$$\mathcal{E} = (\text{c.e. sets}, \subseteq)$$

- Distributive lattice
- Satisfies the reduction property:

  $\forall A \forall B \; \exists \widetilde{A} \subseteq A \exists \widetilde{B} \subseteq B$

  $$\widetilde{A} \cap \widetilde{B} = \emptyset \;\; \wedge \widetilde{A} \cup \widetilde{B} = A \cup B.$$

- continously many automorphisms
- does not interpret an infinite linear order without parameters (more generally, this holds for any structure $\boldsymbol{B}$ of the form $\boldsymbol{A} \times \boldsymbol{A}$).

An approximation to BI for $\mathcal{E}$

A c.e. non recursive set

- $\mathcal{B}(A)$= the Boolean algebra of components of r.e. splittings of A
- $\mathcal{R}(A)$ = the ideal of $\mathcal{B}(A)$ consisting of the recursive subsets of A.
- An ideal I of $\mathcal{B}(A)$ is *k-acceptable* if

  $\mathcal{R}(A) \subseteq I$, and

  $\{e : W_e \in I\}$ is Σ^0_k.

  For instance, $\mathcal{R}(A)$ is 3-acceptable.

The Ideal Definability Lemma

Lemma 7 (Harrington) *Let $k \geq 3$ be odd. Then the k-acceptable ideals of $\mathcal{B}(A)$ can be uniformly defined with parameters.*
In fact there is a fmla $\varphi_k(X; A, \overline{C})$ such that for varying $\overline{C}$,

$$\{X : \mathcal{E} \models \varphi_k(X; A, \overline{C})\}$$

ranges over the $k-$ acceptable ideals, and nothing else.

This is the main tool to prove $\mathrm{Th}(\mathbb{N}) \leq_m \mathrm{Th}(\mathcal{E})$. To obtain a parameter-coded copy of $\mathbb{N}$: represent numbers by certain elements of $\mathcal{B}(A)$, modulo a definable equivalence relation. Use IDL a second time to recognize standardness.

Proving the IDL for $\mathcal{E}$

- Induction start: $k = 3$

  $\varphi_3(X; A, C) \;\equiv$

  $$X \in \mathcal{B}(A) \wedge (\exists R \subseteq A \text{ rec })X \subseteq C \cup R.$$

  Show that for each 3-acceptable I there is a C_I

- Induction step: $k \Rightarrow k+2$. There is a $D \subseteq A$ such that we can reduce the problem of defining a $k{+}2$- acceptable ideal of $\mathcal{B}(A)$ to defining a k-acceptable ideal nof $\mathcal{B}(D)$.

Lattices of Σ^0_n theories

- $\mathcal{B}_n$= the Lindenbaum algebra of $\Pi_{n-1}-\mathrm{Th}(\mathbb{N})$ $(n \geq 2)$
- $\mathcal{B}_1$= the Lindenbaum algebra of Q (say)

$\mathcal{B}_n$ is a Σ^0_n-boolean algebra. Let

$$\mathcal{L}_n = \text{ the lattice of } \Sigma^0_n\text{-filters of } \mathcal{B}_n.$$

- Distributive lattice with the reduction property
- continously many automorphisms
- does not interpret an infinite linear order without parameters
- *dense* (by Rosser's Theorem)

Theorem 8 $Th(\mathbb{N}) \leq Th(\mathcal{L}_n)$.

Applications:

Theorem 9 *Suppose* $D \subseteq A$, *where* $D, A \in \mathcal{E}$. *If* $[D, A]_{\mathcal{E}}$ *is not a boolean algebra, then* $Th(\mathbb{N})$ *can be interpreted in* $Th([D, A])$

Next consider subsets X, Y of $\{0,1\}^{<\omega}$
$X \leq_m^p Y \Leftrightarrow (\exists f \in \mathrm{Ptime}[X = f^{-1}(Y)])$
$X \leq_T^p Y \Leftrightarrow \exists$ polynomial time bounded oracle TM which computes X if the oracle is Y.

Theorem 10 (with Downey) *The theory of* $(DTIME(2^n), \leq_r^p)$ *is undecidable (r=m,T).*

To prove this, find $\boldsymbol{a}$ in DTIME(2^n) and a definable equivalence relation $\approx$ on $[0, \boldsymbol{a}]$ such that

$$[0, \boldsymbol{a}]/_{\approx} \cong \mathcal{L}_2.$$

Proving $\mathbf{Th}(\mathbb{N}) \leq_m \mathbf{Th}(\mathcal{L}_n)$

We need an analog of the ideal definability lemma for $\mathcal{E}$. Consider the case $n = 1$, i.e. the lattice of c.e. extensions of Q closed under inference. Let capital letters range over such theories.

- $A \sqsubset B$ if $\exists C\ (A \cup C)^{\vdash} = B\ \wedge A \cap C = Q^{\vdash}$.
- $\mathcal{B}(A) = \{X : X \sqsubset A\}$
- A is *locally principal* in E if $A \subseteq E$ and

$$\forall \alpha \in E (A \cup \{\alpha\})^{\vdash} \text{ principal.}$$

The Ideal Definability Lemma for $\mathcal{L}_1$

- There is a non-principal E.
- There is A locally principal in E such that $A \not\sqsubset E$.
- Let $\mathcal{R}_E(A) = \{X \sqsubset E : X \subseteq A\}$. ($\mathcal{R}_E(A)$ is a proper Σ^0_3-ideal of $\mathcal{B}(A)$.)

Theorem 11 *For each $k \geq 1$, the class of Σ^0_{2k+1}-ideals of $\mathcal{B}(A)$ containing $\mathcal{R}_E(A)$ is strictly uniformly definable.*

Summary

- Approximations to bi-interpretability with $\mathbb{N}$ can be proved for a variety of structures:
  - Turing degrees of Δ^0_2-sets and c.e. sets
  - C.e. many-one degrees
  - C.e. sets under inclusion
  - Lattices of theories which form Σ^0_n-sets
- The approximations get weaker as the structures deviate more and more from $\mathbb{N}$
- But they are always strong enough to obtain an interpretation of $\mathrm{Th}(\mathbb{N})$ in the theory.