Model theoretic properties of degree structures

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Reducibilities

A central topic of computability theory: studying sets of natural numbers under a notion of relative computability. Let $X, Y \subseteq \mathbb{N}$.

Turing: $X \leq_T Y \Leftrightarrow$

some oracle Turing machine computes X when the oracle is Y

$$\Leftrightarrow$$

X is Δ^0_1 relative to Y.

Many-one: $X \leq_m Y \Leftrightarrow$

$\exists f$ computable $X = f^{-1}(Y)$.

This is a restricted but very natural reducibility: for instance it occurs in interpretations of theories.

Degrees

- For a reducibility $\leq_r$, let

  $X \equiv_r Y \;\Leftrightarrow\; X \leq_r Y \leq_r X.$

- This is an equivalence relation among sets.

- The eq.classes are called *r-degrees*.

- The set $\mathcal{D}_r$ of all r-degrees is an uppersemilattice

- Its least element 0 is the degree consisting of the computable sets.

Degrees structures

Consider a class of sets of limited complexity:

- Computably enumerable (c.e.)
- Δ^0_2, arithmetical, hyperarithmetical, ...

To understand such classes, one studies how their members relate among each other under a reducibility.

c.e.		arithm.		all
$\mathcal{R}_T$	"$\subseteq$"	$\mathcal{A}_T$	$\subseteq$	$\mathcal{D}_T$
$\mathcal{R}_m$	$\subseteq$	$\mathcal{A}_m$	$\subseteq$	$\mathcal{D}_m$

We will mainly consider model theoretic properties of $\mathcal{R}_T$, $\mathcal{R}_m$, but compare with the remaining structures.

A general program

In analyzing a degree structure $\boldsymbol{A}$, often the following is carried out:

1. Understand the basic algebraic properties. This means: determine whether $\boldsymbol{A} \models \varphi$, for certain φ of algebraic significance. Examples: $\mathcal{D}_T, \mathcal{R}_m$ have minimal elements above 0. $\mathcal{R}_T$ is dense.

2. More general algebraic properties of $\boldsymbol{A}$:

- Embeddings, extensions of embeddings for p.o.
- Lattice embeddings (most interesting for $\mathcal{R}_T$)
- Characterizing initial segments (for $\mathcal{D}_T$, $\mathcal{D}_m$, $\mathcal{R}_m$)

3. Properties of Th($\boldsymbol{A}$).

- Prove that Th($\boldsymbol{A}$) is undecidable. Done in all cases.
- Decidable fragments of Th($\boldsymbol{A}$). E.g., all Π_2-theories of the degree structures based on $\leq_m$, in language of usl, are decidable [Degtev 79].
- Undecidable fragments. For instance, in the language of p.o., all Π_3-theories are undecidable.
  - [Schmerl 83] for Δ_2^0 T-degrees and above
  - [Lempp, Nies & Slaman 97] for $\mathcal{R}_T$
  - [Nies 96] for the of degree structures based on $\leq_m$.

4. Model theoretic properties of $\boldsymbol{A}$*.* Now one considers the structure itself, not only its theory.

- Interaction of parts of $\boldsymbol{A}$ with the whole
- Defining a copy of $\mathbb{N}$ without parameters
- Is $\boldsymbol{A}$ a prime model or a minimal model of $\mathrm{Th}(\boldsymbol{A})$?
- Is $\boldsymbol{A}$ bi-interpretable with $\mathbb{N}$ in parameters ?

Characterizations

All three structures $\mathcal{D}_m, \mathcal{A}_m, \mathcal{R}_m$ can be characterized as distributive usl's with a saturation property [Ershov 75] for $\mathcal{D}_m, \mathcal{A}_m$, [Denisov 78] for $\mathcal{R}_m$]

$\boldsymbol{A}$

I

J

Consequences:

- Maximum possible number of automorphisms
- Any two minimal degrees are automorphic
- Each proper final segment $\{\boldsymbol{x} : \boldsymbol{x} \geq \boldsymbol{e}\}$ is isomorphic to the whole.

I. Parts and the whole

We study how definable subsets of a degree structure interact with the whole.

Subsets:

- Initial segments

  $$[0, \boldsymbol{e}] = \{\boldsymbol{x} : \boldsymbol{x} \leq \boldsymbol{e}\}$$

- Final segments $\{\boldsymbol{x} : \boldsymbol{x} \geq \boldsymbol{e}\}$
- Definable subsets (like the minimal degrees)

Interaction:

- Automorphism bases
- Initial segments which are elementary substructures ($\mathcal{R}_m$)

Automorphism bases

Definition 1 *A subset B of a structure $\boldsymbol{A}$ is an automorphism base*

$$\Leftrightarrow$$

the only automorphism of $\boldsymbol{A}$ fixing B pointwise is the identity.

This means that B interacts

strongly with the whole structure.

Example. $B \subseteq (\mathbb{Q}, <)$ is a-base

$$\Leftrightarrow$$

B dense in $\mathbb{Q}$.

Theorem 2 *Each final segment is an a-base of* $\mathcal{D}_m$. *Similarly for* $\boldsymbol{A}_m$ *and* $\mathcal{R}_m$.

Theorem 3 *The minimal degrees do not form an a-base of* $\mathcal{D}_m$, *or of* $\mathcal{A}_m$.

Corollary 4 $Aut(\mathcal{D}_m)$, $Aut(\mathcal{A}_m)$ *are not simple*

Proof. $\{\pi \in \mathrm{Aut}(\boldsymbol{A}) : \pi$

$\quad$ fixes the minimal degrees$\}$

is a proper normal subgroup. $\diamondsuit$

A-bases of $\mathcal{R}_T$

Theorem 5 (Ambos-Spies) *For each* $\boldsymbol{c} \in \mathcal{R}_T$, $\boldsymbol{c} \neq 0$, $[0, \boldsymbol{c}]$ *is an automorphism base.*

A modified proof of this result [N98] uses the following general method to show that B is an a-base of $\boldsymbol{A}$: provide a definable (relative to B)

$$H : \boldsymbol{A} \mapsto \tau B,$$

where τB is the collection of objects of type τ constructed from B.

Example of such a map (not 1-1): $\boldsymbol{A} = \mathcal{R}_T$, $B = [0, \boldsymbol{c}]$,

$$H(\boldsymbol{x}) = \{[0, \boldsymbol{y}] \cap [0, \boldsymbol{c}], [0, \boldsymbol{z}] \cap [0, \boldsymbol{c}] : \boldsymbol{y} \vee \boldsymbol{z} = \boldsymbol{x}\}.$$

A-bases of $\mathcal{R}_m$

Theorem 6 *Each definable $D \subseteq \mathcal{R}_m$, $D \not\subseteq \{0, \mathbf{1}\}$ forms an a-base of $\mathcal{R}_m$.*

Proof. Do it only for the set Min of minimal degrees in $\mathcal{R}_m$. Let

$$\gamma(\boldsymbol{e}) \equiv \forall \boldsymbol{q}[\boldsymbol{e} \not\leq \boldsymbol{q} \Rightarrow \exists \boldsymbol{m} \in Min\,(\boldsymbol{m} \leq \boldsymbol{e} \;\&\; \boldsymbol{m} \not\leq \boldsymbol{q})]$$

If $\widehat{\boldsymbol{e}} \cong \mathcal{R}_m$ effectively, then $\gamma(\boldsymbol{e})$. Let
$H(\boldsymbol{x}) = \{[0, \boldsymbol{y}] \cap Min : \boldsymbol{y} \geq \boldsymbol{x} \;\&\; \gamma(\boldsymbol{y})\}$. $\Diamond$

Little is known about the structure of $\mathrm{Aut}(\mathcal{R}_m)$. From $\mathcal{R}_m - \{\mathbf{1}\} \cong$ the Δ^0_2 m-degrees (Denisov), we can infer the existence of an automorphism of order 2: complementation.

$\boxed{\mathcal{R}_T}$

The property $\gamma(\boldsymbol{e})$ plays a role similar to Ambos-Spies' "downward splitting property": let DSP($\boldsymbol{a}$) $\Leftrightarrow$

$$\forall b \not\leq \boldsymbol{a}\ \forall \boldsymbol{c} \not\leq \boldsymbol{b} \vee \boldsymbol{a}[\widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{a}} \neq \widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{b}}]$$

(where $\widehat{\boldsymbol{x}} = [0, \boldsymbol{x}]$). To prove $\widehat{\boldsymbol{c}}$ is an a-base, use as $H(\boldsymbol{x})$ a complex object built up from ideals $\widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{a}}$, where $DSP(\boldsymbol{a})$.

Initial segments of $\mathcal{R}_m$

Let $\mathcal{R}_m^- = \mathcal{R}_m - \{\mathbf{1}\}$.

Theorem 7 *There is* $\boldsymbol{e} < \mathbf{1}$ *such that* $[0, \boldsymbol{e})$ *is an elementary submodel of* $\mathcal{R}_m^-$.

This means: for each fmla

$$\varphi(x_0, \ldots, x_{n-1}) \text{ , } \boldsymbol{c}_0, \ldots, \boldsymbol{c}_{n-1} < \boldsymbol{e},$$

$$[0, \boldsymbol{e}) \models \varphi(\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{n-1}) \;\Leftrightarrow$$
$$\mathcal{R}_m^- \models \varphi(\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{n-1}).$$

The proof uses Denisov's construction in each successor step of a closure process that takes ω^ω many steps.

Part II:Defining $(\mathbb{N}, +, \times)$

Results about the theory which paved the way for a model theoretic analysis: determine the complexity of the theory (up to $\equiv_m$).

- $\mathcal{D}_T, \mathcal{D}_m$ [Simpson 71; Nerode & Shore 80]:

  the same as the second-order theory of $(\mathbb{N}, +, \times)$

- $\boldsymbol{D}_T(\leq \emptyset')$ [Shore 81]

- $\mathcal{R}_T$ [Harrington, Slaman],

  $\mathcal{R}_m$ [Nies 94]:

  the same as first-order $\mathrm{Th}(\mathbb{N}, +, \times)$.

Coded copies of $\mathbb{N}$

To interpret the appropriate version of arithmetic, use the following:

Definition 8 *A copy of* $\mathbb{N}$ *coded in* $\boldsymbol{A}$ *with params is a structure*

$$\boldsymbol{M} = (D, R_+, R_\times) \cong \mathbb{N}$$

such that D *and the relations* $R_+, R_\times$ *are param-definable in* $\boldsymbol{A}$*, by a fixed collection of formulas (called a scheme).*

Use the letter $\boldsymbol{M}$ for structures coded in this way satisfying Robinson arithmetic.

To interpret $\mathrm{Th}(\mathbb{N}, +, \times)$, we need to single out standard $\boldsymbol{M}$'s by a first-order condition on params.

A definable "projection"

Theorem 9 *For the following degree structures* $\boldsymbol{A}$*, one can define a copy* $N(\boldsymbol{A})$ *of* $\mathbb{N}$ *without params:*

- $\mathcal{R}_m$ *[N93]*
- $\mathcal{R}_T$ *[N,Shore,Slaman 96]*
- *The c.e. weak truth-table degrees [N98]*

To prove this, for $\mathcal{R}_m$ and $\mathcal{R}_T$ proceed as follows:

- for each $\boldsymbol{M}, \boldsymbol{M}'$, define in a uniform way with params maps $f_{\boldsymbol{M},\boldsymbol{M}'}$ extending the isomorphisms between the standard parts
- then: $\boldsymbol{M}$ standard $\Leftrightarrow \forall \boldsymbol{M}'$ $\mathrm{f}_{\boldsymbol{M},\boldsymbol{M}'}$ total (since *some* $\boldsymbol{M}'$ is standard).

$N(\boldsymbol{A})$ consists of definable equivalence classes of tuples $\overline{\boldsymbol{p}}, \boldsymbol{x}$, where $\boldsymbol{M}(\overline{\boldsymbol{p}})$ standard, $\boldsymbol{x} \in \boldsymbol{M}(\overline{\boldsymbol{p}})$, and

$\overline{\boldsymbol{p}}, \boldsymbol{x} \ \sim \overline{\boldsymbol{q}}, \boldsymbol{y} \ \Leftrightarrow$

$$f_{\boldsymbol{M}(\overline{\boldsymbol{p}}),\boldsymbol{M}(\overline{\boldsymbol{q}})}(\boldsymbol{x}) = \boldsymbol{y})$$

Part III: Prime models

Definition 10 $\boldsymbol{A}$ *prime* $\Leftrightarrow$ $\boldsymbol{A}$ *is an elementary substructure of each* $\mathcal{B} \equiv \boldsymbol{A}$.

Equivalent (for countable $\boldsymbol{A}$):

- Each realized n-type is principal
- each orbit under action of $\mathrm{Aut}(\boldsymbol{A})$ on $\boldsymbol{A}^n$ definable.

Thus, a proof that $\boldsymbol{A}$ is prime helps us to understand the orbits.

Some prime models

Theorem 11 *The arithmetical m-degrees form a prime model.*

[Slaman,Woodin] prove the same for $\mathcal{A}_T$.

Question 12 *Is $\mathcal{R}_m$ prime ?*

Theorem 13 *$\mathcal{R}_m$ prime $\Rightarrow \exists k$ [each realized type is principal via a Σ^0_k-formula]*

This stronger fact is actually proved for $\mathcal{A}_m, \mathcal{A}_T$.

Part IV: BI with $\mathbb{N}$

Let $\boldsymbol{A}$ be an arithmetical structure.
$\boldsymbol{A} = (A; R_1, \ldots, R_n)$ is bi-interpretable with $\mathbb{N}$ if there is a structure with the same domain

$$\boldsymbol{A}_{\mathbb{N}} = (A; \oplus, \otimes)$$

such that

- $(A; \oplus, \otimes) \cong (\mathbb{N}, +, \times)$
- $R_1, \ldots, R_n$ and $\oplus, \otimes$ are interdefinable.

For $\boldsymbol{B} = \mathbb{N}$, this coincides with the usual model-theoretic notion of bi-interpretability of structures $\boldsymbol{A}, \boldsymbol{B}$

Example: $\mathbb{Q}$ is b.i. with $\mathbb{N}$.

BI in params

Definition 14 $\boldsymbol{A}$ *is b.i. with* $\mathbb{N}$ *in parameters if* $(\boldsymbol{A}, \overline{\boldsymbol{a}})$ *is b.i. with* $\mathbb{N}$, *for some list of constants* $\overline{\boldsymbol{a}}$ *in* A.

Slaman/Woodin prove this for $\boldsymbol{D}_T(\leq \emptyset')$

Consequences

- Can define a copy $N(\boldsymbol{A})$ of $\mathbb{N}$ without params
- $\boldsymbol{A}$ is a prime model and
  and a minimal model of $\mathrm{Th}(\boldsymbol{A})$. In fact, for $\boldsymbol{B}, \boldsymbol{C} \equiv \boldsymbol{A}$,

$$\boldsymbol{B} \prec \boldsymbol{C} \;\Leftrightarrow\; N(\boldsymbol{B}) \prec N(\boldsymbol{C}).$$

- Arithmetical $\Leftrightarrow$ param-definable
- Each $\pi \in \mathrm{Aut}(\boldsymbol{A})$ is arithmetical. Finite a-base.

$\mathcal{R}_m$ and $\mathcal{R}_T$

BI in params fails for $\mathcal{R}_m$. In fact,

- $|\mathrm{Aut}(\mathcal{R}_m)| = 2^\omega$
- For some $\boldsymbol{e}$, $[0, \boldsymbol{e}) \prec \mathcal{R}_m - \{\mathbf{1}\}$.

Question 15 *Is $\mathcal{R}_T$ biinterpretable with $\mathbb{N}$ in params ?*

- It would be sufficient to find $\boldsymbol{c} \neq 0$, $\boldsymbol{M}$ standard and a param-definable 1-1 $g : [0, \boldsymbol{c}] \mapsto \boldsymbol{M}$:
- Recall that there is a definable 1-1 $H : \mathcal{R}_T \mapsto \tau[0, \boldsymbol{c}]$.
- Then $(\tau g) \circ H$ is a 1-1 map $\mathcal{R}_T \mapsto \tau M$.
- From this would obtain $(\mathcal{R}_T; \oplus, \otimes)$ as desired.

An approximation

Let $\boldsymbol{x}^{(n)}$ be the n-th jump of a Turing degree $\boldsymbol{x}$.
Let $Low_n = \{\boldsymbol{x} : \boldsymbol{x}^{(n)} = 0^{(n)}\}$,
$High_n = \{\boldsymbol{x} : \boldsymbol{x}^{(n)} = 0^{(n+1)}\}$.

We write $\boldsymbol{x} \sim_2 \boldsymbol{y}$ if $\boldsymbol{x}^{(2)} = \boldsymbol{y}^{(2)}$.

Theorem 16 (with Shore and Slaman)

- *The relation $\sim_2$ is definable*

- *There are definable ternary relations $S_\oplus, S_\otimes$ on $\mathcal{R}_T$ which are compatible with $\sim_2$ such that*

$$(\mathcal{R}_T, S_\oplus, S_\otimes)/_{\sim_2} \cong \mathbb{N}.$$

Definability in $\mathcal{R}_T$

Corollary 17 • *if* $R \subseteq \mathcal{R}_T$ *is invariant under* $\sim_2$ *and arithmetical, then* R *is definable.*

- $Low_n, High_n$ *are definable for each* $n \geq 2$

Theorem 18 $High_1$ *is definable in* $\mathcal{R}_T$.

$$\boldsymbol{x} \in High_1 \Leftrightarrow \mathcal{R}_T \models \forall \boldsymbol{y} \exists \boldsymbol{z} \leq \boldsymbol{x} \; [\boldsymbol{z} \sim_2 \boldsymbol{y}].$$