Homogeneity properties of free groups

André Nies,

The University of Chicago

Oct 1997

Slide 1

Questions about free groups

Tarski's question: Does $\mathrm{Th}(F_n)$ equal $\mathrm{Th}(F_m)$ for all m, n $(2 \leq m < n \leq \infty)$

Decidability question: Is $\mathrm{Th}(F_m)$ decidable for each m ?

Program: Understand the model–theoretic properties of F_m. Investigate first–order definability in F_m

Slide 2

The first–order language

The first–order language of groups consists of formulas built up from equations

$$t(x_1, \ldots, x_k) = e \quad (t \in F(x_1, \ldots, x_n)$$

using $\neg, \wedge, \vee, \rightarrow, \exists x, \forall x$.

A sentence is a fmla without free variables.

Th(G) is the set of sentences which hold in G.

Slide 3

The model theoretic program

Let G be a group, $\overline{g}, \overline{h} \in G^k$.

- $\overline{g} \equiv_\exists \overline{h}$ if they satisfy the same $\exists$–formulas.
- $\overline{g} \equiv \overline{h}$ if they satisfy the same formulas.

Slide 4

Orbits

Let $\overline{g}, \overline{h}$ denote tuples of the same fixed length.
$\overline{g} \approx \overline{h} \Leftrightarrow$ *some automorphism maps* $\overline{g}$ to $\overline{h}$.

- In $F(b_1, \ldots, b_n)$, $\mathrm{orbit}(b_1, \ldots, b_n) =$ *all bases.*
- $\mathrm{orbit}(\overline{g})$ is decidable (Whitehead's Theorem).

(say "base" for "free generating set")

Given $\overline{g}$, if $\overline{h}$ is in $\mathrm{orbit}(\overline{g})$, then $\overline{h}$ satisfies the same formulas as $\overline{g}$.

If the converse holds, $\overline{g}$ is called *homogeneous.*

Stronger: $\overline{g}$ is $\exists$–*homogeneous* if for each $\overline{h}$

$$\overline{g} \approx \overline{h} \Leftrightarrow \overline{g} \equiv_{\exists} \overline{h}.$$

Slide 5

An analogy

Let K be an algebraically closed field, and let $P \leq K$ be the prime field.

The orbit of $g \in K$ is determined by

- "$q(x) = 0$", if g is algebraic over P with irreducible polynomial $q \in P[X]$
- "$q(x) \neq 0$" for all polynomials $q(X) \in P[X]$, if g is trancendental over P.

So each $g \in K$ is $\exists$–homogeneous.

Slide 6

Theorem 0.1 *Each tuple in F_2 is $\exists$-homogeneous.*

Thus all the invariant properties of a tuple $\overline{g}$ in F_2 are determined by the $\exists$ formulas it satisfies. (E.g. “the corresponding f.p. group $\langle a, b | \ \overline{g} \ \rangle$ has a solvable WP”.)

Slide 7

Test tuples

Definition 0.2 *The tuple $\overline{g} = g_1, \ldots, g_k \in F_n$ is called a test tuple (Turner) if:*

each endomorphism $\varphi : F_n \to F_n$ fixing all elements g_i is an automorphism.

It is called a test tuple for monomorphisms if each monomorphism $\varphi : F_n \to F_n$ fixing all elements g_i is an automorphism.

Turner’s result

- $\overline{g}$ test tuple in $F_n \Leftrightarrow$
  $\forall R$ proper retract of F_n $\exists i$ $g_i \notin F_n$.
- $\overline{g}$ test tuple for monomorphisms in $F_n \Leftrightarrow$
  $\forall R$ *proper direct factor of* F_n $\exists i$ $g_i \notin F_n$.

Slide 8

Test tuples are ∃-hom

Theorem 0.3 *(i) Suppose $\overline{g}$ is a test tuple in F_n. Then $\overline{g}$ is ∃–homogeneous.*

(ii) Suppose $\overline{g}$ is a test tuple for monomorphisms in F_2. Then $\overline{g}$ is ∃–homogeneous.

Thus, if $\overline{g}$ generates F_n or even a subgroup of finite index of F_n, then $\overline{g}$ is ∃–homogeneous.

Slide 9

Proof. (i). Fix base $a_1, \dots, a_n$ of F, $g_i = t_i(a_1, \dots, a_n)$.

If $\overline{g} \equiv_\exists \overline{h}$, want to show that there is an automorphism of F mapping g_i to h_i.

$\overline{h}$ satifies the existential formula

$$\varphi(y_1, \dots, y_m) = \exists x_1, \dots, x_n \bigwedge y_i = t_i(x_1, \dots, x_n).$$

Let $\alpha(a_i) = x_i$.

Similarly, one obtains a map β such that $\beta(h_i) = g_i$). $\beta \circ \alpha$ fixes each g_i. So $\beta \circ \alpha$ is an automorphism by the definition of test tuples. This implies that α is an automorphism as desired.

(ii) Similar, but require in the definition of φ also that $[x_1, x_2] \neq 1$. $\diamondsuit$

To prove ∃-homogeneity for all tuples in F_2, what's left is the case that g is in a proper direct factor. Use an argument involving Nielsen reductions. $\diamondsuit$

Slide 10

Nondefinability of "base"

Theorem 0.4 *In $F(b_1, \ldots, b_n)$, $2 \leq n < \infty$, the relation "$x_1, \ldots, x_n$ form a base" is not definable (without parameters) via an $\exists\forall$–formula.*

Proof. Use Sacerdote's result: for $2 \leq p < q$, F_p and F_q satisfy the same $\exists\forall$-sentences.

Suppose the $\exists\forall$ formula $\varphi(x_1, \ldots, x_n)$ defines "base ".

If $n \geq 3$: There must be $a_1, \ldots, a_n$ satisfying φ in $F_(a, b)$, since

$$F_n \models \exists x_1, \ldots, x_n \varphi(x_1, \ldots, x_n).$$

which is $\exists\forall$. Pick t_i such that $a_i = t_i(a, b)$. Then $F(a, b)$ and hence F_n satisfy

$$\exists u, v\ \varphi(t_1(u, v), \ldots, t_n(u, v)),$$

contradiction !

Slide 11

$n = 2$

Lyndon, Schupp:

- In $F(a, b)$, orbit($[a, b]$) is the union of the conjugacy classes of $[a, b]$ and of $[b, a]$
- This orbit equals $\{[u, v] : u, v \text{ base}\}$
- In $F_n, n > 2$, each nontrivial orbit splits into infinitely many conjugacy classes.

So $F(a, b)$, but no F_m of higher rk, satisfies

$\exists x, y[\varphi(x, y) \wedge \forall x', y'(\varphi(x', y') \Rightarrow ([x, y] \sim [x', y'] \vee [y, x] \sim [x', y']))]$,

which is $\exists\forall$. $\diamondsuit$

Slide 12

Prime models

Theorem 0.5 *If $m \geq n$, then no map $\alpha : F_m \to F_n$ preserves $\exists$-formulas, unless it is an isomorphism.*

A countable model is *prime* if all n–orbits are definable without parameters.

Example: The algebraic closure of $\mathbb{Q}$ is prime.

Theorem 0.6 *If $Th(F_2) = Th(F_n), n > 2$, then $Th(F_2)$ has no prime model.*

Proof.

$$P \text{ is the prime model } \Rightarrow$$

$$P \prec_{\exists} F_2 \Rightarrow$$

$$P = F_2,$$

since P must be a nonabelian free group.
Now the definability of orbit(c) in F_2 implies
$\mathrm{Th}(F_2) \neq \mathrm{Th}(F_n)$.

Slide 13