Model theoretic properties of structures from computability theory

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The questions

We study structures from computability theory using model theoretic concepts.

- $\mathcal{R}_{wtt}$ (the c.e. wtt degrees), a distributive upper semilattice.
- $\mathcal{R}_T$ (the c.e. Turing degrees) and lattices of ideals of $\mathcal{R}_T$.
- Ideal lattices of c.e. boolean algebras, and the lattice of Π^0_1-classes. They are distributive lattices with the [dual] reduction property.

Questions:

- Restrictions on automorphisms?
- Which natural subsets are (param-) definable?
- Can $(\mathbb{N}, +, \cdot)$ be interpreted? Biinterpretable with $\mathbb{N}$ in params?

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I. C.e. wtt–degrees

$X \leq_{wtt} Y \Leftrightarrow$

$\exists$ oracle Turing machine computing X with oracle Y, and use bounded recursively in the input.

$\mathcal{R}_{wtt}$ = wtt–degrees of c.e. sets.

Denote by $\mathbf{0}$ and $\mathbf{1}$ the least and greatest element, respectively.

The following leads to a large class of uniformly definable finite subsets fo $\mathcal{R}_{wtt}$.

Lemma 1 *Suppose*

- $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n$ *are low, and*
- $\boldsymbol{u}_i \vee \boldsymbol{u}_j = \mathbf{1}$ *for* $i \neq j$.

Then $\{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n\}$ *is uniformly definable from params* $\boldsymbol{g}, \boldsymbol{h}$.

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EN–sets

To prove this:

- Write $nd(\boldsymbol{u})$ if $[\boldsymbol{u}, \mathbf{1}]$ does not embed the diamond preserving least and greatest element.
- Ambos–Spies proved that
$$\boldsymbol{u} \text{ low} \Rightarrow nd(\boldsymbol{u}).$$
- So $\{\boldsymbol{u}_i\}$ = minimal elements in $\{\boldsymbol{x} \in F : nd(\boldsymbol{x})\}$, where F is the end segment
$$\{\boldsymbol{x} : \boldsymbol{x} \text{ upper bd of } [\mathbf{0}, \boldsymbol{u}_0] \cap \ldots \cap [\mathbf{0}, \boldsymbol{u}_n]\}$$
- We can define F from an exact pair, since
$$[\mathbf{0}, \boldsymbol{u}_0] \cap \ldots \cap [\mathbf{0}, \boldsymbol{u}_n] = [\mathbf{0}, \boldsymbol{g}] \cap [\mathbf{0}, \boldsymbol{h}].$$
- Then $\{\boldsymbol{u}_i\}$ unif'ly definable from $\boldsymbol{g}, \boldsymbol{h}$.

Sets parameter–defined in this way are called *EN–sets* (for ENd segment)

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$\mathrm{Th}(\mathcal{R}_{wtt})$ **undecidable**

- Result of Ambos–Spies, N and Shore [1992]
- Lempp and N [1995] improved it:
  $\Pi_4 - \mathrm{Th}(\mathcal{R}_{wtt})$ (as p.o.) is undecidable.

A quick proof of undecidability:

It suffices to code an arbitrary finite bipartite graph

$$(\{0, \dots, n\},\ \{\widetilde{0}, \dots, \widetilde{m}\},\ R).$$

in a uniform way.

Build EN–sets $\{\boldsymbol{u}_0, \dots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_{\widetilde{0}}, \dots, \boldsymbol{v}_{\widetilde{m}}\}$ such that

$$Ri\,\widetilde{j} \;\Leftrightarrow\; \boldsymbol{u}_i \vee \boldsymbol{v}_{\widetilde{j}} = \mathbf{1}.$$

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Interpreting $\mathbb{N}, +, \cdot$

Theorem 2 (APAL ta) *A copy of* $(\mathbb{N}, +, \cdot)$ *can be interpreted in* $\mathcal{R}_{wtt}$ *without parameters.*

Proof.

- To represent $n \in \mathbb{N}$, use params $\boldsymbol{g}, \boldsymbol{h}$ defining EN–sets of size n
- Can define equality and arithmetical operations by extending the coding used in the quick undecidability proof
- Express finiteness of an EN–set defined from $\boldsymbol{g}, \boldsymbol{h}$ by a f.o.–condition

  For this, use a class of unif'ly definable sets dual to EN–sets, constructed as in Lempp & N [1995].

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BI with N in params

Let $\boldsymbol{A}$ be an arithmetical structure.

A copy of $\mathbb{N}$ *coded in* $\boldsymbol{A}$ *with params is a structure*

$$\boldsymbol{M} = (D, R_+, R.) \cong \mathbb{N}$$

such that D *and the relations* $R_+, R.$ *are param-definable in* $\boldsymbol{A}$*.*

$\boldsymbol{A}$ *is biinterpretable with* $\mathbb{N}$ *in params* $\Leftrightarrow$

$\exists M \exists f : A \mapsto M$ $[f\ 1-1, \textit{param-def}\,]$.

Conjecture: BI in params holds for $\mathcal{R}_T$.

Consequences

- One can define a copy of $\mathbb{N}$ without params, and $\boldsymbol{A}$ is the prime model of $\mathrm{Th}(\boldsymbol{A})$.
- For each n, $R \subseteq A^n$,

  R arithmetical $\Leftrightarrow$ R param-definable.
- $\exists k$ each $\pi \in \mathrm{Aut}(\boldsymbol{A})$ has Σ^0_k index relation. Finite automorphism base.

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II. C.e. T–degrees

A collection S_X of fmlas with params to define objects X of a certain type is called a **scheme**.

Theorem 3 *There are schemes* S_M *and* S_f *such that:* $\forall \boldsymbol{d} \neq \mathbf{0}$ $\exists$ *params coding*

M *and a 1-1 map* $f : [\boldsymbol{d}, \mathbf{1}] \to M$*.*

Shore & Slaman [unpubl.] proved this with the restriction $\boldsymbol{d}$ noncappable.

Corollary 4 $\exists k$ *such that, if* $\Phi \in Aut(\mathcal{R}_T)$ *and* $\boldsymbol{d} \neq \mathbf{0}$*, then* $\Phi \restriction [\boldsymbol{d}, \mathbf{1}]$ *is* Σ^0_k*.*

Corollary 5 • $[\boldsymbol{d}, \mathbf{1}]$ *a-base for some* $\boldsymbol{d} \neq \mathbf{0}$ $\Leftrightarrow$ $Aut(\mathcal{R}_T)$ *is ctble.*

- *If proved to be a-base in the standard way, then BI in params holds.*
- *Isom. type of* $[\boldsymbol{d}, \mathbf{1}]$ *unique* $\Rightarrow$ $\boldsymbol{d}$ *definable.*

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SW sets & a gap

The domain of $\boldsymbol{M}$ is a Slaman-Woodin set with lower bd $\boldsymbol{b}$:

$$M = \text{minimal elts in } \{\boldsymbol{x} \in [\boldsymbol{b}, \boldsymbol{c}] \ : \ \boldsymbol{q} \leq \boldsymbol{x} \vee \boldsymbol{p}\},$$

where $\boldsymbol{c}, \boldsymbol{p}, \boldsymbol{q}$ are further params.

The lower bd for the SW set is essential. It allows us to avoid a minimal pair type restraint on P.

A further new idea is to put the params $\boldsymbol{p}, \boldsymbol{q}$ below $\boldsymbol{d}$. Actually $\boldsymbol{b} \leq \boldsymbol{p} \leq \boldsymbol{q} \leq \boldsymbol{d}$.

- In the Boulder volume, I announced something stronger: $\exists M \ \exists f : U \to M$ 1-1 for any u.c.e. set U of nonzero degrees.
- The proof (not published there) had a gap, which has not been closed up to now.
- All the corollaries can be obtained from the weaker statement (as we will see next).

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Low_1

Theorem 6 *Low_1 is parameter definable.*

Proof. Cholak, Slaman & Groszek showed there is a degree $\boldsymbol{d} \neq \mathbf{o}$ such that

$$\forall \boldsymbol{x} \ [\boldsymbol{x} \in \text{Low}_1 \Leftrightarrow \ \boldsymbol{x} \vee \boldsymbol{d} \in \text{Low}_1].$$

The class $[\boldsymbol{d}, \mathbf{1}] \cap \text{Low}_1$ is arithmetical, and hence param-definable by Theorem **??**. $\diamondsuit$

An interesting definable class: Let

$$\mathcal{F} = \{\boldsymbol{d} : \exists \boldsymbol{c} \text{ cappable } \boldsymbol{c} \vee \boldsymbol{d} = \mathbf{1}\}.$$

Since low-cuppable = non-cappable, $\mathcal{F} \cap \text{Low}_1 = \emptyset$.

Theorem 7 *$\mathcal{F} \cap Low_2 \neq \emptyset$.*

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Definable ideals

If $D \subseteq \mathcal{R}_T$, let $[D]_{id}$ be the ideal of the usl $\mathcal{R}_T$ generated by D.

Corollary 8 *$[D]_{id}$ is definable without params in the structure $(\mathcal{R}_T, D)$.*

Proof. If $D \neq \{0\}$, then $[D]_{id} =$

$$\{\boldsymbol{x} : \exists \boldsymbol{d} \in D \ \exists F \subseteq [\boldsymbol{d}, \boldsymbol{1}] \text{ finite}$$

$$\boldsymbol{x} \leq \bigvee F \ \& \ \forall \ \boldsymbol{y} \in F \ \exists \boldsymbol{z} \in D \ \boldsymbol{z} \vee \boldsymbol{d} = \boldsymbol{y}\}. \qquad \diamondsuit$$

Let NB $\subseteq$ CAP be the (definable) class of nonbounding degrees.

Theorem 9
- *$[NB]_{id} \subset \ CAP$*
- *$[NB]_{id} \neq \ NCup.$*

Thus $[\mathrm{NB}]_{id}$ is a new definable ideal. This answers Shore's question 2.8 in Boulder volume.

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Ideal lattices

Let

$\mathcal{L}_k$ = lattice of Σ^0_k–ideals $(4 \leq k)$

$\mathcal{H}$ = lattice of all ideals.

$\mathcal{R}_T$ is embedded via $\boldsymbol{a} \rightarrow [\mathbf{o}, \boldsymbol{a}]$. Identify.

Some Facts.

- $\mathcal{R}_T$ is automorphism invariant in $\mathcal{H}$, since $\mathcal{R}_T$ = set of compact elements
- In $\mathcal{L}_k, \ k \geq 6$ and $\mathcal{H}$:

  NCup is intersection of all maximal ideals (and hence NCup is a nontrivial definable element).

Corollary 10 *Let $k \geq 7$. Then*

$(\mathcal{L}_k, \mathcal{R}_T)$ is biinterpretable with $\mathcal{R}_T$.

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Initial intervals

The ability to code SW–sets (with lower bd) in $\mathbf{o}, \boldsymbol{d}$ for any $\boldsymbol{d} \neq \mathbf{o}$ leads to

Theorem 11 *Let $\boldsymbol{d} \neq \mathbf{o}$. Then a copy of $(\mathbb{N}, +, \cdot)$ can be interpreted in $[\mathbf{o}, \boldsymbol{d}]$ without parameters.*

Ambos–Spies, Hirschfeldt & Shore proved each nontrivial interval has an undec. theory.

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III. $\mathcal{I}(\mathcal{B})$; Π^0_1-classes

Σ^0_k**–boolean algebras.**

A structure $(\mathbb{N}, \preceq, \wedge, \vee)$

is *c.e. boolean algebra* if:

- $\vee, \wedge$ are computable
- $\preceq$ is c.e.
  Let $x \approx y$ if $x \preceq y \preceq x$.
- $\mathcal{B} = (\mathbb{N}, \preceq, \wedge, \vee)/\approx$
  is a boolean algebra

More generally, for Σ^0_k-boolean algebras, require that $\preceq$ be Σ^0_k and that $\wedge, \vee$ be recursive in $\emptyset^{(k-1)}$.

Examples. Let $\mathcal{B}_n$= the Lindenbaum algebra of $\Pi_{n-1} - \mathrm{Th}(\mathbb{N}, +, \cdot)$ (of Q for $n = 1$).

$\mathcal{B}_n$ is a Σ^0_n-boolean algebra.

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Effective density

A c.e. boolean algebra $\mathcal{B}$ is called **effectively dense** if there is a computable F such that $\forall x\ [F(x) \preceq x]$ and

$$\forall x \not\approx 0 \ \ [0 \prec F(x) \prec x].$$

For a Σ^0_k boolean algebra $\mathcal{B}$, require that $F \leq_T \emptyset^{(k-1)}$.

This notion was first formulated by Montagna & Sorbi [1985] for any c.e. preordering.

We give examples of e.d. BA's from two opposite ends of the effectivity spectrum:

- $\mathcal{D}$, the recursive dense BA
- An effectively inseparable c.e. BA, defined next.

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E.i. boolean algebras

Disjoint Σ^0_k sets E_0, E_1 are $\emptyset^{(k-1)}$–effectively inseparable if

$\forall D_0, D_1$ disjoint Σ^0_k–sets $\exists\ f \leq_T \emptyset^{(k-1)}$

$$x \in D_i \ \Leftrightarrow\ f(x) \in E_i \ (i = 0, 1).$$

A Σ^0_k $\mathcal{B}$ is called $\emptyset^{(k-1)}$-e.i. if the sets

$$\{n \in \mathbb{N} : n \approx 0\},\ \{n \in \mathbb{N} : n \approx 1\}$$

(i.e. the sets of names for $0^{\mathcal{B}}, 1^{\mathcal{B}}$) are $\emptyset^{(k-1)}$–effectively inseparable.

- The Lindenbaum algebra of $\Pi_{k-1} - \mathrm{Th}(\mathbb{N}, +, \cdot) \cup Q$ is $\emptyset^{(k-1)}$–e.i.
- A $\emptyset^{(k-1)}$–e.i. BA is unique up to $\emptyset^{(k-1)}$–isomorphism. (Montagna & Sorbi [1985], using Kripke & Pour-El [1967])

Fact 12 *$\mathcal{B}$ e.i. $\Rightarrow$ $\mathcal{B}$ effectively dense.*

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Splittings of $A \in \mathcal{E}$

Let A be c.e. non-computable.

- $\mathcal{B}(A)$= the boolean algebra of halves of c.e. splittings of A
- $\mathcal{R}(A)$ = the ideal of $\mathcal{B}(A)$ consisting of the recursive subsets of A.
- $\mathcal{B}_{\mathcal{R}}(A) = \mathcal{B}(A)/_{\mathcal{R}(A)}$ is Σ^0_3.

Proposition 13 *The Σ^0_3–boolean algebra $\mathcal{B}_{\mathcal{R}}(A)$ is $\emptyset^{(2)}$–e.i.*

Thus, for A_0, A_1 non-computable,

$$\mathcal{B}_{\mathcal{R}}(A_0) \cong_{\Delta^0_3} \mathcal{B}_{\mathcal{R}}(A_1).$$

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$\mathcal{I}(\mathcal{B})$

For a Σ^0_k–boolean algebra $\mathcal{B}$, let
$\mathcal{I}(\mathcal{B}) :=$ lattice of Σ^0_k–ideals of $\mathcal{B}$.

- $\mathcal{I}(\mathcal{B})$ is a distributive lattice with least and greatest elements
- $\mathcal{I}(\mathcal{B})$ has the reduction property
- All principal ideals $[0, b]_{\mathcal{B}}$ are in $\mathcal{I}(\mathcal{B})$
- An ideal is principal iff complemented

$\mathcal{B}$ e.d. $\Rightarrow$ to some extent $\mathcal{I}(\mathcal{B})$ behaves similar to $\mathcal{E}^k$.

Theorem 14 (N, Trans. AMS 2000) *If $\mathcal{B}$ is effectively dense, then Th$(\mathbb{N}, +, \cdot) \leq \mathrm{Th}(\mathcal{I}(\mathcal{B}))$.*

The proof works by interpreting $\mathrm{Th}(\mathcal{E}^7)$. The latter theory interprets $\mathrm{Th}(\mathbb{N}, +, \cdot)$ (Harrington).

Previously I had proved that $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$ is undecidable [Bull. LMS 1997], by an easier argument.

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Coding $\mathcal{I}(\mathcal{B})$

1. *Intervals of* $\mathcal{E}^*$ [N 2000]. *If* $[D, A]_{\mathcal{E}^*}$ *is not a boolean algebra, then* $Th(\mathbb{N}, +, \cdot)$ *can be interpreted in* $\mathrm{Th}([D^*, A^*])$.

   Proof. If $D \subseteq_m A$, let $\mathcal{B}$ = the Σ_3–BA of complemented elements in $[D^*, A^*]$.

   Represent Σ^0_3 a ideal I of $\mathcal{B}$ by param $C_I^* \in [D^*, A^*]$:

   $$I = \{X^* : (X \cap C_I)^* = D^*\}.$$

   (Reduce general case to the $D \subseteq_m A$ case.)

2. *PTIME reducibilities on complexity classes low down* [Downey,N 1997]

   $X \leq^p_m Y \Leftrightarrow \exists f \in \mathrm{PTIME}\ [X = f^{-1}(Y)]$

   $X \leq^p_T Y \Leftrightarrow\ \exists$ pol. bounded oracle TM which computes X with oracle Y.

   Theorem 15 $\mathrm{Th}(DTIME(2^n), \leq^p_r)$ *is undecidable (r=m,T).*

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3. Computable sets with parameterized reducibilities (Coles, Downey, Sorbi).

4. The Solovay degrees of c.e. reals have an undecidable theory (Coles, Downey, Hirschfeldt & LaForte)

5. The lattice of Π_1–sentences mod provability in Peano arithmetic has an undecidable theory (work in progress, Lindström & Shavrukov)

6. $\mathrm{Th}(\mathbb{N}, +, \cdot)$ can be interpreted in $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$, if $\mathcal{B}$ is *recursive* and some ideal in $\mathcal{I}(\mathcal{B})$ is non-principal (N & Cenzer).

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Π^0_1–classes

- A Π^0_1–class is the set $[T]$ of paths through a recursive subtree $T \subseteq 2^\omega$.
- $\mathcal{L}$ = lattice of Π^0_1–classes
- $\mathcal{L}$ is distributive and satisfies dual reduction property:
  $\forall a, b\ \exists \widehat{a} \geq a \exists \widehat{b} \geq b\ [\widehat{a} \vee \widehat{b} = 1\ \&\ \widehat{a} \wedge \widehat{b} = a \wedge b]$.
- Effective Stone duality provides anti-isomorphism

  $$\mathcal{L} \rightarrow \mathcal{I}(\mathcal{D}):$$

  (Fix rec. free generating set $d_0, d_1, \ldots$.

  Map $[T]$ to ideal generated by $\{d_0^{\sigma(0)} \wedge \ldots d_0^{\sigma(n-1)} : \sigma \notin T\}$.)

Letters P, Q, R denote Π^0_1–classes.

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Definability in $\mathcal{L}$

Finite differences

- $P - Q$ finite $\Rightarrow$ all paths in $P - Q$ are computable
- Write $P =^* Q$ if $\exists R$ finite $[P \vee R = Q \vee R]$.
- Duality: in $\mathcal{I}(\mathcal{D})$, $A =^* B$ if
  $\exists R\ \mathcal{D}/_R$ finite $\ \&\ A \cap R = B \cap R$.

Theorem 16 (w. Cenzer) *$\{P \in \mathcal{L} : P$ finite$\}$, and hence $=^*$, is definable in $\mathcal{L}$.*

More generally, for each $n \geq 1$, $\{P \in \mathcal{L} : rk(P) = n$ is definable. Answers Question 6.1 of Cenzer & Jockusch in Boulder volume.

Proof uses a uniform definability lemma for $\mathcal{I}(\mathcal{B})$, $\mathcal{B}$ eff'ly dense, from [N 2000].

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Initial intervals of $\mathcal{L}^*$

Consider a finite distributive lattice D with the dual reduction property. Examples: finite BA's, linear orders, ...

Theorem 17 (w. Cenzer)
$\forall\ D\ \exists\ P\ [\emptyset, P]/_{=^*} \cong D$.

By previous Thm, the set of such P is definable in $\mathcal{L}$.

Remark. Cholak & Downey announced no orbit, for $D = 2$.

Contrast: finite end intervals of $\mathcal{E}^*$ are boolean algebras, by the Owings Splitting Thm.

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End intervals

Easy: each end interval of $\mathcal{L}$ is eff'ly isom. to one of:

- $\mathcal{L}$
- $\{P : 0^\omega \in P\}$.

Dual statement: each initial interval of $\mathcal{I}(\mathcal{D})$ is eff'ly isom. to $\mathcal{I}(\mathcal{D})$ or $[\{0\}, E]_{\mathcal{I}(\mathcal{D})}$, where E is non principal.

In Boulder volume, Herrmann and Cenzer/Jockusch asked whether the two possible end intervals of $\mathcal{L}$ (or the two initial intervals of $\mathcal{I}(\mathcal{D})$) are isomorphic.

This would be in analogy to the case of nontrivial initial intervals of $\mathcal{E}$.

Theorem 18 (w. Cenzer) *Let $E \in \mathcal{I}(\mathcal{D})$ be nonprincipal. Then*

$$\mathcal{I}(\mathcal{D}) \ncong [\{0\}, E]_{\mathcal{I}(\mathcal{D})}.$$

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1. If $\mathcal{B}$ is e.d. Σ^0_3–BA, then the lattice $\mathcal{I}(\mathcal{B})$ does *not* have representation recursive in $\emptyset^{(3)}$ (only recursive in $\emptyset^{(4)}$).

2.

- Since E is nonprincipal, can choose A locally principal in E:

$$A \not\subseteq E \text{ and } \forall e \in E[A \cap e \text{ principal}].$$

- Define e.i. Σ^0_3–BA $\mathcal{B} = \mathcal{B}_E(A)$, an analog to $\mathcal{B}_{\mathcal{R}}(A)$ in the case of $\mathcal{E}$. Here, splits of E play role of recursive sets.
- Can code $\mathcal{IB}$ in $[\{0\}, E]_{\mathcal{I}(\mathcal{D})}$ with few quantifiers and A as a param.
- If $\mathcal{I}(\mathcal{D}) \cong [\{0\}, E]_{\mathcal{I}(\mathcal{D})}$, then there would be similar coding of $\mathcal{I}(\mathcal{B})$ in $\mathcal{I}(\mathcal{D})$. This gives representation of $\mathcal{I}(\mathcal{D})$ recursive in $\emptyset^{(3)}$.

Conjecture. The two possible nontrivial initial intervals of $\mathcal{I}(\mathcal{D})$ are not even elementarily equivalent.

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Summary

The method of coding with first-order fmlas yields model theoretic results about structures from computability theory.

- $\mathrm{Th}(\mathbb{N}, +, \cdot)$ can be interpreted in $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$. $(\mathbb{N}, +, \cdot)$ can be coded without params in $\mathcal{R}_{wtt}$ and initial intervals of $\mathcal{R}_T$.
- In $\mathcal{R}_T$, for each $\boldsymbol{d} \neq \mathbf{o}$ there is a definable setting $f : [\boldsymbol{d}, \mathbf{1}] \to M$. This implies Low_1 is param definable, and that restrictions of auto's to end intervals are Σ^0_k for some fixed k. Moreover we obtained a new definable ideal $[\mathrm{NB}]_{id}$.
- The method of interpreting ideal lattices of e.d. BA's is useful in a wide variety of contexts. It proves the theory is complex: Moreover, it gives structural and definability results about the lattice of Π^0_1–classes.

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Cereals

Joint with Hirschfeldt and Downey

In the following all reals are > 0. A real α is **c.e.** if there is an increasing effective sequence (α_s) of rationals converging to α.

Use α, β, γ for c.e. reals. They come equipped with such a sequence.

Examples.

- $\sqrt{2}, e, \pi$ are in fact recursive reals
- $.A$, where A is a c.e. set
- The halting probability
$$\Omega = \sum\{2^{|\sigma|} : U(\sigma, \lambda)\downarrow\},$$
where U is a universal Chaitin machine.

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Randomness

The binary representation of Ω is **random** in the sense of Martin–Löf.

Solovay introduced a red'ty $\leq_S$ to measure the relative randomness of c.e. reals.

$\beta \leq_S \alpha \;\Leftrightarrow$

$\exists C$ rational $\exists f$ computable incr

$\forall s\ \beta - \beta f(s) \leq C(\alpha - \alpha_s)$.

He proved that
$\beta \leq_S \alpha \;\Rightarrow\; \exists k\ \forall n\ H(\beta \upharpoonright n) \leq H(\alpha \upharpoonright n) + k$,

where $H(\sigma)$ is the Kolmogorov complexity based on the self delimiting machine U.

Ω is $\leq_S$–complete. Slaman proved that, for c.e. reals, $\leq_S$–complete = random.

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$\le_S$ and arithmetic

$\beta \le_S \alpha \;\Leftrightarrow\; \exists C$ rational $\exists\gamma\; C(\beta+\gamma)=\alpha$.

From this formulation, it is esp. easy to see that

$$\beta \le_S \alpha \;\Rightarrow\; \beta \le_T \alpha.$$

Let $\mathcal{S}$ be the degree structure induced on the c.e. reals. We investigate the algebraic properties of $\mathcal{S}$. $\mathcal{S}$ is an usl where the sup is given by the usual addition.

Proposition 19 *$\mathcal{S}$ is a distributive usl*

Among degree structures on c.e. sets, $\mathcal{R}_m$ and $\mathcal{R}_{wtt}$ are distributive.

Is $\mathcal{S}$ more like $Rwtt$, or more like $\mathcal{R}_m$?

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Density

Using standard coding and preservation strategies, we obtain upward density.

Theorem 20 *Let $\gamma <_S \Omega$. Then there is β such that $\gamma <_S \beta <_S \Omega$.*

If $\alpha <_S \Omega$, we prove that in a sense any sequence for Ω converges much slower than one for α. This gives combined splitting and density below α.

Theorem 21 *Let $\gamma <_S \alpha <_S \Omega$. There are β^0 and β^1 such that $\gamma <_S \beta^0, \beta^1 <_S \alpha$ and $\beta^0+\beta^1=\alpha$.*

Combining the two, we obtain a non-uniform proof of density. So, $\mathcal{S}$ is like $\mathcal{R}_{wtt}$?

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Splitting random

Theorem 22 *If $\alpha + \beta$ is random, then one of α, β is random.*

Using Slaman's Thm, this implies

Corollary 23 *In $\mathcal{S}$, the greatest element is join irreducible.*

This is a property of $\mathcal{S}$ shared with $\mathcal{R}_m$.

Further directions.

- More f.o. properties, eg existence of noncappable
- Interpret $\mathrm{Th}(\mathbb{N}, +, \cdot)$
- Find automorphisms
- Connections to analysis

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