SUPERHIGHNESS AND STRONG JUMP TRACEABILITY

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Abstract. Let A be a c.e. set. Then A is strongly jump traceable if and only if A is Turing below each superhigh Martin-Löf random set. The proof combines priority with measure theoretic arguments.

1. Introduction

A lowness property of a set $A \subseteq \mathbb{N}$ specifies a sense in which A is computationally weak.

(I) Usually this means that A has limited strength when used as an oracle. An example is superlowness, $A' \leq_{tt} \emptyset'$. Further examples are given by traceability properties of A. Such a property specifies how to effectively approximate the values of certain functions (partial) computable in A. For instance, A is *jump traceable* [12] if $J^A(n)\downarrow$ implies $J^A(n) \in T_n$ for some uniformly c.e. sequence $(T_n)_{n\in\mathbb{N}}$ of computably bounded size. Here J is the jump functional: If $X \subseteq \mathbb{N}$, we write $J^X(n)$ for $\Phi_n^X(n)$.

(II) A further way to be computationally weak is to be easy to compute. A lowness property of this kind specifies a sense in which many oracles compute A. For instance, consider the property to be a base for ML-randomness, introduced in [8]. Here the class of oracles computing A is large enough to admit a set that is ML-random relative to A. By [6] this property coincides with the type (I) lowness property of being low for ML-randomness.

As our main result, we show a surprising further coincidence of a type (I) and a type (II) lowness property. The type (I) property is strong jump traceability, introduced in [3], and studied in depth in [1]. We say that a computable function $h\colon \mathbb{N} \to \mathbb{N}$ is an *order function* if h is nondecreasing and unbounded.

Definition 1.1. *$A \subseteq \mathbb{N}$ is* strongly jump traceable *(s.j.t.) if for each order function h, there is a uniformly c.e. sequence $(T_n)_{n\in\mathbb{N}}$ such that*

- $\forall n\, |T_n| \leq h(n)$
- $\forall n\, [J^A(n)\downarrow \to\ J^A(n) \in T_n]$.

Figueira, Nies and Stephan [3] built a promptly simple set that is strongly jump traceable. Cholak, Downey and Greenberg [1] showed that the strongly jump traceable c.e. sets form a proper subideal of the K-trivial c.e. sets under Turing reducibility.

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We say that a set $Y \subseteq \mathbb{N}$ is *superhigh* if $\emptyset'' \le_{tt} Y'$. This notion was first studied by Mohrherr [10] for c.e. sets. For more background see [11, 7]. The type (II) property is to be Turing below each superhigh ML-random set. Thus our main result is that *a c.e. set A is strongly jump traceable $\Leftrightarrow$ A is Turing below each superhigh Martin-Löf random set.*

The property to be Turing below each superhigh ML-random set can be put into a more general context. For a class $\mathcal{H} \subseteq 2^\omega$, we define the corresponding diamond class

$$\mathcal{H}^\diamond = \{A\colon\ A \text{ is c.e. } \&\ \forall Y \in \mathcal{H} \cap \mathsf{MLR}\, [A \le_T Y]\}.$$

Here MLR is the class of ML-random sets. Note that $\mathcal{H}^\diamond$ determines an ideal in the c.e. Turing degrees. By a result of Hirschfeldt and Miller (see [11, 5.3.15]), for each null Σ^0_3 class, the corresponding diamond class contains a promptly simple set A . Their proof is a non-adaptive cost-function construction. As argued in [11, Section 5.3], this means that the construction of A can be viewed as injury-free. In contrast, the direct construction of a promptly simple strongly jump traceable set in [3] is a variant of Post's construction of a low simple set, which therefore has injury.

In [4] a result similar to our main result was obtained when $\mathcal{H}$ is the class of superlow sets Y (namely, $Y' \le_{tt} \emptyset'$). Both results derive from earlier work of Hirschfeldt and Nies who obtained such a coincidence for the class $\mathcal{H}$ of ω-c.e. sets Y (namely, $Y \le_{tt} \emptyset'$).

In all cases, to show that a c.e. strongly jump traceable set A is in the required diamond class, one finds an appropriate collection of benign cost functions; this key concept was introduced by Greenberg and Nies [5]. The set A obeys each benign cost function by the main result of [5]. This implies that A is in the diamond class.

It is harder to prove the converse inclusion: each c.e. set in $\mathcal{H}^\diamond$ is s.j.t., suppose an order function h is given. For one thing, similar to proving the analogous inclusion in [4], we use a variant of the golden run method introduced in [2, 13]. One wants to restrict the changes of A to the extent that A is strongly jump traceable. To this end, one defines a "bad set" $Z \in \mathcal{H} \cap \mathsf{MLR}$. It exploits the changes of A in order to avoid being Turing above A. The number of levels in the golden run construction is infinite, with the e-the level based on the Turing functional Φ_e. If the golden run fails to exist at level e, then $A \neq \Phi_e^Z$. Then, as $A \in \mathcal{H}^\diamond$, the golden run must exist. Since it is golden, it successfully builds the required trace for J^A with bound h.

A further ingredient in our proof stems from ideas that started in Kurtz [9] and were elaborated further, for instance, in Nies [13, 14]: mixing priority arguments and measure theoretic arguments. In contrast, the proof in [4] is not measure theoretic. (Indeed, they prove, more generally, that for *each* non-empty Π^0_1 class P, each c.e. set Turing below every superlow member of P must be strongly jump traceable.) Here we need to make the bad set Y superhigh. This is done by coding of $\emptyset''$ (see [11, 3.3.2]) in the style of Kučera, but not quite into Y: the coding strings change due to the activity of the tracing procedures. Their number of changes is computably bounded. So the coding merely yields $\emptyset'' \le_{tt} Y'$.

Notation. Suppose f is a unary function and $\widetilde{f}$ is binary. We write

$$\forall n\, f(n) = \lim_s^{\text{comp}} \widetilde{f}(n,s)$$

if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all n, the set

$$\{s > 0\colon\ \widetilde{f}(n,s) \neq \widetilde{f}(n,s-1)\}$$

has cardinality less than $g(n)$, and $\lim_s \widetilde{f}(n,s) = f(n)$.

We let $X' = \{n\colon\ J^X(n)\downarrow\}$, and $X'_t = \{n\colon\ J^X_t(n)\downarrow\}$.

2. Benign cost functions and $\mathsf{Shigh}^\diamond$

Note that a function f is d.n.c. relative to $\emptyset'$ if $\forall x\, \neg f(x) = J^{\emptyset'}(x)$. Let P be the $\Pi^0_1(\emptyset')$ class of $\{0,1\}$-valued functions that are d.n.c. relative to $\emptyset'$. By [11, 8.5.12] relative to $\emptyset'$, the class $\{Z\colon\ \exists f \leq_T Z \oplus \emptyset'\, [f \in P]\}$ is null. Then, since $\mathrm{GL}_1 = \{Z\colon\ Z' \equiv_T Z \oplus \emptyset'\}$ is conull, the following class, suggested by Simpson,

$$\mathcal{H} = \{Z\colon\ \exists f \leq_{\mathrm{tt}} Z'\, [f \in P]\} \tag{1}$$

is also null. This class clearly contains Shigh.

Since $\mathcal{H}$ is Σ^0_3, by a result of Hirschfeldt and Miller (see [11, 5.3.15]) the class $\mathcal{H}^\diamond$ contains a promptly simple set. We strengthen this:

Theorem 2.1. *Let A be a c.e. set that is strongly jump traceable. Then $A \in \mathcal{H}^\diamond$.*

Proof. For each truth table reduction Γ we define a benign cost function c such that for each Δ^0_2 set A, and each ML-random set Y,

$$A \text{ obeys } c \text{ and } \Gamma^{Y'} \text{ is } \{0,1\}\text{-valued d.n.c. relative to } \emptyset' \Rightarrow A \leq_T Y.$$

Let (I_e) be the sequence of consecutive intervals of length e. Thus $\min I_e = e(e+1)/2$. We define a function $\alpha \leq_T \emptyset'$. We are given a partial computable function p and think of p as a reduction function for α, namely, p is total, increasing, and $\forall x\ \ \alpha(x) \simeq J^{\emptyset'}(p(x))$.

At stage s of the construction we define the approximations $\alpha_s(x)$. Let $\alpha_s(x) = 0$ unless $p(y)$ is defined at stage s for each $y \in I_e$. In this case, let

$$\mathcal{C}_{e,s} = \{Y\colon\ \exists t_{\, v \leq t \leq s} \forall x \in I_e\, [1 - \alpha_t(x) = \Gamma(Y'_t, p(x))]\},$$

where $v \leq s$ is greatest such that $v = 0$ or $\alpha_v \upharpoonright I_e \neq \alpha_{v-1} \upharpoonright I_e$. (Thus, $\mathcal{C}_{e,s}$ is the set of oracles Y such that Y' computes α correctly at some stage t after the last change of $\alpha\upharpoonright_{I_e}$.)

Construction of α.

Stage $s > 0$. For each $e < s$, if $\lambda\mathcal{C}_{e,s-1} \leq 2^{-e+1}$ let $\alpha_s \upharpoonright I_e = \alpha_{s-1} \upharpoonright I_e$. Otherwise change $\alpha\upharpoonright_{I_e}$: define $\alpha_s \upharpoonright I_e$ in such a way that $\lambda\mathcal{C}_{e,s} \leq 2^{-e}$.

Claim. *$\alpha(x) = \lim_s \alpha_s(x)$ exists for each x.*

We use a measure theoretic fact suggested by Hirschfeldt in a related context (see [11, 1.9.15]). Suppose $N, e \in \mathbb{N}$, and for $1 \leq i \leq N$, the class $\mathcal{B}_i$ is measurable and $\lambda\mathcal{B}_i \geq 2^{-e}$. If $N > k2^e$ then there is a set $F \subseteq \{1,\ldots,N\}$ such that $\#F = k+1$ and $\bigcap_{i\in F} \mathcal{B}_i \neq \emptyset$. For instance, if $N = 5$ classes of measure at least $1/2$ are given, then the intersection of three of them is non-empty.

Suppose now that $0 = v_0 < v_1 < \ldots < v_N$ are consecutive stages at which $\alpha \upharpoonright I_e$ changes. Thus $p \upharpoonright I_e$ is defined. Then $\lambda\mathcal{B}_i \geq 2^{-e}$ for each $i \leq N$, where

$$\mathcal{B}_i = \{Y\colon\ Y'_{v_{i+1}}\upharpoonright_k \neq Y'_{v_i}\upharpoonright_k\},$$

and $k = \mathsf{use}\,\Gamma(\max p(I_e))$, because $\lambda\mathcal{C}_e$ increased by at least 2^{-e} from v_i to v_{i+1}. Note that the intersection of any $k+1$ of the $\mathcal{B}_i$ is empty. Thus $N \leq 2^e k$ by the measure theoretic fact. $\diamond$

Since α is Δ^0_2, by the Recursion Theorem, we can now assume that p is a reduction function for α. Then in fact we have a computable bound g on the number of changes of $\alpha \upharpoonright I_e$ given by $g(e) = 2^e \mathsf{use}\,\Gamma(\max p(I_e))$.

To complete the proof, let A be a c.e. set that is strongly jump traceable. We define a cost function c by $c(x,s) = 2^{-x}$ for each $x \geq s$; if $x < s$, and $e < x$ is least such that $\alpha_s \upharpoonright I_e \neq \alpha_{s-1} \upharpoonright I_e$ let

$$c(x,s) = \max(c(x,s-1), 2^{-e}).$$

Note that the cost function c is benign as defined in [5]: if $x_0 < \ldots < x_n$ and $c(x_i, x_{i+1}) \geq 2^{-e}$ for each i, then $\alpha_s \upharpoonright I_e \neq \alpha_{s-1} \upharpoonright I_e$ for some s such that $x_i < s \leq x_{i+1}$. Hence $n \leq g(e)$ where g is defined after the claim.

By [5] fix a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A that obeys c. (The rest of the argument actually works for a computable approximation $(A_s)_{s\in\mathbb{N}}$ of a Δ^0_2 set A.)

We build a Solovay test $\mathcal{G}$ as follows: when $A_{t-1}(x) \neq A_t(x)$, we put $\mathcal{C}_{e,t}$ into $\mathcal{G}$ where e is largest such that $\alpha \upharpoonright I_e$ has been stable from x to t. Then $2^{-e} \leq c(x,t)$. Since $\lambda\mathcal{C}_{e,t} \leq 2^{-e+1} \leq 2c(x,t)$ and the computable approximation of A obeys c, $\mathcal{G}$ is indeed a Solovay test.

Choose s_0 such that $\sigma \not\preceq Y$ for each $[\sigma]$ enumerated into $\mathcal{G}$ after stage s_0. To show $A \leq_T Y$, given an input $y \geq s_0$, using Y as an oracle, compute $s > y$ such that $\alpha_s(x) = \Gamma(Y'_s; x)$ for each $x < y$. Then $A_s(y) = A(y)$: if $A_t(y) \neq A_{t-1}(y)$ for $t > s$, let $e \leq y$ be largest such that $\alpha \upharpoonright I_e$ has been stable from y to t. Then by stage $s > y$ the set Y is in $\mathcal{C}_{e,s} \subseteq \mathcal{C}_{e,t}$, so we put Y into $\mathcal{G}$ at stage t, contradiction. $\square$

In the following we give a direct construction of a null Σ^0_3 class containing the superhigh sets. Note that the class $\mathcal{H}$ defined in (1) is such a class. However, the proof below uses techniques of independent interest. For instance, they might be of use to resolve the open question whether superhighness itself is a Σ^0_3 property.

Proposition 2.2. *There is a null Σ^0_3 class containing the superhigh sets.*

Proof. For each truth-table reduction Φ, we uniformly define a null Π^0_2 class $\mathcal{S}_\Phi$ such that $\emptyset'' = \Phi(Y') \ \rightarrow\ Y \in \mathcal{S}_\Phi$.

We build a Δ^0_2 set D_Φ. Then, by the Recursion Theorem we have a truth-table reduction Γ_Φ such that $\emptyset'' = \Phi(Y') \ \rightarrow\ D_\Phi = \Gamma(Y')$. We define D_Φ in such a way that $\mathcal{S}_\Phi = \{Y\colon\ D_\Phi = \Gamma(Y')\}$ is null. Also, $\mathcal{S}_\Phi$ is Π^0_2 because

$$Y \in \mathcal{S}_\Phi \ \leftrightarrow\ \forall w\, \forall i > w \exists s > i\, D_\Phi(w,s) = \Gamma(Y'_s; w).$$

Claim. *For each string σ, the real number $r_\sigma = \lambda\{Z\colon\ \sigma \prec Z'\}$ is difference left-c.e. (see [11, 1.8.15]) uniformly in σ.*

To see this, note that for each finite set F the class $\mathcal{C}_F = \{Z\colon F \subseteq Z'\}$ is uniformly Σ^0_1. Let $F(\sigma) = \{j < |\sigma|\colon\ \sigma(j) = 1\}$, then

$$r_\sigma = \lambda(\mathcal{C}_{F(\sigma)} - \bigcup_{r<|\sigma|\&\sigma(r)=0} \mathcal{C}_{\{r\}\cup F(\sigma)}).$$

This proves the claim. Now, for each τ let

$$b_\tau = \lambda\{Z\colon \tau \prec \Gamma(Z')\}.$$

Then $b_\tau = \sum_\sigma r_\sigma \llbracket \tau = \Gamma^\sigma \rrbracket$ is uniformly difference left-c.e.

We define the Δ^0_2 set $D = D_\Phi$ in such a way that $b_{D\restriction n+1} \leq b_{D\restriction n}/2$ for each n. Then $2^{-n} \geq \lambda\{Y\colon D_\Phi \restriction_n = \Gamma(Y') \restriction_n\}$ for each n, so $\mathcal{S}_\Phi$ is null. □

3. Each set in $\mathsf{Shigh}^\diamond$ is strongly jump traceable

Theorem 3.1. *Let A be a c.e. set that is Turing below all ML-random superhigh sets. Then A is strongly jump traceable.*

Proof. Let h be an order function. We will define a ML-random superhigh set Z such that $A \leq_T Z$ implies that A is jump traceable via bound h. In fact for an arbitrary given set G we can define Z such that $G \leq_{\mathrm{tt}} Z'$. If also $G \geq_{tt} \emptyset''$, then Z is superhigh.

Preliminaries. We will need a lower bound on the measure of a non-empty Π^0_1 class of ML-random sets. This bound is given uniformly in an index for the class (Kučera; see [11, 3.3.3]). Let $Q_0 \subseteq \mathsf{MLR}$ be the complement $2^\omega - \mathcal{R}_1$ of the second component of the standard universal ML-test.

Lemma 3.2. *Given an effective listing $(P^v)_{v\in\mathbb{N}}$ of Π^0_1 classes, $P^v \subseteq Q_0$, there is a constant c_0 such that $\lambda P^v \leq 2^{-K(v)-c_0} \to P^v = \emptyset$.*

We assume an indexing of all the Π^0_1 classes. Given an index for a Π^0_1 class P we have an effective approximation $P = \bigcap_t P_t$ where P_t is a clopen set ([11, Section 1.8]).

The basic set-up. For each e, a procedure R^e (with further parameters to be discussed later) builds a c.e. trace $(T_x)_{x\in\mathbb{N}}$ with bound h. Either for almost all x, $J^A(x)\downarrow$ implies $J^A(x) \in T_x$, or R^e shows that $A \neq \Phi^Z_e$. Since Z is superhigh, the first alternative must hold for some e.

When a new computation $w = J^A(x)\downarrow$ with use u appears, R^e activates a sub-procedure S^e_x. This sub-procedure waits for evidence that $A\restriction_u$ is stable before putting w into the trace set T_x. By first waiting long enough, it makes sure that an $A\restriction_u$ change after this tracing can happen for at most $h(x)$ times, so that $\#T_x \leq h(x)$. S^e_x also calls an instance of the next procedure R^{e+1}. Thus, during the construction we can have many runs of each of the procedures R^e and S^e_x.

The environment of a procedure. Each R^e has as further parameters a Π^0_1 class P and a number $r \in \mathbb{N}$. It assumes that $Z \in P$ and $2^{-r} < \lambda P$. Each S^e_x activated by $R^e(P, r)$ will specify an appropriate subclass $Q \subseteq P$ and a number $q \in \mathbb{N}$, and call $R^{e+1}(Q, q)$.

Initially we call $R^0(Q_0, 2)$

The two phases of S^e_x. A procedure S^e_x alternates between Phases I, and II. When changing phases it returns control to R^e. In our first approximation to describing the construction, once a computation $w = J^A(x)\downarrow$ with use u appears, S^e_x enters Phase I. It considers the Σ^0_1 class $C = \{Z\colon \Phi^Z_e \restriction_u = A\restriction_u\}$.

It calls $R^{e+1}(Q, q)$ where $Q = P - C$ and q is obtained by Lemma 3.2. If it stays here then, because $Z \in Q$, its outcome is that $\Phi_e^Z \neq A$.

For a threshold δ depending only on r and x, once $\lambda(P_s \cap C_s) > \delta$ at stage s it lets $D = C_s$ and puts w into T_x. Now the outcome is that $J^A(x)$ has been traced. So S_x^e can return and stay inactive unless $A \upharpoonright_u$ changes.

Once $A \upharpoonright_u$ has changed, S_x^e enters Phase II by calling $R^{e+1}(Q, q)$ where now $Q = P \cap D$ and q is obtained by Lemma 3.2. Its outcome is again that $\Phi_e^Z \neq A$, this time because $\Phi_e^Z \upharpoonright_u$ is the previous value of $A \upharpoonright_u$ (here we use that A is c.e.).

If, later on, $P \cap D$ becomes empty, then S_x^e returns. It is now turned back to the beginning and may start again in Phase I when a new computation $J^A(x)$ appears. Note that P has now lost a measure of δ. So S_x^e can go back to Phase I for at most $1/\delta$ times.

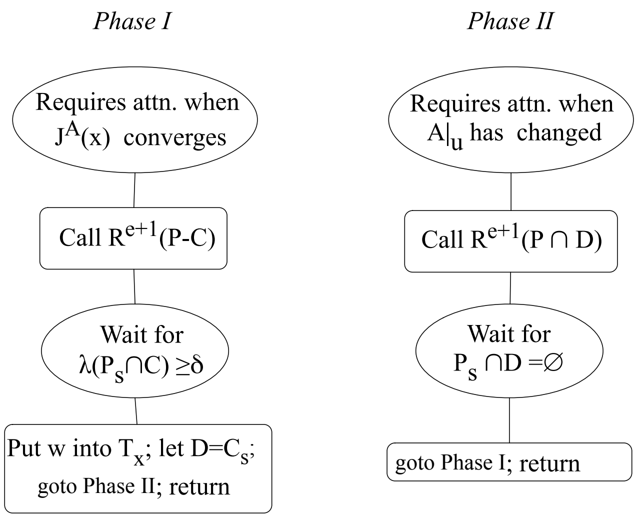

FIGURE 1. Diagram for the procedure S_x^e

The golden run. For some e we want a run of R^e such that each sub-procedure S_x^e it calls returns. For then, the c.e. trace $(T_x)_{x \in \mathbb{N}}$ this run of R^e builds is a trace for J^A. If no such run R^e exists then each run of R^e eventually calls some S_x^e which does not return, and therefore permanently runs a procedure R^{e+1}. If $Y \in \bigcap P_e$ where P_e is the parameter of the final run of a procedure R^e, then $A \not\leq_T Y$. So we have a contradiction if we can define a set $Z \in \bigcap_e P_e$ such that $G \leq_{tt} Z'$.

Ensuring that $G \leq_{tt} Z'$. For this we have to introduce new parameters into the procedures S_x^e.

Note that $G \leq_{tt} Z'$ iff there is a binary function $f \leq_T Z$ such that $\forall x\, G \upharpoonright_x = \lim_s^{\text{comp}} f(x, s)$ (namely, the number of changes is computably bounded). We will define Z such that Z' encodes G. We use a variant of Kučera's method to code into ML-random sets. We define strings $z_\gamma = \lim_s^{\text{comp}} z_{\gamma,s}$ and let $Z = \bigcup_{\gamma \prec G} z_\gamma$. The strings $z_{\gamma,s}$ are given effectively, and for each s they are pairwise incomparable. Then we let $f(x,s)=\gamma$ if $z_{\gamma,s}\prec Z$, and $f(x,s)=\varnothing$ if there is no such γ.

Firstly, we review Kučera's coding into members of a Π^0_1-class of positive measure.

Lemma 3.3 (Kučera; see [11], 3.3.1). *Suppose that P is a Π^0_1 class, x is a string, and $\lambda(P|x)\geq 2^{-l}$ where $l\in\mathbb{N}$. Then there are at least two strings $w\succeq x$ of length $|x|+l+1$ such that $\lambda(P|w)>2^{-l-1}$. We let w_0 be the leftmost and w_1 be the rightmost such string.*

In the following we code a string β into a string y_β on a Π^0_1 class P.

Definition 3.4. Given a Π^0_1 class P, a string z such that $P\subseteq[z]$, and $r\in\mathbb{N}$ such that $2^{-r}<\lambda P$, we define a string

$$y_\beta=\mathsf{kuc}(P,r,z,\beta)$$

as follows: $y_\varnothing=y$; if $x=y_\beta$ has been defined, let $l=r+|\beta|$, and let $y_{\beta\frown b}=w_b$ for $b\in\{0,1\}$, where the strings w_b are defined as in Lemma 3.3.

Note that for each β we have $\lambda(P\mid y_\beta)\geq 2^{-r-|\beta|}$ and

$$|y_\beta|\leq|z|+|\beta|(r+|\beta|+1). \tag{2}$$

At stage s we have the approximation $y_{\beta,s}=\mathsf{kuc}(P_s\cap[z],r,z,\beta)$. While $y_{\beta,s}$ is stable, the string w_b in the inductive definition above changes at most 2^l times. Thus, inductively, $y_{\beta,s}$ changes at most $2^{|\beta|(r+|\beta|+1)}$ times.

For each e,η we may have a version of R^e denoted $R^{e,\eta}(P,r,z_\eta)$. It assumes that η has already been coded into the initial segment z_η of Z, and works within $P\subseteq[z_\eta]$. It calls procedures $S^{e,\eta\alpha}_x(P,r,z)$ for certain x,α. In this case we let $z_{\eta\alpha}=y_\alpha=\mathsf{kuc}(P,r,z_\eta,\alpha)$.

For each x, once $J^A(x)\downarrow$, $R^{e,\eta}$ wishes to run $S^{e,\eta\alpha}$ for all α of a certain length m defined in (4) below, which increases with $h(x)$. Thus, as x increases, more and more bits beyond η are coded into Z. The trace set T_x will contain all the numbers enumerated by procedures $S^{e,\eta\alpha}_x$ where $|\alpha|=m$. We ensure that m is small enough so that $\#T_x\leq h(x)$.

Formal details. Some ML-random set $Y\not\geq_T\emptyset'$ is superhigh by [11, 3.4.13]. Since $A\leq_T Y$ and A is c.e., A is a base for ML-randomness by [11, 5.1.18], and therefore superlow. Hence there is an order function g and a computable enumeration of A such that $J^A(x)[s]$ becomes undefined for at most $g(x)$ times.

We build a sequence of Π^0_1 classes $(P^n)_{n\in\mathbb{N}}$ as in Lemma 3.2. If $n=\langle e,\gamma,x,i\rangle$, then since $K(n)\leq^+ 2\log\langle e,\gamma\rangle+2\log x+2\log i$, we have

$$P^{\langle e,\gamma,x,i\rangle}\neq\emptyset\ \Rightarrow\ \lambda P^{\langle e,\gamma,x,i\rangle}\geq 2^{-q} \tag{3}$$

where $q=2\log\langle e,\gamma\rangle+2\log x+2\log i+c$ for some fixed $c\in\mathbb{N}$. By the Recursion Theorem we may assume that we know c in advance.

The construction starts off by calling $R^{0,\varnothing}(Q_0,3,\varnothing)$.

Procedure $R^{e,\eta}(P,r,z)$, where $z\in 2^{<\omega}$, $P\subseteq\mathsf{MLR}\cap[z]$ is a Π^0_1 class and $r\in\mathbb{N}$. This procedure enumerates a c.e. trace $(T_x)_{x\in\mathbb{N}}$. (It assumes that $2^{-r}<\lambda P$.)

For each string α of length at most s, see whether some procedure $S_x^{e,\eta\alpha}(P)$ requires attention or is at (b), (e) and no procedure $S_y^{e,\eta\beta}(P)$ for $\beta \prec \alpha$ satisfies the same condition. If so, choose x least for α and activate $S_x^{e,\eta\alpha}(P)$. (This suspends any runs $S_z^{e,\delta}$ for $\eta\alpha \preceq \delta$. Such a run may be resumed later.)

Procedure $S_x^{e,\eta\alpha}(P,r,z)$, *where* $|\alpha|$ *is the greatest* m *such that, if* $n = m(r+m+1)$, *we have*

$$2^{|\eta\alpha|}2^{2n+r+2} \le h(x). \tag{4}$$

There only is such a procedure if x is so large that m exists.

Let $y_{\alpha,s} = \mathsf{kuc}(P_s, \alpha, r, z)$. Let

$$\delta = 2^{-|y_{\alpha,s}|-m-r-1}.$$

(Comment: $S_x^{e,\eta\alpha}(P,r,z)$ cannot change $y_{\alpha,s}$. It only changes "by itself" as P_s gets smaller. This makes the procedure go back to the beginning. So in the following we can assume y_α is stable.)

Phase I.

(a) $S_x^{e,\eta\alpha}$ *requires attention* if $w = J^A(x)\downarrow$ with use u. Let
$$C = [y_\alpha] \cap \{Z\colon\ \Phi_e^Z\restriction_u = A\restriction_u\},$$
a Σ_1^0 class. Let $C_s = [y_{\alpha,s}] \cap \{Z\colon\ \Phi_e^Z\restriction_u = A\restriction_u [s]\}$ be its approximation at stage s, which is clopen.

(b) WHILE $\lambda(P_s \cap C_s) < \delta$ run in case $e < s$ the procedure
$$R^{e+1,\eta\alpha}(Q, q, y_{\alpha,s});$$
here Q is the Π_1^0 class $P \cap [y_{\alpha,s}] - C$, and
$$q = 2\log\langle e, \eta\alpha\rangle + 2\log x + 2\log i + c,$$
where i is the number of times $S_x^{e,\eta\alpha}$ has called $R^{e+1,\eta\alpha}$ (the constant c was defined after (3) at the beginning of the formal construction). Then $2^{-q} < \lambda Q$ unless $Q = \emptyset$. Meanwhile, if $y_{\alpha,s} \neq y_{\alpha,s-1}$ put w into T_x, cancel all sub-runs, GOTO (a), and RETURN. Otherwise, if $A_s\restriction_u \neq A_{s-1}\restriction_u$ cancel all sub-runs, GOTO (a) and RETURN. (Comment: if the run $S_x^{e,\eta\alpha}$ stays at (b) and $Z \in Q$, then $A\restriction_u = \Phi_e^Z\restriction_u$ fails, so we have defeated Φ_e.)

(c) Put w into T_x, let $D = C_s$, GOTO (d), and RETURN. (Thus, the next time we call $S_x^{e,\eta\alpha}(P)$ it will be in Phase II.)

Phase II.

(d) $S_x^{e,\eta\alpha}$ requires attention again if $A\restriction_u$ has changed.

(e) WHILE $P_s \cap D \neq \emptyset$ RUN in case $e < s$
$$R^{e+1}(P \cap D, q, y_{\alpha,s})$$
where $q \in \mathbb{N}$ is defined as in (b). Meanwhile, if $y_{\alpha,s} \neq y_{\alpha,s-1}$ cancel all sub-runs, GOTO (a), and RETURN.
(Comment: if the run $S_x^{e,\eta\alpha}$ stays at (e) and $Z \in Q$ then again $A\restriction_u = \Phi_e^Z\restriction_u$ fails, this time because $Z \in D$ and $\Phi_e^Z\restriction_u$ is an old version of $A\restriction_u$.)

(f) GOTO (a) and RETURN.

Verification. The function g was defined at the beginning of the formal proof. First we compute bounds on how often a particular run $S_x^{e,\eta\alpha}$ does certain things.

Claim 1. *Consider a run $S_x^{e,\eta\alpha}(P,r,z)$ called by $R^{e,\eta}(P,r,z)$. As in the construction, let $m=|\alpha|$ and $n=m(r+m+1)$.*

(i) *While $y_{\alpha,s}$ does not change, the run passes (f) for at most 2^{m+r+1} times.*

(ii) *The run enumerates at most 2^{2n+r+2} elements into T_x.*

(iii) *It calls a run $R^{e+1,\eta\alpha}$ at (b) or (e) for at most $2^{n+1}g(x)$ times.*

To prove (i), as before let $\delta=2^{-|y_\alpha|-m-r-1}$. Note that each time the run passes (f), the class $P\cap[y_\alpha]$ loses $\lambda D\geq\delta$ in measure. This can repeat itself at most 2^{m+r+1} times. (This argument allows for the case that the run of $S_x^{e,\eta\alpha}$ is suspended due to the run of some $S_z^{e,\eta\beta}$ for $\beta\prec\alpha$. If $S_z^{e,\eta\beta}$ finishes then $S_x^{e,\eta\alpha}$, with the same parameters, continues from the same point on where it was when it was suspended.)
(ii) There are at most 2^n values for y_α during a run of $S_x^{e,\eta\alpha}$ by the remarks after Definition 3.4. Therefore this run enumerates at most $2^n2^{n+r+1}+2^n$ elements into T_x where at most 2^n elements are enumerated when y_α changes.
(iii): for each value y_α there are at most $2g(x)$ calls, namely, at most two for each computation $J^A(x)$ (g is defined at the beginning of the formal proof). ◇

Note that $\#T_x\leq h(x)$ by (ii) of Claim 1 and (4).

The strings $z_{\gamma,s}$, $\gamma\in 2^{<\omega}$ are used for coding the given set G into Z'. Let $z_{\varnothing,s}=\varnothing$.

- If $z_{\eta,s}$ has been defined and $R^{e,\eta}(P,r,z_\eta)$ is running at stage s, then for all β such that no procedure $S^{e,\eta\alpha}$ is running for any $\alpha\prec\beta$, let $z_{\eta\beta}=\mathsf{kuc}(P,r,z_\eta,\beta)$.
- If α is maximal under the prefix relation such that $z_{\eta\alpha,s}$ is now defined, it must be the case that $R^{e+1,\eta\alpha}(Q,q,z_{\eta\alpha})$ runs. So we may continue the recursive definition.

Claim 2 *For each γ, $z_\gamma=\lim_s z_{\gamma,s}$ exists, with the number of changes computably bounded in γ.*

We say that a run of $S_x^{e,\delta}$ is a *k-run* if $|\delta|\leq k$. For each number parameter p we will let $\overline{p}(k,v)$ denote a computable upper bound for p computed from k,v. Such a function is always chosen nondecreasing in each argument.

To prove Claim 2, we think of k as fixed and define by simultaneous recursion on $v\leq k$ computable functions $\overline{r}(k,v),\overline{x}(k,v),\overline{b}(k,v)$, $\overline{c}(k,v)$ with the following properties:

(i) *$\overline{r}(k,v)$ bounds r in any call $R^{e,\eta}(Q,r)$ where $|\eta|\leq k$ and $e\leq v$.*

(ii) *$\overline{x}(k,v)$ bounds the largest x such that some k-run $S_x^{e,\eta\alpha}$ is started where $e\leq v$.*

(iii) *For each x, $\overline{b}(k,v)$ bounds the number of times a k-run $S_x^{e,\eta\alpha}$ for $e\leq v$ requires attention.*

(iv) *For each x, $\overline{c}(k,v)$ bounds the number of times a run $R^{e+1,\eta\alpha}$ is started by some k-run $S_x^{e,\eta\alpha}$ for $e\leq v$.*

Fix γ such that $|\gamma| = k$. In the following we may assume that $\eta\alpha \preceq \gamma$, because then the actual bounds can be obtained by multiplying with 2^k.

Suppose now $k \geq v \geq 0$ and we have defined the bounds in (i)–(iv) for $v-1$ in case $v > 0$. We define the bounds for v and verify (i)–(iv). We may assume $e = v$, because then the required bounds are obtained by adding the bounds for $k, v-1$ to the bounds now obtained for $e = v$.

(i). First suppose that $v = 0$. Then $\eta = \varnothing$, so let $\overline{r}(k, 0) = 3$. If $v > 0$, we define a sequence of Π^0_1 classes as in Lemma 3.2: if for the i-th time a run $S^{e-1,\delta}_x$ calls a run $R^{e,\delta}(Q, q)$ we let $P^{\langle e,\delta,x,i\rangle} = Q$. By the inductive hypothesis (iii) and (iv) for $v-1$ we have a bound $\overline{i}(v, x)$ on the largest i such that a class $P^{\langle v,\eta\alpha,x,i\rangle}$ is defined (when $S^{v-1,\eta}_x$ in (b) or (e) starts a run $R^{v,\eta}$). Thus let $\overline{r}(k, v) = 2 \log\langle v, \gamma\rangle + 2 \log \overline{x}(k, v-1) + 2 \log \overline{i}(v, \overline{x}(k, v-1)) + c$. To prove (ii) and (iii), suppose $R^{e,\eta}(Q, r)$ calls $S^{e,\eta\alpha}_x$. Let $m = |\alpha|$ and $n = m(r + m + 1)$. Then $n \leq k(\overline{r}(k, v) + k + 1)$.

(ii) We have $h(x) < 2^{k+2k(\overline{r}(k,v)+k+1)+3}$ because m is chosen maximal in (4). Since h is an order function, this gives the desired computable bound $\overline{x}(k, v)$ on x.

(iii). By Claim 1(i), for each value of y_α, the run can pass (f) for at most $2^{k+\overline{r}(k,v)+1}$ times. Further, it can require attention $2^n + g(\overline{x}(k, v))$ more times because y_α changes or because $J^A(x)$ changes. This allows us to define $\overline{b}(k, v)$.

(iv). By Claim 1(iv) a run $R^{v+1,\eta\alpha}$ is started for at most $\overline{b}(k, v)2^{k+1}g(\overline{x}(k, v))$ times.

This completes the recursive definition of the four functions.

Now, to obtain Claim 2, fix γ. One reason that z_γ changes is that

(A) some run $S^{e,\delta}_y$ for $\delta \preceq \gamma$, calls $R^{e+1,\delta}$ in (e). This run is a k-run for $k = |\gamma|$. By (ii) and (iii), the number of times this happens is computably bounded by $\overline{b}(k, k)\overline{x}(k, k)$. While it does not happen, z_γ can also change because

(B) for some $\eta\alpha \preceq \gamma$ as in the construction, y_α changes because some P_s, which defines y_α, decreases. Since there is a computable bound $\overline{l}(k)$ on the length of z_γ by (i) of this claim and (2), while (A) does not apply this can happen for at most $2^{\overline{l}(k)}$ times. Thus in total z_γ changes for at most $\overline{b}(k, k)\overline{x}(k, k)2^{\overline{l}(k)}$ times. ◇

Now let $Z = \bigcup_{\gamma \prec G} z_\gamma$. By Claim 2 we have $G \leq_{\mathrm{tt}} Z'$.

Claim 3 (Golden Run Lemma) *For some $\eta \prec G$ and some e, there is a run $R^{e,\eta}(P, r)$ (called a golden run) that is not cancelled such that, each time it calls a run $S^{e,\eta\alpha}_x$ where $\eta\alpha \prec G$, that run returns.*

Assume the claim fails. We verify the following for each e.

(i) There is a run $R^{e,\eta}$ that is not cancelled; further, $S^{e,\eta\alpha}_x(P)$ is running for some x, where $\eta\alpha \prec G$, and eventually does not return.

(ii) $A \neq \Phi^Z_e$.

(i) We use induction. For $e = 0$ clearly the single run of $R^{0,\varnothing}$ is not cancelled. Suppose now that a run of $R^{e,\eta}$ is not cancelled. Since we assume the claim fails, some run $S^{e,\eta\alpha}_x$, $\eta\alpha \prec G$, eventually does not return. From then on

the computation $J^A(x)$ it is based on and y_α are stable. So the run calls $R^{e+1,\eta\alpha}$ and that run is not cancelled.
(ii) Suppose the run $S_x^{e,\eta\alpha}(P,r,z)$ that does not return has been called at stage s. Suppose further it now stays at (b) or (e), after having called $R^{e,\eta\alpha}(Q,q,y_\alpha)$. Since $y_{\eta\alpha}$ is stable by stage s, we have $Z \in Q$. Hence $A \neq \Phi_e^Z$ by the comments in (b) or (e). ◇

Let $(T_x)_{x\in\mathbb{N}}$ be the c.e. trace enumerated by this golden run.
Claim 4 *(T_x) is a trace for J^A with bound h.*

As remarked after Claim 1, we have $\#T_x \le h(x)$. Suppose x is so large that m in (4) exists. Suppose further that the final value of $w = J^A(x)$ appears at stage t. Let $\eta\alpha \prec G$ such that $|\alpha| = m$.

As the run is golden and by Claim 1(i), eventually no procedure $S_y^{e,\eta\beta}(P)$ for $\beta \prec \alpha$ is at (b) or (e). Thus, from some stage $s > t$ on, the run $S_x^{e,\eta\alpha}$ is not suspended. If y_α has not settled by stage s then w goes into T_x. Else $\lambda(P \mid y_{\alpha,s}) > 2^{-r-|\alpha|}$. Since $S_x^{e,\eta\alpha}$ returns each time it is called, the run is at (a) at some stage after t. Also, $P_s \cap C_s$ must reach the size $\delta = 2^{-|y_\alpha|-|\alpha|-r-1}$ required for putting w into T_x. □

As a consequence, we separate highness properties within the ML-random sets. See [11, Def. 8.4.13] for the weak reducibility $\le_{JT}$, and [5] for the highness property "$\emptyset'$ is c.e. traceable by Y". Note that JT-hardness implies both this highness property and superhighness.

Corollary 3.5. *There is a ML-random superhigh Δ^0_3 set Z such that $\emptyset'$ is not c.e. traceable by Z. In particular, Z is not JT-hard.*

Proof. By [11, Lemma 8.5.19] there is a benign cost function c such that each c.e. set A that obeys c is Turing below each ML-random set Y such that $\emptyset'$ is c.e. traceable by Y. By [11, Exercise 8.5.8] there is an order function h such that some c.e. set A obeys c but is not jump traceable with bound h. Then by the proof of Theorem 3.1 there is a ML-random superhigh set $Z \le_T \emptyset''$ such that $A \not\le_T Z$. Hence Z is not JT-hard. □

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