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Lowness Properties of Reals and Hyper-Immunity

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Abstract

Ambos-Spies and Kučera [1, Problem 4.5] asked if there is a non-computable set A which is low for the computably random reals. We show that no such A is of hyper-immune degree. Thus, each $g \leq_T A$ is dominated by a computable function. Ambos-Spies and Kučera [1, Problem 4.8] also asked if every S-low set is S_0-low. We give a partial solution to this problem, showing that no S-low set is of hyper-immune degree.

Keywords: Randomness, S-lowness, hyper-immunity.

1 Introduction

The formalization of the intuitive notions of computability and randomness has been studied in order to provide a mathematical foundation to computer science. Since 1936, several equivalent models of computability have been proposed to capture the intuitive sense of computability (Church-Turing thesis). The formalization of the intuitive notion of randomness has also motivated several mathematicians and computer scientist to study the subject. Because randomness in an absolute sense does not exist [1], some restrictions must be

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imposed to capture better the intuitive notion of randomness. This leads to a hierarchy of notions of randomness. In 1940, Alonso Church [2] suggested that intuitive randomness should be defined as algorithmic randomness proposing a formal notion of computable randomness. Church's proposal was widely accepted, but still was deficient from a statistical point of view. Kolmogoroff intended to define randomness in terms of his complexity notion. This motivated Per Martin-Löf in 1966 [5] to propose a new formal definition for algorithmic randomness without the statistical problem of Church's concept. Martin Löf randomness is very restrictive since the tests are r.e. rather than computable objects. So Schnorr [9] provided a broader randomness notion based on computable tests, which still does not have the Church statistical problems.

Infinite sequences of 0's and 1's will be called *reals* and are identified with sets of natural numbers. A lowness property of a real says that, in some sense, it has a low computational power when used as oracle [7]. For instance, a real B is *low for random* if each Martin-Löf random real is already Martin-Löf random relative to B, i.e. using B as an oracle does not help to detect regularities in any random set. When considering Schnorr randomness instead, because of the absence of a universal test, we obtain two lowness notions. An oracle B is S-low if each Schnorr random real is already Schnorr random relative to B. B is S_0-low if it does not even change the power of Schnorr tests. Then S_0-lowness implies S-lowness. In [1, Problem 4.5] it is asked if S_0-low $= S$-low. Another open problem, pointed out by Ambos-Spies and Kučera [1, Problem 4.8], is the following: "Are there non computable sets which are low for the class of computably random sets? If so, what is the relation between these sets and the ML-low (S-low) sets?".

Some Martin-Löf random reals have hyper-immune free degree. Our first result is that there is no low for computably random real which has hyper-immune degree. So, we partially answer [1, Problem 4.8]. The proof can be adapted to the case of S-lowness. Thus all S-low sets are of hyper-immune free degree as well. Terwijn and Zambella [12] proved that A is S_0-low if and only if A is recursively traceable, a property which implies being of hyper-immune free degree. Our result gives some positive evidence that S-low $= S_0$-low: we show that S-low is at least close, as it lies somewhere between S_0-low and hyper-immune free.

2 Basic Notions

A real is an infinite binary sequence of 0's and 1's, identified with a set of natural numbers. Let 2^ω be the set of reals. A real A is computable relative to a real B, or is Turing reducible to B, denoted by $A \leq_T B$, if we have an access to B then we can compute A, that is, if we can compute χ_A [3] using

[3] χ_A is the characteristic function of the set A.

χ_B as oracle. Let $\mathcal{C}$ be a relativizable class of reals. For an oracle A, the relativization is denoted by $\mathcal{C}^A$. An oracle A is called *low for* $\mathcal{C}$ if $\mathcal{C}^A = \mathcal{C}$. For all classes we consider the lowness property is downward closed under $\leq_T$. For instance, if $\mathcal{C}$ is the class of Δ^0_2 sets (the class of real s which can be computed with the halting problem), then lowness for $\mathcal{C}$ coincides with the usual lowness $A \leq_T \emptyset'$, that is A is low for $\mathcal{C}$ if and only if A is Turing reducible to $\emptyset'$. If $\mathcal{C}$ is a randomness notion, then the intuitive meaning of "A is low for $\mathcal{C}$", is that the oracle A does not help to detect further regularities in the sense of $\mathcal{C}$.

An oracle A is *hyper-immune* if there is a computable function g relative to A, $g \leq_T A$, which is not dominated by any recursive function and is *hyper-immune free* if it is not hyper-immune [10]. In other words, an oracle A is hyper-immune free if each total function, recursive in A, is majorized by a recursive function, that is, if for each $g \leq_T A$ there exists a computable function f such that for all x, $g(x) \leq f(x)$ [6].

A set A is *low for random*, if each random real X is already random relative to A (that is, X passes all A-recursive enumerable test, where an A-recursive enumerable test is a set $U \subseteq \omega \times 2^{<\omega}$ [4] which is recursively enumerable relative to A). A Schnorr-test U is a recursive set $U \subseteq \omega \times 2^{<\omega}$ such that $\mu U_n = 2^{-n}$ for each $U_n = \{x : (x, n) \in U\}$, where μU is the usual Lebesgue measure of a open set U in Cantor space 2^ω (as usual we identify U_n with the corresponding open set). A class of reals is Schnorr null if it is contained in $\bigcap_{n\in\omega} U_n$ for some Schnorr test (U_n). An oracle A is called S_0*-low*, if for each Schnorr test (V_n) relative to A, there is an unrelativized Schnorr test $\bigcap_{n\in\omega} U_n$ such that $\bigcap_{n\in\omega} V_n \subseteq \bigcap_{n\in\omega} U_n$

A real is *Schnorr random* if it does not belong to any Schnorr null set, i.e. if for each Schnorr-test U, $R \notin \bigcap_n U_n$. If $\mathcal{C}$ is the set of all Schnorr random reals, then A is *S-low* if $\mathcal{C}^A = \mathcal{C}$. Clearly each S_0-low oracle is S-low, but the converse is unknown [1]. Terwijn and Zambella in [12] classified the oracles which satisfy the stronger property of S_0-low and showed that an oracle A is S_0-low if, and only if, A is recursively traceable, where an oracle A is recursively traceable if there is a recursive bound $h : \omega \longrightarrow \omega$ such that each total function $g \leq_T A$ has a recursive trace T bounded by h, that is $\parallel T^{[k]} \parallel \leq h(k)$ [5], for each $k \geq 0$. A computable set $T \subseteq \omega \times \omega$ is a recursive trace for a function $f : \omega \longrightarrow \omega$, if for each section $T^{[k]} = \{m : (k, m) \in T\}$ of T we have that $f(k) \in T^{[k]}$, $T^{[k]}$ is finite and the function mapping k into the canonical index of $T^{[k]}$ is computable.

The concept of martingales, proposed by P. Levy, has been widely applied in the study of stochastic processes [4], learning [10] and randomness [13]. A martingale allows us to calculate the gambling-account of a player who always tries to predict the next value of a function [10]. The idea is that martingales capture *betting strategies* to predict the next digit in a binary sequence.

[4] $2^{<\omega}$ denoted the set of all binary finite strings.

[5] $\parallel T^{[k]} \parallel$ is the cardinality of $T^{[k]}$.

For our purposes, a *martingale* (MG in short) is a function $M : 2^{<\omega} \mapsto \mathbb{Q}$ such that $dom(M)$ is $2^{<\omega}$, or $2^{\leq n}$ for some n, $M(\lambda) \leq 1$, and M has the martingale property $M(x0) + M(x1) = 2M(x)$ whenever the strings $x0, x1$ belongs to the domain. A MG M succeeds on a sequence Z if

$$\limsup_{n\to\infty} M(Z \upharpoonright n) = \infty,$$

where $Z \upharpoonright n$ is the prefix of n bits of Z. A real is *computably random* if no computable MG succeeds.

A MG M *effectively succeeds* on a sequence Z if there is a nondecreasing and unbounded computable function $f : \omega \longrightarrow \omega$ such that

$$\limsup_{n\to\infty} M(Z \upharpoonright n) - f(n) > 0.$$

It is possible to provide a characterization of Schnorr randomness in terms of martingales. A sequence Z is Schnorr random if and only if no computable MG effectively succeeds on Z.

3 Main result

Theorem 3.1

(i) No low for computably random real A has hyper-immune degree.

(ii) Each S-low set is of hyper immune-free degree.

Proof: (i) Suppose A has hyper-immune degree, so there is a function $g \leq_T A$ not dominated by a computable function. Thus for each computable f, $\exists^\infty x\ f(x) \leq g(x)$. We will define a computably random real R and an A-computable $\mathbb{Q}$-valued MG L which succeeds on R, so A is not low for computably random. In the following α, β, γ denote finite subsets of $\mathbb{N}$, and $n_\alpha = \sum_{i\in\alpha} 2^i$ (here $n_\emptyset = 0$).

Let M_e be an effective listing of partial recursive martingales with range included in $[1/2, \infty)$. At stage t, we have a finite portion $M_e[t]$ whose domain is of the form $2^{\leq n}$ for some n. If R is not computably random, then $M_e(R) = \infty$ for some total M_e [9]. Let

$$TMG = \{e : M_e \text{ total }\}.$$

For certain α, and all those included in TMG, we will define strings x_α, in a way that $\alpha \subseteq \beta \Rightarrow x_\alpha \preceq x_\beta$, that is x_α is a prefix of x_β. We chose the strings in a way that $M_e(x_\alpha)$ is bounded by a fixed constant, for each total M_e and each α containing e. Then the real

$$R = \bigcup_{\alpha\subseteq TMG} x_\alpha$$

is computably random. On the other hand we are able to define an A-computable MG L which succeeds on R. We give an inductive definition of the strings x_α, "scaling factors" $p_\alpha \in \mathbb{Q}^+$ and partial computable MGs M_α

such that, if x_α is defined then

$$M_\alpha(x_\alpha) \text{ converges in } g(|x_\alpha|) \text{ steps and } M_\alpha(x_\alpha) < 2. \tag{1}$$

It will be clear that A can decide if $y = x_\alpha$ given inputs y and α.

Let $x_\emptyset$ be the empty string, and $M_\emptyset = 0$. Now suppose $\alpha = \beta \cup \{e\}$ where $e > \max(\beta)$, and inductively suppose that (1) holds for β. Let

$$p_\alpha = \tfrac{1}{2} 2^{-|x_\beta|}(2 - M_\beta(x_\beta)),$$

and let $M_\alpha = M_\beta + p_\alpha M_e$. Since M_e is a MG on its domain, $M_e(z) \leq 2^{|z|}$ for any z. So $M_\alpha(x_\beta) < 2$ if defined.

To define x_α, we look for a sufficiently long extension x of x_β such that M_α does not increase from x_β to x and $M_\alpha(x)$ converges in $g(|x|)$ steps. In detail, for larger and larger $m > x_\beta$, $m \geq 4n_\alpha$, if no string y, $|y| < m$ has been designated to be x_α as yet, and if $M_\alpha(z)$ (i.e., each $M_e(z), e \in \alpha$) converges in $g(m)$ steps, for each string of length $\leq m$, then choose x_α of length m, $x_\beta \prec x_\alpha$ such that M_α does not increase from x_β to x_α.

Lemma 3.2 *If $\alpha \subseteq TMG$, then x_α and p_α are defined.*

Proof: The lemma is trivial for $\alpha = \emptyset$. Suppose it holds for β, and $\alpha = \beta \cup \{e\}$ where $e > \max(\beta)$. Since the function

$$f(m) = \mu s \forall e \in \alpha \forall x [|x| \leq m \Rightarrow M_e(x) \textit{ converges in } s \textit{ steps}]$$

is computable, there is a least $m \geq 4n_\alpha$, $m > |x_\beta|$ such that $g(m) \geq f(m)$. Since there is a path down the tree starting at x_β where M_α does not increase, we are able to choose x_α. ◇

Lemma 3.3 *R is computably random.*

Proof: Suppose M_e is total. Let $\alpha = TMG \cap [0, e]$. If $\alpha \subseteq \gamma$, $\gamma' = \gamma \cup \{i\}$, $\max(\gamma) < i$ and $\gamma' \subseteq TMG$, then for each x, $x_\gamma \preceq x \prec x_{\gamma'}$,

$$p_\alpha M_e(x) \leq M_\gamma(x) \leq M_\gamma(x_\gamma) < 2,$$

Thus $M_e(x) < 2/p_\alpha$ for each $x \prec R$. ◇

Lemma 3.4 *There is a MG $L \leq_T A$ which succeeds on R. In fact,*

$$\exists^\infty x \prec R \; L(x) \geq \lfloor |x|/4 \rfloor$$

Proof: For a string z, let $r(z) = \lfloor |z|/2 \rfloor$. We let $L = \sum_\alpha L_\alpha$, where L_α is a MG with initial capital $L_\alpha(\lambda) = 2^{-n_\alpha}$ which bets everything along x_α from $x_\alpha \upharpoonright r(x_\alpha)$ on. More precisely, if x_α is undefined then L_α is constant with value 2^{-n_α}. Otherwise, let $x = x_\alpha \upharpoonright 2r(x_\alpha)$, and

- let $L_\alpha(y) = 2^{-n_\alpha}$ unless $x \upharpoonright r(x) \preceq y$
- in that case, if x, y are incompatible, let $L(y) = 0$
- else let $L(y) = 2^{-n_\alpha} 2^{\min(|y| - r(x), r(x))}$

Then $L_\alpha(x_\alpha) = r(x_\alpha) - n_\alpha$. Since $r(x_\alpha) \geq 2n_\alpha$, this implies $L_\alpha(x_\alpha) \geq \lfloor |x_\alpha|/4 \rfloor$.

It remains to check that $L \leq_T A$. Given input y, it suffices to determine $L_\alpha(y)$ for each α such that $n_\alpha \leq |y|$. Using g, see if some string x, $|x| \leq 2|y|$ is x_α. If not, $L_\alpha(y) = 2^{-n_\alpha}$. Else we determine $L_\alpha(y)$ from x using the definition of L_α. $\diamond$

(ii) Note that, in the proof of (i), the MG L succeeds effectively on R. Thus R is Schnorr random, but not Schnorr random relative to A. Hence, if A is of hyper-immune degree, then A is not S-low. $\square$

4 Final Remarks

The theorem 3.1 proved two lowness properties for reals, namely: all real A which is low for computably random has hyper-immune free degree and all real A which is S-low has hyper-immune free degree. These results partially solves the problems 4.5 and 4.8 enunciated by Ambos-Spies and Kučera in [1]. Nies in very recent work has announced a negative solution to 4.8, namely all low for computably random oracles are computable.

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