RANDOMNESS NOTIONS AND PARTIAL RELATIVIZATION

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Abstract. We study weak 2 randomness, weak randomness relative to $\emptyset'$ and Schnorr randomness relative to $\emptyset'$. One major theme is characterizing the oracles A such that $\mathsf{ML}[A] \subseteq \mathcal{C}$, where $\mathcal{C}$ is a randomness notion and $\mathsf{ML}[A]$ denotes the Martin-Löf random reals relative to A. We discuss the connections with LR-reducibility and also study the reducibility associated with weak 2-randomness.

1. Introduction

Martin-Löf randomness has been criticized for not being strong enough to appropriately formalize our intuition of a random set. For instance, left-c.e. sets, like Ω, and superlow sets can be Martin-Löf random. On the other hand, it is the randomness notion that interacts best with computability theoretic concepts. Many examples of such interaction are given in [Nie09] (see beginning of Chapter 4 for an overview).

This paper serves two purposes, the second being the principal:

(1) We study randomness notions between Martin-Löf randomness and 2-randomness.

(2) We provide some new interactions of these randomness notions with computability theoretic concepts.

Purpose (1). In Section 2 we consider Martin-Löf randomness, Schnorr randomness relative to $0'$, weak randomness relative to $0'$, and weak 2-randomness. We study the computational complexity and provide various separations of these classes. In particular, we show that within the Martin-Löf random sets, weak randomness relative to any oracle can be separated from weak 2-randomness.

The notions of randomness we study are displayed in Table 1, together with the symbols for them. Recall that a set is Martin-Löf random if it passes all Martin-Löf tests. That is, it is not a member of any class of the form $\bigcap_j U_j$ such that (U_j) is a uniformly c.e. sequence of open sets with $\mu(U_j) \leq 2^{-j}$. Here μ denotes the usual product measure on Cantor space. In the following, Martin-Löf random sets may be referred to simply as "random". Also recall that 2-randomness is Martin-Löf randomness relative to $\emptyset'$. A set is weakly random (or Kurtz random) if it is not a member of any null Π^0_1 class. Similarly, it is weakly 2-random if it is not a member of any null Π^0_2 class. A set is Schnorr random if no computable martingale succeeds on it quickly. Equivalently, it passes all Martin-Löf tests with the special

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property that their members have uniformly computable measure. The following implications hold:

$$\mathsf{ML}[\emptyset'] \Rightarrow \mathsf{SR}[\emptyset'] \Rightarrow \mathsf{W2R} \Rightarrow \mathsf{Kurtz}[\emptyset'] \cap \mathsf{ML} \Rightarrow \mathsf{ML} \tag{1.1}$$

None of the implications in (1.1) can be reversed; see Section 2.

Martin-Löf randomness	ML
weak randomness relative to $\emptyset'$	$\mathsf{Kurtz}[\emptyset']$
weak 2-randomness	$\mathsf{W2R}$
Schnorr random relative to $\emptyset'$	$\mathsf{SR}[\emptyset']$
2-randomness	$\mathsf{ML}[\emptyset']$

TABLE 1. Randomness notions and the symbols used to denote them.

For more background on algorithmic randomness and unexplained notions we refer to Chapter 3 of [Nie09].

Purpose (2). We provide some new interactions of the randomness notions in Table 1 with computability theoretic concepts.

Given two classes $\mathcal{M}$ and $\mathcal{N}$, define High$(\mathcal{M}, \mathcal{N})$ to be the class containing all oracles A such that $\mathcal{M}^A \subseteq \mathcal{N}$. For instance, High$(\mathsf{ML}, \mathsf{SR}[\emptyset'])$ is the set of oracles A that are computationally complex in the sense that each set that is Martin-Löf random in A is already $\mathsf{SR}[\emptyset']$. The results are summarized in the following table. We prove the characterizations in (a)–(d) and observe (f). The equivalence (e) is due to Kjos-Hanssen/Miller/Solomon [KHMSxx] (also see [Nie09] or [Sim07] for a proof). Recall from [DS04] that A is uniformly almost everywhere dominating (u.a.e. dominating, for short) if it computes a function that dominates all Turing functionals on almost all oracles.

(a)	$A \in$ High$(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$	$\emptyset'$ is non-d.n.c. by A
(b)	$A \in$ High$(\mathsf{ML}, \mathsf{W2R})$	
(c)	$A \in$ High$(\mathsf{ML}, \mathsf{SR}[\emptyset'])$	$\emptyset'$ is c.e. traceable by A
(d)	$A \in$ High$(\mathsf{W2R}, \mathsf{ML}[\emptyset'])$	A is u.a.e. dominating
(e)	$A \in$ High$(\mathsf{ML}, \mathsf{ML}[\emptyset'])$	
(f)	$A \in$ High$(\mathsf{Kurtz}, \mathsf{ML})$	impossible

TABLE 2. Highness classes with respect to randomness notions and their equivalent computability-theoretic characterizations.

Some of the properties on the right column of Table 2 are obtained by partial relativization, indicated with the preposition "by", from standard notions. This means that we only relativize certain components of the notions, rather than all of them as in complete relativization. For example, we say that Y is c.e. traceable *by* A if there is a computable function h such that for each function $f \leq_T Y$ there is an A-c.e. trace for f with bound h. Recall that a sequence of sets (T_i) is a trace for a function f if $f(n) \in T_n$ for all $n \in \mathbb{N}$. Also, (T_i) has bound h if $|T_n| < h(n)$ for all $n \in \mathbb{N}$. By the method of [TZ01], if Y is c.e. traceable by A, then the bound of the required trace can be any non-decreasing unbounded computable function. See [Nie09, 8.2.3].

Let $\mathsf{DNC}[A]$ be the class of diagonally non-computable functions relative to A. That is, the functions g such that $g(e) \neq \Phi_e^A(e)$ for all e such that $\Phi_e^A(e)\downarrow$ (where Φ_e is the e-th Turing functional). We say that Y is non-d.n.c. by A if Y does not compute any function in $\mathsf{DNC}[A]$.

There are earlier examples of notions that were obtained by partial relativization,[1] such as the LR relation that was defined in [Nie05]. A set A is LR reducible to a set B (denoted by $A \leq_{LR} B$) if every B-random (i.e., Martin-Löf random relative to B) is A-random. This notion was obtained by partially relativizing the notion of *low for random* from [KT99].[2] The notion of u.a.e. dominating set (also in Table 2) can also be obtained in this way: it was shown in Kjos-Hanssen/Miller/Solomon [KHMSxx] that a set A is u.a.e. dominating iff $\emptyset' \leq_{LR} A$. Note that, by definition, $A \in \mathrm{High}(\mathsf{ML}, \mathsf{ML}[\emptyset'])$ iff $\emptyset' \leq_{LR} A$.

Generalizing the process that led to $\leq_{LR}$, for each randomness notion $\mathcal{C}$ we have an associated reducibility $\leq_{\mathcal{C}}$ given by

$$A \leq_{\mathcal{C}} B \Leftrightarrow \mathcal{C}^A \supseteq \mathcal{C}^B.$$

Namely if A can find "regularities" in a set in the sense of $\mathcal{C}$, then so can B. In Section 4 we study the reducibility associated with weak 2-randomness, denoted it by $\leq_{W2R}$, and its connections with $\leq_{LR}$. We show that $\leq_{LR}$ and $\leq_{W2R}$ coincide on the Δ_2^0 sets, as well as the low for Ω sets. Recall that a set is *low for* Ω if Ω is Martin-Löf random relative to it. Given that the low for Ω sets are downward closed with respect to $\leq_{LR}$, it follows that the two reducibilities have interesting common initial segments. On the other hand, we show that they differ on the class of Δ_3^0 sets. These (weak) reducibilities induce equivalence relations $\equiv_{LR}$ and $\equiv_{W2R}$ respectively, and therefore degree structures. We show that $\equiv_{LR}$, $\equiv_{W2R}$ coincide on all sets. Hence, although the degree structures differ as partially ordered sets, the actual degrees as equivalence classes coincide. Barmpalias/Lewis/Soskova [BLS08a] proved that there are continuum many sets $\leq_{LR} \emptyset'$. We finish Section 4 with a similar result, proving that there are continuum many sets $\leq_{W2R} \emptyset''$. Therefore, $A \leq_{W2R} B$ does not imply that $A \leq_T B'$, which is interesting because it follows from a result of Kjos-Hanssen/Miller/Solomon [KHMSxx] that $A \leq_{LR} B$ and $A \leq_T B'$ together imply that $A \leq_{W2R} B$.

Weak 2-randomness is a very natural notion of randomness and it has a very simple definition. Its exact relation with Martin-Löf randomness was clarified by a result of Hirschfeldt/Miller (see Section 5.3 in [Nie09]) as follows: a set is weakly 2-random iff it is Martin-Löf random and it forms a minimal pair with $\emptyset'$. This characterization provides some evidence that weak 2-randomness is much closer to Martin-Löf randomness than it is to 2-randomness. Further evidence for this claim can be seen in Table 2, in particular the second and the fourth line of the table. The complexity required for an oracle A to lift Martin-Löf randomness to

[1] In [CS07, Sim07] some notions obtained by partial relativization were shown to play an important role in the study of mass problems and the degrees of difficulty. For example, we mention the notion of *bounded limit recursiveness* and *jump-traceability*. However we will not study these notions in this paper.

[2] We say that A is *low for random* if every A-random is random. The full relativization of this notion is as follows: A is low for random relative to B if every B-random is $A \oplus B$ random. A classic example of full relativization is the class GL_1. It contains the oracles A such that $\emptyset'$ is complete relative to A.

weak 2-randomness is just the property '$\emptyset'$ non-d.n.c. by A' (which coincides with non-lowness in the Δ^0_2 degrees). This is much less than u.a.e. domination, which is the complexity required for A to lift weak 2-randomness to 2-randomness.

Clearly, there is no weakly 2-random Δ^0_2 set. In Section 5 we show that there is a weakly 2-random set $Z \leq_{LR} \emptyset'$. In fact, we show that there is a weakly 2-random that is K-trivial relative to $\emptyset'$. This shows that some weakly 2-random sets are very close to $\emptyset'$. On the other hand, we show that there is no weakly 2-random set LR-below a low Δ^0_2 set. We also show that no weakly 2-random has a non-K-trivial Δ^0_2 set LR-below it. In particular, no weakly 2-random can have complete LR degree.

2. Notions between 1 and 2-randomness

Consider the notions of randomness in Table 1. The first implication of (1.1) follows from the definitions. The second follows from the observation that every null Π^0_2 class is contained in a Schnorr test. The third follows in a similar way (every $\Pi^0_1[\emptyset']$ class is a Π^0_2 class) and the fourth is trivial.

The strictness of the first implication follows by relativizing the well known fact that some Schnorr random is not Martin-Löf random. Recall that B is *generalized low* (GL_1) if $A' \leq_T \emptyset' \oplus A$. For the second, notice that there is a weakly 2-random set that is not GL_1 by [LMN07], while all sets Schnorr random relative to $\emptyset'$ are GL_1 by Proposition 2.1. The strictness of the third implication in (1.1) is shown in Theorem 2.3 below. Finally for the strictness of the fourth implication, notice that some Martin-Löf random set is computable from $\emptyset'$, and hence a $\Pi^0_1[\emptyset']$ singleton.

Proposition 2.1. $\mathsf{SR}[\emptyset'] \subseteq \mathrm{GL}_1$.

Proof. Uniformly in e, $\emptyset'$ can compute a stage s so large that e goes into A' after stage s for at most measure 2^{-e} oracles A. Let f be the $\emptyset'$ computable function that computes s from e. Given e and $s = f(e)$, the oracles A such that e goes into A' after stage s form a Σ^0_1 class V_e. Since $\mu V_e < 2^{-e}$, $\emptyset'$ can uniformly form a $\Sigma^0_1[\emptyset']$ class U_e that contains V_e and has measure exactly 2^{-e}. Then $W_i = \bigcup_{e>i} U_e$ is a $\mathsf{SR}[\emptyset']$ test, and if A is not covered by this test, then (except finitely often) $e \in A' \leftrightarrow e \in A'_{f(e)}$. □

Another strengthing of Martin-Löf randomness is Demuth randomness. See [Nie09, 3.6.24]. Note that Demuth randomness is incomparable with weak 2-randomness. Also $\mathsf{SR}[\emptyset']$ is contained in both the weakly 2-random and the Demuth random sets. A more complex argument than in the previous result shows that each Demuth random set is in GL_1 (see [Nie09, 3.6.26]).

We next investigate the dubious power of $\mathsf{Kurtz}[A]$ randomness, for an arbitrary oracle A. It is easy to see that there is no oracle A such that every Kurtz random relative to A is Martin-Löf random. In other words $\mathrm{High}(\mathsf{Kurtz}, \mathsf{ML}) = \emptyset$, hence clause (f) of Table 2. This follows from purely topological considerations. The universal Martin-Löf test makes it clear that the non-ML random reals form a comeager class. On the other hand, for any A, the union of all measure zero $\Pi^0_1[A]$ classes is meager. Hence by the Baire category theorem, there is a Kurtz random relative to A that is not Martin-Löf random. One must work harder to answer the following question: what does Kurtz randomness relative to A imply if Z is already Martin-Löf random? We show that there is no oracle A such that Martin-Löf randomness and $\mathsf{Kurtz}[A]$ randomness is enough to imply weak 2-randomness. First, we need the following lemma.

Lemma 2.2. *Let $P \subseteq 2^\omega$ be a nowhere dense Π^0_1 class. There is a null Π^0_2 class Q such that $Q \cap P$ is dense in P.*

Proof. We will define Q to cover the left endpoints of maximal open intervals in $\overline{P} = 2^\omega - P$. Since P is nowhere dense, these points are dense in P. It will be helpful to use the euclidean metric on 2^ω; that is, for $X, Y \in 2^\omega$ we take $|X - Y|$ to be distance between the reals numbers in $[0, 1]$ whose binary expansions are given by X and Y.[3] We also use the natural order on 2^ω and let $\mathcal{F} \subseteq 2^\omega$ represent the sequences with finitely many ones. For $s \in \omega$, let

$$V_s = \{X : (\exists t \geq s)(\exists A, B \in \mathcal{F})\ X \in P_s \text{ and } X < A < B \text{ and } [A, B] \cap P_t = \emptyset \text{ and } |A - X| < |B - A|/s\}.$$

It should be clear that V_s is a Σ^0_1 class. It is also easy to see that if X is the left endpoints of a maximal open interval in $\overline{P}$, then $X \in V_s$. Hence, letting $Q = \bigcap_{s\in\omega} V_s$, we have $X \in Q$. All that remains to prove is that $\mu(Q) = 0$, for which it is sufficient to show that $\lim_s \mu(V_s) = 0$.

Fix $s \in \omega$. Let (Y, Z) be a maximal interval in $\overline{P}$ and let $\ell = |Z - Y|$ be its length. Say that X is added to V_s with witnesses $A, B \in (Y, Z)$. If $X \notin (Y, Z)$, then it must be the case that $X < Y$ and $|Y - X| < |A - X| < \ell/s$. Thus we have $\mu(V_s) \leq (1 + 1/s)\mu(\overline{P})$. On the other hand, this estimate includes the measure of all the sequences in $\overline{P_s}$, but these have been excluded in the definition of V_s. So in fact, we have $\mu(V_s) \leq (1 + 1/s)\mu(\overline{P}) - \mu(\overline{P_s})$. But both $(1 + 1/s)\mu(\overline{P})$ and $\mu(\overline{P_s})$ approach $\mu(\overline{P})$ as s goes to infinity. Therefore, $\lim_s \mu(V_s) = 0$. □

Now we are ready to separate $\mathsf{Kurtz}[A]$ from the weakly 2-randoms within the class of Martin-Löf randoms, for an arbitrary oracle A.

Theorem 2.3. *For any A, there is a Martin-Löf random Z that is $\mathsf{Kurtz}[A]$ but not weakly 2-random.*

Proof. Let P be a Π^0_1 class containing only Martin-Löf random reals. Let Q be the measure zero Π^0_2 class from the lemma. We will, as in the remarks before Lemma 2.2, use the Baire category theorem, but this time with respect to the compact subspace P. Note that $Q \cap P$ is a G_δ set relative to P and it is dense in P, hence it is comeager in P. Next, consider a measure one $\Sigma^0_1[A]$ class V. Let $\sigma \in 2^{<\omega}$. If $[\sigma] \cap P \neq \emptyset$, then it is a nonempty Π^0_1 class containing a Martin-Löf random, so $\mu([\sigma] \cap P) > 0$ (see, for example, [Nie09]). Hence $V \cap [\sigma] \cap P$ is nonempty. Therefore, $V \cap P$ is dense in P. Since it is an open set relative to P, it is also comeager in P. By the Baire category theorem relative to P, there is a $Z \in P$ in the intersection of Q with (the countable collection of) all measure one $\Sigma^0_1[A]$ classes. Clearly Z is Kurtz random relative to A. Since $Z \in P$, it is Martin-Löf random. Finally, $Z \in Q$ implies that it is not weakly 2-random. □

[3] Strictly speaking, this is not a metric on 2^ω since the two distinct sequences representing a dyadic rational have distance zero from each other.

Note that we could recast the proof of Theorem 2.3 as an initial segment construction, were we so inclined.

3. Characterizing highness notions

In this section we show the equivalences between highness properties (a)–(c) of Table 2 and the corresponding computability theoretic notions. We begin with High(ML, W2R). We show that, despite the fact that $\mathsf{W2R}$ is a stronger randomness notion than $\mathsf{ML} \cap \mathsf{Kurtz}[\emptyset']$, the complexity that is required for an oracle A to turn $\mathsf{ML}[A]$ into a subclass of $\mathsf{Kurtz}[\emptyset']$ is the same as the complexity required to turn it into a subclass of $\mathsf{W2R}$. Also, we give a characterization of this highness property High($\mathsf{ML}, \mathsf{W2R}$) in computability theoretic terms. We start with the following lemma, which is a partial relativization of a result from [GM09]; the proof is due to the second author.

Lemma 3.1. *If $A \in \mathrm{High}(\mathsf{ML}, \mathsf{Kurtz}[Y])$, then Y does not compute a $\mathsf{DNC}[A]$ function.*

Proof. Assume that $f \leq_T Y$ is a $\mathsf{DNC}[A]$ function. We show that Y computes an infinite subset D of a set that is ML-random in A. This shows that there is a set that is Martin-Löf random in A but is in a null $\Pi^0_1[Y]$ class, thus not in $\mathsf{Kurtz}[Y]$. Let Q be a non-empty $\Pi^0_1[A]$ class of ML[A]-random sets. By a well known lemma of Kučera [Kuč85], we may assume that if $P \subseteq Q$ is a nonempty $\Pi^0_1[A]$ class, then we can compute, uniformly from an index for P, a k such that $2^{-k} < \mu P$.

Using f we compute a sequence $d_0 < d_1 < \cdots$ such that, for each n, the $\Pi^0_1[A]$ class $\{Z \in Q\colon\ d_0, \ldots, d_{n-1} \in Z\}$ is non-empty. Let $D = \{d_0, d_1, d_2, \ldots\}$. By compactness $\{Z \in Q\colon\ D \subseteq Z\}$ is non-empty. Suppose we have determined $d_0 < \cdots < d_{n-1}$ such that the $\Pi^0_1[A]$ class

$$P_n = \{Z \in Q\colon\ d_0, \ldots, d_{n-1} \in Z\}$$

is non-empty. The set $G = \{m\colon\ \forall Z \in P_n\,[Z(m) = 0]\}$ is c.e. in A uniformly in an index for P_n. We will determine $d_n \notin G$. Since $P_n \subseteq Q$ is nonempty, compute k such that $2^{-k} < \mu P_n$ and hence $|G| \leq k$.

Let $\omega^{<\omega}$ be the set of finite sequences of natural numbers. We denote concatenation of strings by $*$. Let $(S_\sigma)_{\sigma \in \omega^{<\omega}}$ be a uniformly computable sequence of sets such that $S_\varnothing = \mathbb{N}$ and for each σ, $(S_{\sigma * i})_{i \in \mathbb{N}}$ is an infinite partition of S_σ into non-empty sets. Define a Turing functional Ψ as follows. Let $\Psi^A(\sigma) = i$ if i is the first number such that some element of $S_{\sigma * i}$ is enumerated in G. Let α be a computable function such that $J^A(\alpha(\sigma)) \simeq \Psi^A(\sigma)$ for all $\sigma \in \omega^\omega$, where J is the jump functional (i.e., $J^A(e) \simeq \Phi^A_e(e)$, where (Φ_e) is an effective list of all Turing functionals). Since f is d.n.c. relative to A, we have $f(\alpha(\sigma)) \neq \Psi^A(\sigma)$ for each σ.

Now let $\sigma_0 = \varnothing$ and $\sigma_{i+1} = \sigma_i * f(\alpha(\sigma_i))$ for $i < k$. Clearly $G \cap S_{\sigma_k} = \emptyset$ since for each $i < k$ some element of G is in some $S_{\sigma_i * r}$ for $r \neq f(\alpha(\sigma_i))$ (unless already $G \cap S_{\sigma_i} = \emptyset$). Choose $d_n > d_{n-1}$ in S_{σ_k}. Then d_n is as desired, and the sequence (d_i) is computable in Y. So $\bigcap_i P_i$ is a non-empty $\Pi^0_1[Y]$ class of measure 0 and it is contained in Q. Therefore there is a Martin-Löf random set relative to A that is not in $\mathsf{Kurtz}[Y]$. □

Theorem 3.2. *For $A \in 2^\omega$, the following are equivalent:*

(i) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$,

(ii) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$,
(iii) $\emptyset'$ *does not compute a* $\mathsf{DNC}[A]$ *function.*

Proof. (iii)⇒(i) Assume that $\{V_n\}_{n\in\omega}$ is an effective sequence of Σ^0_1 classes such that $\mu(V_n) \to 0$. It suffices to show that $\bigcap_n V_n$ is contained in a Martin-Löf test relative to A. Note that $\emptyset'$ computes a function f such that $\mu(V_{f(k)}) \leq 2^{-k}$, for all $k \in \omega$. For a Σ^0_1 class V and rational $\varepsilon > 0$, let $(V)_\varepsilon$ denote the Σ^0_1 class uniformly obtained by enumerating V as long as the measure does not exceed ε. Since $\emptyset'$ does not compute a $\mathsf{DNC}[A]$ function, there are infinitely many k such that $f(k) = J^A(k)$, where J denotes the jump functional. Therefore, $W_m = \bigcup_{k>m}(V_{J^A(k)})_{2^{-k}}$ covers $\bigcap_{n\in\omega} V_n$, for each m (where $V_{J^A(k)}$ is taken to be empty if $J^A(k)\uparrow$). By definition, $\mu(W_m) \leq 2^{-m}$, so $\{W_m\}_{m\in\omega}$ is a Martin-Löf test relative to A that covers $\bigcap_n V_n$. Hence $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$.

Since every $\Pi^0_1[\emptyset']$ class is a Π^0_2 class, we have (i)⇒(ii). Finally, (ii)⇒(iii) follows by Lemma 3.1 for $Y = \emptyset'$. □

Note that by the Arslanov completeness criterion relative to A, $\emptyset' \oplus A$ computes a function in $\mathsf{DNC}[A]$ iff $\emptyset'$ is Turing complete relative to A, that is, A is GL_1. Thus, by (iii) of Theorem 3.2, if A is not GL_1, then it is in $\mathrm{High}(\mathsf{ML}, \mathsf{W2R})$. For the special case where A is Δ^0_2 we get the following.

Corollary 3.3. *If A is Δ^0_2, the following are equivalent:*
(i) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$,
(ii) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$,
(iii) A *is not low.*

Next, we consider the class $\mathrm{High}(\mathsf{ML}, \mathsf{SR}[\emptyset'])$. Recall that Y is c.e. traceable by A if there is a computable function h such that for each $f \leq_T Y$ there is an A-c.e. trace for f with bound h. The next theorem with $Y = \emptyset'$ characterizes the condition that $A \in \mathrm{High}(\mathsf{ML}, \mathsf{SR}[\emptyset'])$, clause (c) in Table 2. First, we need the following consequence of the Lebesgue density theorem.

Lemma 3.4 ([Nie09], Lemma 8.3.4). *Suppose that $\bigcap_i U_i \subseteq R$ for open sets U_i, R with $\mu(R) < q < 1$. Then there is a string τ and $d \in \mathbb{N}$ such that $\mu_\tau(R) < q$ and $\mu_\tau(U_d - R) = 0$.*

Recall that $\mu_\tau(S)$ is the measure of S relative to $[\tau] = \{X \mid \tau \subset X\}$. That is, $\mu_\tau(S) = \mu([\tau] \cap S)/2^{-|\tau|}$.

Theorem 3.5. *For $A, Y \in 2^\omega$, the following are equivalent:*
(i) $\mathsf{ML}^A \subseteq \mathsf{SR}[Y]$, *and*
(ii) Y *is c.e. traceable by* A.

Proof. For (ii) ⇒ (i) it suffices to show that every Schnorr test relative to Y is contained in a Martin-Löf test relative to A. Let (V_i) be a Schnorr test relative to Y, i.e., a Martin-Löf test relative to Y where the sequence $(\mu(V_i))$ is Y-computable. Without loss of generality we can assume that $\mu(V_n) = 2^{-n-1}$ for each $n \in \mathbb{N}$. Now let (D_i) be an effective sequence of all finite sets. There is a Y-computable function f such that $V_n = \bigcup_i D_{f(n,i)}$ and $\mu(D_{f(n,i)}) \leq 2^{-n-i}$ for all $n, i \in \mathbb{N}$. Now consider a trace of $f(n,i)$ which is computable in A with bound $n + i$. That is, an A-c.e. sequence $(T_{n,i})$ such that $|T_{n,i}| \leq n + i$ and $f(n,i) \in T_{n,i}$ for all $n, i \in \mathbb{N}$.

Without loss of generality we can assume that $T_{n,i}$ only contains numbers j such that $\mu(D_j) \leq 2^{-n-i}$. Define $U_n = \bigcup_i \bigcup_{j \in T_{n,i}} D_j$. Clearly $V_n \subseteq U_n$ for all $n \in \mathbb{N}$. Also,

$$\mu(U_n) \leq \sum_i (n+i) \cdot 2^{-n-i}$$

which means that (U_n) is a Martin-Löf test relative to A (modulo a computable shift of the indices).

For (i) $\Rightarrow$ (ii), suppose $f \leq_T Y$ and we wish to build an A-c.e. trace for f. It suffices to build an A-c.e. trace for $g(n) := nf(n) + n$. Let $B_{k,n}$ be the set of reals that have n consecutive 0s after the kth digit. Clearly, $\mu(B_{k,n}) = 2^{-n}$ for all $k, n \in \mathbb{N}$. It is easy to check that the sets $U_d = \bigcup_{n>d} B_{g(n),n}$ form a Schnorr test relative to Y. Let R be the second member of the universal Martin-Löf test relative to A, so that $\mu(R) < 2^{-2}$. Since $\mathsf{ML}^A \subseteq \mathsf{SR}[Y]$ we have $\bigcap_d U_d \subseteq R$. By Lemma 3.4 there is a string τ and $d \in \mathbb{N}$ such that $\mu_\tau(R) < 2^{-2}$ and $\mu_\tau(B_{g(n),n} - R) = 0$ for all $n > d$. Now let $n\mathbb{N}$ denote the multiples of n and consider the following trace:

$$T_n = \{k \in n\mathbb{N} \mid \mu_\tau(B_{k,n} - R) < 2^{-k-3}\}. \tag{3.1}$$

Since $B_{k,n}$ clopen and R is $\Sigma^0_1[A]$, the sequence (T_n) is uniformly c.e. in A. On the other hand, $g(n) \in T_n$ for all $n > d$, by the choice of d, τ.

It remains to show that the sequence $|T_n|$ is computably bounded. By (3.1) we have $\mu_\tau(\bigcup_{k \in T_n} B_{k,n} - R) < 2^{-2}$, which implies that

$$\mu_\tau(2^\omega - \bigcup_{k \in T_n} B_{k,n}) + \mu_\tau(R) \geq 1 - 2^{-2}.$$

Since $\mu_\tau(R) < 2^{-2}$, this means that $\mu_\tau(2^\omega - \bigcup_{k \in T_n} B_{k,n}) > 2^{-1}$. On the other hand, $\mu_\tau(B_{k,n}) = 2^{-n}$ for $n > |\tau|$. Since T_n consists of multiples of n, the sets $B_{k,n}$ are independent and

$$\mu_\tau(2^\omega - \bigcup_{k \in T_n} B_{k,n}) = (1 - 2^{-n})^{T_n}$$

for $n > |\tau|$. Hence $(1 - 2^{-n})^{|T_n|} > 2^{-1}$ which shows that $|T_n| < 2^n$, for $n > |\tau|$.[4] □

We note that the proof of Theorem 3.5 is an adaptation of the proof of Theorem 8.3.3 in [Nie09].

4. The reducibility associated with weak 2-randomness

We say that a class $\mathcal{C} \subseteq 2^\omega$ is *bounded* if $\mu\mathcal{C} < 1$. Kjos-Hanssen [KH07] proved that the following are equivalent for $X, Y \in 2^\omega$: (a) $X \leq_{LR} Y$; (b) there exists a bounded $\Sigma^0_1[Y]$ class V such that $U^X - V = \emptyset$, where U is a member of a universal oracle Martin-Löf test. Instead of the relevant classes being empty, one can equivalently state that they are null.

Lemma 4.1. *The following are equivalent for $X, Y \in 2^\omega$:*

(a) $X \leq_{LR} Y$,
(b) *There exists a bounded $\Sigma^0_1[Y]$ class V such that $\mu(U^X - V) = 0$, where U is a member of a universal oracle Martin-Löf test.*

[4] This follows from the fact that $(1 - 1/k)^k < e^{-1}$ for any $k \geq 1$.

Proof. We have (a) $\Rightarrow$ (b) from the Theorem of Kjos-Hanssen, so it suffices to show that (b) $\Rightarrow$ (a). Choose a bounded $\Sigma^0_1[Y]$ class V such that $\mu(U^X - V) = 0$, and a rational $q < 1$ such that $\mu(V) < q$. We claim that $\mu_\sigma(V) = 1$ for all σ such that $[\sigma] \subseteq U^X$. Otherwise there is a σ such that $\mu_\sigma(V) < 1$ and $[\sigma] \subseteq U^X$. This implies

$$\mu(U^X - V) \geq \mu([\sigma] - V) > 2^{-|\sigma|}(1 - \mu_\sigma(V)) > 0$$

which contradicts the hypothesis (b). Now if we let

$$F = \{\tau \mid \tau \text{ is minimal such that } \mu_\tau(V) > q\},$$

then we have $U^X \subseteq [F]$ and $[F]$ is a $\Sigma^0_1[Y]$ class. If (ρ_i) is a list of the strings in F, then

$$q \cdot \mu([F]) \leq \sum_i 2^{-|\rho_i|}\mu_{\rho_i}(V) = \mu(V \cap [F]) \leq \mu(V) < q,$$

which implies that $\mu(F) < 1$, proving (a). □

Theorem 4.2. *The following are equivalent for $X, Y \in 2^\omega$:*

(a) $X \leq_{LR} Y$,

(b) *Every weakly 2-random relative to Y is Martin-Löf random relative to X.*

Hence, $\leq_{W2R}$ implies $\leq_{LR}$.

Proof. By definition of $\leq_{LR}$ we have (a) $\Rightarrow$ (b). For (b) $\Rightarrow$ (a) suppose that $X \not\leq_{LR} Y$. We construct a sequence Z that is weakly 2-random relative to Y but not Martin-Löf random relative to X.

Let (U_i) be a universal oracle Martin-Löf test. By Lemma 4.1 we know that for every $\tau \in 2^{<\omega}$, every $\Sigma^0_1[Y]$ class V^Y and every $i \in \mathbb{N}$, if $\mu([\tau] - V^Y) > 0$ then there exists $[\sigma] \subseteq U_i^X$ such that $\tau \subset \sigma$ and $\mu([\sigma] - V^Y) > 0$. Otherwise, $(2^\omega - [\tau]) \cup V^Y$ would satisfy part (b) of Lemma 4.1. Let (S_j^e) be a double sequence of $\Sigma^0_1[Y]$ classes such that $S^e_{j+1} \subseteq S^e_j$ and every $\Pi^0_2[Y]$ class is of the form $\bigcap_j S^e_j$ for some e. We build $Z = \bigcup_s \sigma_s$ and a sequence of open sets (R_s) in stages.

Let $\sigma_0 = \emptyset$ and R_0 be S^0_j for the least j such that $\mu(S^0_j) < 2^{-2}$ if there is such, and $\emptyset$ otherwise. Inductively assume that $\mu([\sigma_s] - R_s) > 0$ and at stage $s+1$ we choose some $\sigma \supset \sigma_s$ such that $\sigma \in U^X_s$ and $\mu([\sigma] - R_s) > 0$. Let $\sigma_{s+1} = \sigma$. Let $q > 0$ be a rational such that $\mu([\sigma_{s+1}] - R_s) > q$ and let $R_{s+1} = R_s \cup S$ where S is S^{s+1}_j for the least j such that $\mu(S^{s+1}_j) < q$ if there is such, and $\emptyset$ otherwise. Notice that $\mu([\sigma_{s+1}] - R_{s+1}) > 0$.

The construction is well defined since R_s is $\Sigma^0_1[Y]$ for all $s \in \mathbb{N}$, so the required string σ will be found at every stage $s+1$. Moreover $[\sigma_s] \not\subseteq R_s$ for all $s \in \mathbb{N}$ and $R_t \subseteq R_s$ for all $t < s$. So $Z = \bigcup_s \sigma_s$ is not in any R_t, which shows that it is not in any null $\Pi^0_2[Y]$ class. On the other hand $Z \in \bigcap_i U^X_i$ so it is not 1-random relative to X. □

Notice that $A \in \mathrm{High}(\mathsf{W2R}, \mathsf{ML}[\emptyset'])$ iff $\emptyset' \leq_{LR} A$ by Theorem 4.2. This yields line (d) in Table 2. By the remarks before Theorem 3.2 we have $\overline{GL_1} \subseteq \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$, while Theorem 4.2 states that $\mathrm{High}(\mathsf{W2R}, \mathsf{ML}[\emptyset']) =$ LR-complete. As explained in the introduction, this gives evidence that the class ML is closer to W2R than W2R is to ML[$\emptyset'$].

The sets that are low for Ω are, by definition, closed downward with respect to $\leq_{LR}$; in other words, they form an initial segment of the LR degrees.

Theorem 4.3. *The relations $\leq_{W2R}$ and $\leq_{LR}$ coincide on the LR initial segment of sets that are low for Ω.*

Proof. Let X, Y be low for Ω reals such that $X \leq_{LR} Y$. In view of Theorem 4.2 it suffices to show that $X \leq_{W2R} Y$. By a theorem in [Mil] we have that $X \leq_T Y'$. But by a theorem of Kjos-Hanssen/Miller/Solomon [KHMSxx] (see 5.6.9. in [Nie09]) we have that if $X \leq_{LR} Y$ and $X \leq_T Y'$ then every $\Pi^0_2[X]$ class is contained in a $\Pi^0_2[Y]$ class of the same measure. This means that every weakly 2-random real relative to Y is also weakly 2-random relative to X, i.e., $X \leq_{W2R} Y$. □

By the theorem of Kjos-Hanssen/Miller/Solomon mentioned in the proof of Theorem 4.3, we also get the following.

Corollary 4.4. *The relations $\leq_{LR}, \leq_{W2R}$ coincide on the class of Δ^0_2 sets.*

In contrast, the relations $\leq_{W2R}$ and $\leq_{LR}$ differ on every LR-lower cone of a non-K-trivial Δ^0_2 set.

Theorem 4.5. *If Y is Δ^0_2 and $Y \not\leq_{LR} \emptyset$, then for all $Z \geq_T \emptyset'$ there exists $X \leq_{LR} Y$ such that $X \oplus \emptyset' \equiv_T Z$.*

Proof. By [Bar] we know that there is a perfect Π^0_1 class P such that $A \leq_{LR} Y$ for all $A \in P$. We use P and a standard coding to define $X \in P$ in stages s by finite extensions σ_s. Let $\sigma_0 = \emptyset$ and if σ_s is defined, find (with oracle $\emptyset'$) the least node $\tau \supset \sigma_s$ such that both $\tau * 0, \tau * 1$ are extendible in P. Then define $\sigma_{s+1} = \tau * Z(s)$. Clearly $Z \equiv_T X \oplus \emptyset'$ and $X \leq_{LR} Y$ since X belongs to P. □

Corollary 4.6. *If Y is Δ^0_2 and $Y \not\leq_{LR} \emptyset$, then there exists $X \leq_{LR} Y$ such that $X \not\leq_{W2R} Y$.*

Proof. Let X be as in Theorem 4.5 for $Z = \emptyset'''$. It suffices to find a set A that is not weakly 2-random relative to X but is weakly 2-random relative to Y. Let A be a 3-random that is recursive in $\emptyset'''$. Since $Y \leq_T \emptyset'$, the set A is (weakly) 2-random relative to Y (i.e., $A \in \mathsf{ML}[Y']$). However,

$$A \leq_T \emptyset''' \leq_T X \oplus \emptyset' \leq_T X'.$$

so A belongs to a null $\Pi^0_2[X]$ class; in fact, $\{A\}$ is $\Pi^0_2[X]$. Hence, A is not weakly 2-random relative to X. □

The following result contrasts with Corollary 4.4. It follows by using lowness in the proof of Corollary 4.6.

Corollary 4.7. *The relations $\leq_{LR}, \leq_{W2R}$ do not coincide on the class of Δ^0_3 sets.*

Proof. Notice that the set X separating $\leq_{LR}$, $\leq_{W2R}$ that was constructed in the proof of Theorem 4.5 is computable in $\emptyset' \oplus Z$. Now in the statement of Corollary 4.6, pick Y such that $Y' \equiv_T \emptyset'$ and $Y \not\leq_{LR} \emptyset$. We modify the proof so that we separate $\leq_{LR}, \leq_{W2R}$ within Δ^0_3. Consider the X given by Theorem 4.5 for $Z = \emptyset''$. Let A be 2-random and computable in $\emptyset''$. Since Y is low, the set A is 2-random relative to Y, i.e., $A \in \mathsf{ML}[Y']$. In particular, it is weakly 2-random relative to Y. However

$$A \leq_T \emptyset'' \leq_T X \oplus \emptyset' \leq_T X'.$$

so A belongs to a null $\Pi^0_2[X]$ class. Hence, it is not weakly 2-random relative to X. Finally, note that Y is Δ^0_2 and $X \leq_T \emptyset' \oplus Z \equiv_T \emptyset''$ is Δ^0_3. □

From [Nie05] (also see [Sim07] for a relevant discussion) we know that if $A \equiv_{LR} B$ then $A' \equiv_{tt} B'$. This, combined with the theorem of Kjos-Hanssen/Miller/Solomon that was mentioned above gives the following.

Corollary 4.8. *For all sets A, B we have $A \equiv_{LR} B$ if and only if $A \equiv_{W2R} B$.*

Hence the equivalence classes induced by $\leq_{W2R}$ coincide with those induced by $\leq_{LR}$, but the ordering of them differs as was shown in Corollary 4.6. Despite the above results, we do not have a characterization of $\leq_{W2R}$. One possibility is given by the observation that if every $\Pi^0_2[A]$ null class is contained in a $\Pi^0_2[B]$ null class, then $A \leq_{W2R} B$. The converse is open:

Question 4.9. Does $A \leq_{W2R} B$ imply that every $\Pi^0_2[A]$ null class is contained in some $\Pi^0_2[B]$ null class?

Kjos-Hanssen/Miller/Solomon [KHMSxx] studied a stronger condition, that every $\Pi^0_2[A]$ class is contained in a $\Pi^0_2[B]$ class of the same measure. They proved that this condition is equivalent to $A \leq_{LR} B$ and $A \leq_T B'$. We can separate this stronger condition from $A \leq_{W2R} B$ by proving that $A \leq_{W2R} B$ does *not* imply $A \leq_T B'$. Recall from [BLS08a] that there are uncountably many sets $\leq_{LR} \emptyset'$. Since every lower Turing cone is countable, $A \leq_{LR} B$ does not imply $A \leq_T B'$. We follow a similar approach for $\leq_{W2R}$.

Theorem 4.10. *The class of sets $\{X\colon\ X \leq_{W2R} \emptyset''\}$ is uncountable.*

Proof. It suffices to build a perfect tree T and a Martin-Löf test (U_i) relative to $\emptyset''$ with the following property: for all $X \in [T]$, every null $\Pi^0_2[X]$ is contained in $\bigcap_i U_i$. Recall that a perfect tree is a function from strings to strings that preserves the prefix and incompatibility relations. Level n of T is the set of strings $T(\sigma)$ such that $|\sigma| = n$. We build T level by level, computably in $\emptyset''$. At stage e we define level e and enumerate into the open sets S_i, $i \leq e$. We ensure that the total measure of S_i is at most 2^{-i}. Our Martin-Löf test relative to $\emptyset''$ will be $U_j := \bigcup_{i>j} S_i$.

Consider a double sequence $(V_{e,j})$ of oracle Σ^0_1 classes such that $V^X_{e,j+1} \subseteq V^X_{e,j}$ for all $e, j \in \mathbb{N}$ and all sets X. Notice that every $\Pi^0_2[X]$ class is of the form $\bigcap_j V^X_{e,j}$ for some $e \in \mathbb{N}$. We refer to the map $X \to \bigcap_j V^X_{e,j}$ as the oracle Π^0_2 class with index e (the eth oracle Π^0_2 class). Level e of T will be devoted to dealing with the eth oracle Π^0_2 class. For each string σ, let T_σ be the full subtree of T above node $T(\sigma)$.[5] We consider a countable set of requirements that are sufficient for the proof. For each $e \in \mathbb{N}$ and each T_σ for σ of length e, we require that one of the following holds:

- for all $X \in [T_\sigma]$ the eth $\Pi^0_2[X]$ class is not null, or
- for some $j \in \mathbb{N}$ and all $X \in [T_\sigma]$ we have $V^X_{e,j} \subseteq S_e$.

To see that this is sufficient, suppose that $X \in [T]$ and let $F = \bigcap_j V^X_{e,j}$ be a null $\Pi^0_2[X]$ class. Then we can show that $F \subseteq S_k$ for infinitely many k. Indeed, let $k_0 \in \mathbb{N}$ be given and let $e > k_0$ be an index of F. Let σ be the string of length e such that $X \in [T_\sigma]$. Since F is null, the construction will ensure that $V^X_{e,j} \subseteq S_e$ for some $j \in \mathbb{N}$ and all $X \in [T_\sigma]$. In particular, $F \subseteq S_e$.

[5] Our trees are 'growing' upward.

The requirements can be written as follows:

$$R_e: \ \forall\sigma\ \forall X\ \Big[|\sigma| = e\ \wedge\ X \in [T_\sigma] \Rightarrow \Big(\mu\Big(\bigcap_j V^X_{e,j}\Big) > 0\ \vee\ \exists j\ V^X_{e,j} \subseteq S_e\Big)\Big].$$

At level/stage e we first define splittings of the strings in the previous level, in order to ensure that T is perfect. After this preliminary step, we make a decision about how to deal with the eth Π^0_2 class (above each string of this level). In particular, for each node $T(\rho)$ on the eth level of T, we check if we can force $\bigcap_j V^X_{e,j}$ to be non-null for all $X \in [T_\rho]$. That is, for an appropriately small value 2^{-t}, we check if for all $\tau \supseteq T(\rho)$ and all $i \in \mathbb{N}$ there exists $\gamma \supseteq \tau$ such that $\mu(V^\gamma_{e,i}) > 2^{-t}$. In that case we let $f(\rho) = 0$ to declare this fact. In later stages we define T above ρ to ensure that $\mu(\bigcap_j V^X_{e,j}) \geq 2^{-t}$ for all $X \in [T_\rho]$.

Otherwise for some $n > 1$ and $\zeta \supseteq T(\rho)$, the oracle class $V_{e,n}$ has the uniform bound 2^{-t} on the measure of $V^X_{e,n}$ for all X extending ζ. To declare this fact, we let $f(\rho) = n$ and move $T(\rho)$ to ζ.[6] By choosing appropriate extensions in later stages, under this hypothesis we will be able to enumerate into S_e all $V^X_{e,n}$ for $X \in T_\rho$ while keeping the measure of S_e small.[7]

To sum up, at stage e the following actions determine level e:

- split the strings of the previous level,
- define extensions of the current paths according to the decisions that have been made in previous stages about R_i, for $i < e$, and
- make a decision about how to satisfy R_e above each node of level e.

Construction. At stage 0 define $T(\emptyset) = \emptyset$ (where $\emptyset$ is the empty sequence here). At stage $e > 0$ we can assume that all previous levels of T have been defined. Given σ of length e we define $T(\sigma)$ in e substages, corresponding to the indices of the first e oracle Π^0_2 classes (starting from index 1). We define $\tau_0, \dots, \tau_{e-1}$ successively, and set $T(\sigma) \supseteq \tau_{e-1}$. Define τ_0 so that incompatibility is met: let i be the last digit of σ and define $\tau_0 := T(\sigma^-) * i$, where σ^- is the predecessor of σ. Now if τ_j, $j < k$ have been defined and $k < e$, let $\rho_k = \sigma \upharpoonright k$. If $f(\rho_k) = 0$, let τ_k be an extension of τ_{k-1} such that $\mu(V^{\tau_k}_{k,e}) > 2^{-2k-1}$. Otherwise let τ_k be an extension of τ_{k-1} such that

$$\mu(V^{\tau}_{k,f(\rho_k)} - V^{\tau_k}_{k,f(\rho_k)}) \leq 2^{-2(e+1)-1} \text{ for all } \tau \supseteq \tau_k. \tag{4.1}$$

When τ_{e-1} is defined, using $\emptyset''$ as an oracle determine if the following is true:

$$\forall i \forall \rho \supseteq \tau_{e-1} \exists \gamma \supseteq \rho\ [\mu(V^\gamma_{e,i}) > 2^{-2e-1}]. \tag{4.2}$$

If (4.2) holds, set $f(\sigma) = 0$ and $T(\sigma) = \tau_{e-1}$. Otherwise choose an $n \in \mathbb{N}$ and $\rho \supseteq \tau_{e-1}$ such that $\mu(V^\gamma_{e,n}) \leq 2^{-2e-1}$ and

$$\mu(V^\gamma_{e,n} - V^\rho_{e,n}) \leq 2^{-2(e+1)-1}, \tag{4.3}$$

for all $\gamma \supseteq \rho$. Let $T(\sigma) = \rho$ and $f(\sigma) = n$.

Finally, for all $k < e$ such that $f(\rho_k) > 0$ enumerate $V^{T(\sigma)}_{k,f(\rho_k)}$ into S_k.

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[6] The final value of $T(\rho)$ is only fixed at the *end* of stage e.

[7] The method in this case is the same as in the proof in [BLS08a] that the class of sets $\leq_{LR} \emptyset'$ is uncountable.

Verification. First we note that the construction is well defined. That is, when the construction defines a string according to (4.1) or (4.3), the search halts. Otherwise, we could inductively push up the measure of $V^{\tau}_{k,f(\rho_k)}$ (or $V^{\gamma}_{e,n}$) as high as we would like, which is impossible.

Second, we show that $\mu(S_e) \leq 2^{-e}$ for all $e \in \mathbb{N}$. Notice that the only 'strategies' that enumerate into S_e are the nodes $T(\rho)$ with $|\rho| = e$ and $f(\rho) > 0$. There are at most 2^e such nodes ρ, so fix one. Let $S_e(\rho)$ be the part of S_e that is enumerated by T_ρ. Consider the full subtree T_ρ of T above $T(\rho)$.

By the construction, $\mu(V^{\tau}_{e,f(\rho)}) \leq 2^{-2e-1}$ for all strings $\tau \in T_\rho$. In particular, $\mu\big(V^{T_\rho(\emptyset)}_{e,f(\rho)}\big) \leq 2^{-2e-1}$. Also, by the way we define T_ρ we have

$$\mu\left(V^{T_\rho(\eta)}_{e,f(\rho)} - V^{T_\rho(\eta^-)}_{e,f(\rho)}\right) \leq 2^{-2(e+|\eta|)-1} \text{ for all } \eta \in 2^{<\omega} \text{ with } |\eta| > 0.$$

Hence,

$$\mu(S_e(\rho)) \leq 2^{-2e-1} + \sum_{i>0} 2^i \cdot 2^{-2(e+i)-1} = 2^{-2e},$$

and so $\mu(S_e) \leq 2^e \cdot 2^{-2e} = 2^{-e}$.

Third, we argue for the satisfaction of R_e. At stage e the construction defines $f(\rho)$ for all strings ρ of length e. Fix such a string ρ. If $f(\rho) = 0$ the subtree T_ρ is defined such that $\mu(V^X_{e,i}) > 2^{-2(e+1)-3}$ for all $X \in [T_\rho]$ and all $i \in \mathbb{N}$. Therefore $\mu(\bigcap_i V^X_{e,i}) > 0$ for all $X \in [T_\rho]$. On the other hand, if $f(\rho) > 0$ the construction enumerates $V^X_{e,f(\rho)}$ into S_e, for all $X \in [T_\rho]$. □

Corollary 4.11. *$A \leq_{W2R} B$ does not imply $A \leq_T B'$.*

5. Weakly 2-random sets and $\leq_{LR}$

Note that $\leq_{LR}$ is a Σ^0_3 relation implied by $\leq_T$. In many ways $\leq_{LR}$ is similar to $\leq_T$ [BLS08a, BLS08b]. In this section we relate $\leq_{LR}$ to weak 2-randomness. Since the LR upper cone above a Δ^0_2 set $X \not\leq_{LR} \emptyset$ is null and Σ^0_3 we have the following.

Proposition 5.1. *If Z is weakly 2-random then every Δ^0_2 set $A \leq_{LR} Z$ is K-trivial.*

Proof. Assume otherwise. Then there is a bounded oracle Σ^0_1 class V such that $U^A \subseteq V^Z$, where U is a member of the universal oracle Martin-Löf test. But then Z is a member of the class

$$\{X \mid U^A \subseteq V^X\} = \bigcap_{n,s_0} \bigcup_{s>s_0} \{X \mid U^{A\upharpoonright n}[s] \subseteq V^X\}$$

which is Π^0_2 and it is null since non-trivial LR upper cones are null by a theorem of Stephan (see [BLS08a]). This is a contradiction. □

The LR lower cone below a Δ^0_2 set is only Σ^0_4 in general, and it turns out to be possible to have a weakly 2-random set LR-below a Δ^0_2 set (see Theorem 5.3). If the oracle (the top of the lower cone) is low then its lower cone has lower complexity, thus allowing us to show the following.

Proposition 5.2. *If $B' \leq_T \emptyset'$ then there is no Z in $W2R$ (and in fact no Z in* $\mathsf{Kurtz}[\emptyset']$*) such that $Z \leq_{LR} B$.*

Proof. If $Z \leq_{LR} B$ then Z belongs to

$$\{X \mid \forall n \exists s\ U^{X\restriction n} \subseteq V^B[s]\} \tag{5.1}$$

for some oracle Σ^0_1 class V (where U is a member of the universal oracle Martin-Löf test) which is a $\Pi^0_1[B']$ class. If $B' \leq_T \emptyset'$ then (5.1) is a $\Pi^0_1[\emptyset']$ class (and so a Π^0_2 class). Also it is null since lower LR cones have measure 0 [BLS08a], so Z cannot be Kurtz random relative to $\emptyset'$ (or weakly 2-random). □

Recently there has been an interest in understanding the class of oracles $\leq_{LR} \emptyset'$, see for example Section 5.6 of [Nie09]. In [BLS08a] it was shown that it is uncountable and in [BLS08b] it was shown that it contains sets of hyperimmune-free Turing degree. In the following we show that it contains a weakly 2-random set. Notice that by definition of $\leq_{LR}$ it does not contain 2-random sets.

Theorem 5.3. *There is a weakly 2-random Z that is K-trivial relative to $\emptyset'$. Thus $Z \oplus \emptyset' \leq_{LR} \emptyset'$. Moreover, Z can be chosen of hyperimmune-free Turing degree.*

Proof. By Nies [Nie09], a set Z is K-trivial relative to $\emptyset'$ iff $Z \oplus \emptyset' \leq_{LR} \emptyset'$. In particular, this notion is closed downward with respect to $\leq_T$. Kučera and Nies (see [Nie09, Exercise 1.8.46 and its solution]) have shown the following. Let P be a non-empty Π^0_1 class. Suppose that $B >_T \emptyset'$ is Σ^0_2. Then there is a set $Z \in P$ of hyperimmune-free Turing degree such that $Z' \leq_T B$.

Now let P be a non-empty Π^0_1 class of ML-randoms. The members of P that form a minimal pair with $\emptyset'$ are weakly 2-random (see [Nie09, Section 5.3]). Let $B >_T \emptyset'$ be a Σ^0_2 set that is K-trivial relative to $\emptyset'$. This exists by a relativization of the well known construction of a non-computable c.e. K-trivial set. By applying the above theorem we get Z is as required. Indeed, since Z is of hyperimmune degree, it forms a minimal pair with $\emptyset'$. Hence it is weakly 2-random. Moreover it is computable from B, therefore it is K-trivial relative to $\emptyset'$. □

Theorem 5.3 does not hold if we replace 'weakly 2-random' with $\mathsf{SR}[\emptyset']$. Indeed, exercise 5.5.10 in [Nie09] shows that no Schnorr random set is K-trivial; the relativization of this argument to $\emptyset'$ shows that no set in $\mathsf{SR}[\emptyset']$ is K-trivial relative to $\emptyset'$. Also, notice that any K-trivial relative to $\emptyset'$ is computable from $\emptyset''$. This follows by relativization of the fact from [Nie09] that every K-trivial is Δ^0_2.

We note that if for some A there is a weakly 2-random $\leq_{LR} A$ this does not necessarily mean that there is a weakly 2-random in the same LR degree as A. For example, Exercise 5.6.22 in [Nie09] shows that the only c.e. LR degree that contains a Martin-Löf random set is the LR degree of $\emptyset'$. Also notice that by Theorem 5.1 there is no weakly 2-random in the LR degree of $\emptyset'$. We do not know whether the property of LR bounding a weakly 2-random is an LR-completeness criterion for Δ^0_2 sets; in other words, if the condition that 'B is low' in Theorem 5.2 can be replaced with 'B is Δ^0_2 and not LR complete'.

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