Eliminating concepts

André Nies
Department of Computer Science
University of Auckland, New Zealand

Four classes of sets have been introduced independently by various researchers: low for K, low for ML-randomness, basis for ML-randomness and K-trivial. They are all equal. This survey serves as an introduction to these coincidence results, obtained in [24] and [10]. The focus is on providing backdoor access to the proofs.

1. Outline of the results

All sets will be subsets of $\mathbb{N}$ unless otherwise stated. $K(x)$ denotes the prefix free complexity of a string x. A set A is K-trivial if, within a constant, each initial segment of A has minimal prefix free complexity. That is, there is $c \in \mathbb{N}$ such that

$$\forall n \; K(A \upharpoonright n) \leq K(0^n) + c.$$

This class was introduced by Chaitin [5] and further studied by Solovay (unpublished). Note that the particular effective epresentation of a number n by a string (unary here) is irrelevant, since up to a constant $K(n)$ is independent from the representation.

A is low for Martin-Löf randomness if each Martin-Löf random set is already Martin-Löf random relative to A. This class was defined in Zambella [28], and studied by Kučera and Terwijn [17].

In this survey we will see that the two classes are equivalent [24]. Further concepts have been introduced: to be a basis for ML-randomness (Kučera [16]), and to be low for K (Muchnik jr, in a seminar at Moscow State, 1999). They will also be eliminated, by showing equivalence with K-triviality. All the equivalent definitions show different aspects of the same notion. In particular, while low for K, low for random and basis for ML-randomness are forms of computational weakness, K-trivial intuitively means being far from random.

Solovay (1975) proved the existence of a non-computable K-trivial. Kučera and Muchnik each showed the existence of a non-computable set in the class introduced. For the class of low for ML-random sets, existence was only shown in 1997 [17]. All examples were c.e., except for Solovay's example of a K-trivial, which was only Δ^0_2. Later this was improved to a c.e. example by Kummer (unpublished), and Calude & Coles [3].

The main purpose of this paper is to survey the coincidence results obtained in [24] and [10] and to present the proof ideas in an accessible way. However, in Subsection 3.2 we provide some new facts about the cost function construction of a K-trivial set. We also include a sketch of a proof that each K-trivial is low via the golden run method, which is simplest application of this method. Facts quoted without reference can be found in [8], or in my forthcoming book [20]. However, for the ease of the reader we recall some facts here. Throughout, "Martin-Löf" will be abbreviated by "ML". Schnorr's Theorem states that Z is ML-random iff for some c, $\forall n\, K(Z \upharpoonright n) \geq n - c$. Thus Z is ML-random if for each n, $K(Z \upharpoonright n)$ is near its maximal value $n + K(0^n)$. To say that A is K-trivial means that A is far from ML-random, because $K(A \upharpoonright n)$ is minimal (all up to constants).

An example of a ML-random set is Chaitin's halting probability,

$$\Omega = \textstyle\sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is the reference universal prefix free machine.

If x, y are expressions, then $x \leq^+ y$ denotes that $x \leq y + c$ for a constant c independent of the values of x and y.

2. Computational weakness

2.1. *Three classes*

First we will discuss the low for K sets, the low for ML-randomness sets, and the bases for ML-randomness.

In general, adding an oracle A to the computational power of the universal machine decreases $K(y)$. A is low for K if this is not so. In other words,

$$\forall y\, K(y) \leq^+ K^A(y).$$

Let $\mathcal{M}$ denote this class, introduced by Andrej A. Muchnik in 1999, who proved that there is a c.e. noncomputable $A \in \mathcal{M}$. We defer a proof of existence till later. Here is a useful observation.

Proposition 2.1: *If A is low for K, then A is GL_1, namely, $A' \leq_T A \oplus \emptyset'$.*

Proof: If t is least such that $e \in A'_t$, then $K^A(t) \leq^+ 2\log e$. To see this, recall that the number e has the prefix free code $0^{|\sigma|}1\sigma$ where σ is the string representing e in binary. Consider the prefix free machine with oracle A that, on input a prefix free code for e searches for t such that $J^A(e)$ converges at stage t. This machine shows that $K^A(t) \leq^+ 2\log e$. Also $K(t) \leq^+ K^A(t)$ by hypothesis on A, so $K(t) \leq 2\log e + c$ for some constant c. Now $\emptyset'$ can compute $s = \max\{U(\sigma) : |\sigma| \leq 2\log e + c\}$, where U is the reference universal prefix free machine. Then $e \in A' \Leftrightarrow e \in A'_s$. $\square$

Let MLR denote the class of Martin-Löf-random sets. Because an oracle A increases the power of tests, $\mathsf{MLR}^A \subseteq \mathsf{MLR}$. In general one would expect this inclusion to be proper. Zambella [28] defined A to be low for ML-randomness if

$$\mathsf{MLR}^A = \mathsf{MLR}. \tag{2.1}$$

In 1997, Kucera and Terwijn proved that there is a non-computable c.e. set that is low for ML-randomness [17]. To see that low for K implies low for ML-randomness, first note that Schnorr's Theorem relativizes: Z is Martin-Löf random relative to A iff for some c, $\forall n\ K^A(Z \upharpoonright n) \geq n - c$. Now, since MLR can be defined in terms of K, and MLR^A in terms of K^A, low for K implies low for ML-randomness. Thus, the existence of a non-computable c.e. set that is low for ML-randomness also follows from Muchnik's result.

Kučera [16] introduced a further concept expressing computational weakness. He studied sets A such that

$$A \leq_T Z \text{ for some } Z \in \mathsf{MLR}^A.$$

That is, A can be computed from a set that is ML-random relative to A. While Kučera used the term "basis for 1-RRA", we will call such a set a basis for ML-randomness. There is no connection to basis theorems.

If A is low for ML-randomness then A is a basis for ML-randomness. For, by the Kučera-Gács Theorem there is a ML-random Z such that $A \leq_T Z$. Then Z is ML-random relative to A.

2.2. *The existence and equivalence theorems*

In the following We will discuss two theorems:
Theorem 2.2: There is a c.e. non-computable basis for ML-randomness [16].
Theorem 2.5: Each basis for ML-randomness is low for K [10].

Now two concepts are gone. For, we have already obtained the easy inclusions

$$\text{low for } K \Rightarrow \text{low for ML-randomness} \Rightarrow \text{basis for ML-randomness.}$$

Then, by the second Theorem, all three classes are the same. In particular, the Theorems together imply the result of Muchnik that there is a non-computable c.e. low for K set.

How about the fourth concept, K-triviality? While the implication "low for $K \Rightarrow K$-trivial" is immediate, the converse, "K-trivial $\Rightarrow$ low for K", is hard. The proof is carried out separately from all of the above and will be discussed in Section 3.

Chaitin [5] proved that each K-trivial set is Δ^0_2, by an elegant short argument involving the coding theorem. Thus a set that is low for ML-randomness is Δ^0_2. This answers an open question of Kučera and Terwijn [17]. I first gave a direct proof of this [21], introducing techniques which I later extended in order to prove Theorem 2.6 below.

We now proceed to the first Theorem.

Theorem 2.2: ***Kučera*** [16]**.** *There is a c.e. non-computable set A that is a basis for ML-randomness.*

The proof sketched here differs a bit from Kučera's original one. One combines the following two results. The first comes from Kučera's priority free solution to Post's problem. A function $f : \mathbb{N} \mapsto \mathbb{N}$ is called *diagonally non-computable* if $\forall e \, \neg \, f(e) = \Phi_e(e)$.

Theorem 2.3: ***Kučera*** [15]**.** *Let Z be Δ^0_2 and diagonally non-computable. Then there is a simple set $A \leq_T Z$.*

Each ML-random set is diagonally non-computable. So we may apply Theorem 2.3 to a low ML-random set Z (say), and then use the following lemma of Hirschfeldt, Nies and Stephan in order to obtain a simple basis for ML-randomness.

Lemma 2.4: [10]**.** *If $Z <_T \emptyset'$ is ML-random and $A \leq_T Z$ is c.e., then Z is already ML-random relative to A.*

To prove the Lemma, one argues that one can turn a ML-test relative to A which Z fails into a plain ML-test. This uses the incompleteness of Z.

Theorem 2.3 is easier to prove under the stronger hypothesis that Z is ML-random, and this is the only case we need here. Under this stronger hypothesis, it can be proved without using the Recursion Theorem. In fact, Kučera had first thought of this special case, and only later he generalized it to diagonally non-computable Z, where the Recursion Theorem is needed. We sketch the proof of Theorem 2.3 for a ML-random set Z.

Proof: A *Solovay test* $\mathcal{G}$ is given by an effective enumeration of strings $\sigma_0, \sigma_1, \ldots$, such that $\sum_i 2^{-|\sigma_i|} < \infty$. It is not hard to see that Z is ML-random iff for each Solovay test $\mathcal{G} = \sigma_0, \sigma_1, \ldots$, for almost all i, $\sigma_i \not\preceq Z$.

We will enumerate A and a Solovay test $\mathcal{G}$. To make A simple, we meet the requirements

$$S_e: \ |W_e| = \infty \Rightarrow A \cap W_e \neq \emptyset.$$

Construction. At stage $s > 0$, if S_e is not satisfied yet, see if there is an x, $2e \leq x < s$, such that

$$x \in W_{e,s} - W_{e,s-1} \ \ \& \ \ \forall t_{x<t<s}\, Z_t \upharpoonright e = Z_s \upharpoonright e.$$

If so, put x into A. Put the string $\sigma = Z_s \upharpoonright e$ into $\mathcal{G}$. Declare S_e satisfied.

Clearly A is simple (in fact, A can even be made promptly simple). Also, $\mathcal{G}$ is a Solovay test since the requirement S_e contributes at most 2^{-e} to $\mathcal{G}$.

To see $A \leq_T Z$, choose s_0 such that $\sigma \not\preceq Z$ for any σ enumerated into $\mathcal{G}$ after stage s_0. Given an input $x \geq s_0$, using Z, compute $t > x$ such that $Z_t \upharpoonright x = Z \upharpoonright x$. Then $x \in A \Leftrightarrow x \in A_t$, for if we put x into A at a stage $s > t$ for the sake of S_e, then $e < x$, so we also put σ into $\mathcal{G}$ where $\sigma = Z_s \upharpoonright e = Z \upharpoonright e$. This contradicts the fact that $\sigma \not\preceq Z$. $\square$

Note that the enumeration into A is heavily restrained. If $Z \upharpoonright e$ changes another time at stage s, then no $x < s$ can be enumerated after s for the sake of S_e. Z can restrict S_e in this way as late and as often as it wants.

Now we discuss the second result, which we call the hungry sets theorem.

Theorem 2.5: ***Hirschfeldt, Nies, Stephan*** [10]**.** *If A is a basis for ML-randomness, then A is low for K.*

We actually obtained the conclusion that A is K-trivial, by a very similar proof. A full proof of the present version is in [20].

The Kraft-Chaitin Theorem. We use the KC-Theorem as a tool. A c.e. set $W \subseteq \mathbb{N} \times 2^{<\omega}$ is a Kraft-Chaitin set (KC set) if

$$\sum_{\langle r,y\rangle \in W} 2^{-r} \leq 1.$$

(Note that some values of r can occur several times in this sum.) The KC-Theorem states that, from a Kraft-Chaitin set L, one can effectively obtain a prefix free machine M such that

$$\forall r, y[\langle r, y\rangle \in L \Leftrightarrow \exists w\ (|w| = r \ \&\ M(w) = y)].$$

Thus, we enumerate requests $\langle r, y\rangle$ ("give a description of y that has length r"). The weight of this request is 2^{-r}. If their total weight is at most 1, then each request will be fulfilled: there actually is an M-description of length r. A critical point in proofs applying the KC-theorem is to verify that the set of requests is in fact KC, namely the total weight is at most 1. We will now outline proof of the hungry sets theorem, and reveal the reason for its culinary name. To ensure the sets enumerated are KC-sets, we use the method of accounting.

Proof: Given a Turing functional Φ, we define a ML-test $(V_d^X)_{d\in\mathbb{N}^+}$ relative to oracle X (later, we use this test for $X = A$). Suppose $A = \Phi^Z$. The goal is this: if $Z \notin V_d^A$ then A is low for K, with constant $d + \mathcal{O}(1)$.

To realise this goal, we also build a uniformly c.e. sequence $(L_d)_{d\in\mathbb{N}^+}$ of KC sets. For each computation of the universal prefix free machine $\mathbf{U}$,

$$\mathbf{U}^\eta(\sigma) = y \text{ where } \eta \preceq A,$$

(that is, whenever y has a description σ with oracle A), we want to ensure there is a description without an oracle that is only by a constant longer. Thus we want to put a request

$$\langle |\sigma| + d + 1, y\rangle$$

into L_d. The problem is that we don't know A, so we don't know which η's to take. So L_d could fail to be KC. To avoid this, the description $\mathbf{U}^\eta(\sigma) = y$ first has to prove itself worthy.

Recall that we are building an auxiliary ML-test (V_d) relative to A. If $Z \notin V_d$ then L_d works. We effectively enumerate open sets $C^\eta_{d,\sigma}$ and let

$$V_d^A = \bigcup_{\eta\prec A} C^\eta_{d,\sigma}.$$

While $\mu(C^\eta_{d,\sigma}) < 2^{-|\sigma|-d}$, $C^\eta_{d,\sigma}$ is hungry. We feed it with fresh oracle strings α, where $\eta \prec \Phi^\alpha$.

All the open sets $[C^\eta_{d,\sigma}]^\preceq$ are disjoint. When $\mu(C^\eta_{d,\sigma})$ exceeds $2^{-|\sigma|-d-1}$, we put the request $\langle|\sigma|+d+1, y\rangle$ into L_d. We can account the weight of those requests against the measure of the sets $C^\eta_{d,\sigma}$, since the measure of $C^\eta_{d,\sigma}$ is greater than the weight of the request. This shows that each L_d is a KC set. Because Z is ML-random relative to A, there is some d such that whenever $\mathbf{U}^\eta(\sigma) = y$ in the relevant case that $\eta \preceq A$, the request $\langle|\sigma|+d+1, y\rangle$ we are after is enumerated into L_d. Thus the following fact (named to honor the Brazilian president Lula) can be verified.

Fome Zero Lemma. *Suppose $Z \notin V_d$. Then for each description $\mathbf{U}^\eta(\sigma) = y$, where $\eta \prec A$, $\mu(C^\eta_{d,\sigma}) = 2^{-|\sigma|-d}$. In other words, $\mu(C^\eta_{d,\sigma})$ is hungry no more at the end of time.* □

Discussion. For each partial computable functional Φ, let $S^\Phi_A = \{Z : A = \Phi^Z\}$. For each $n > 0$, let $S^\Phi_{A,n} = [\{\sigma : A \restriction n = \Phi^\sigma\}]^\preceq$. Then $S^\Phi_{A,n}$ is open and c.e. relative to A, uniformly in n. Moreover, $S^\Phi_A = \bigcap_n S^\Phi_{A,n}$. Thus S^Φ_A is a Π^0_2 class relative to A, and $\mu S^\Phi_A = 0$ is equivalent to $\lim_n \mu S^\Phi_{A,n} = 0$.

Let us compare a few facts related to this.

- If A is non-computable, then $\mu S_A = 0$ [26].
- If A is ML-random, then (after leaving out the first few components), $(S^\Phi_{A,n})_{n\in\mathbb{N}}$ is a ML-test relative to A, a fact from [19]. In other words, there is c such that $\forall n\, \mu S_{A,n} \leq 2^{-n+c}$.
- By the proof of the hungry sets theorem, based on Φ, one can build an oracle ML-test (V_d) such that, whenever A is not low for K, then $S_A \subseteq \bigcap_d V^A_d$. Then, since there is a universal ML test, the whole class $\{Z : A \leq_T Z\}$ is ML-null relative to A. (The converse holds as well: if A is low for K then $\Omega \geq_T A$ is ML-random relative to A, so the class is not ML-null relative to A.)

These facts suggest that, the more random A is, the fewer sets compute it, where "fewer" is taken in the sense of how effective the null set S^Φ_A is. The only case where S^Φ_A is merely a Π^0_2 null set is when A is low for K, or equivalently, K-trivial.

2.3. *Lowness for other randomness notions*

Next, we digress a bit in order to study lowness for randomness notions implied by ML-randomness (for more details, see [8]). The definition of those classes is the exact analog of (2.1). Each computable set is low for the randomness notion in question. The question is whether there are others, and if so, how to characterize the class. Unexpected things happen.

For my purposes, a *martingale* is a function $M : \{0,1\}^* \mapsto \mathbb{Q}_0^+$ such that

$$M(x0) + M(x1) = 2M(x)$$

(here $\mathbb{Q}_0^+$ is the set of non-negative rationals). M *succeeds* on Z if $\limsup_n M(Z \upharpoonright n) = \infty$, and the class of the sets where M succeeds is denoted $\mathsf{Success}(M)$. Z is *computably random* if no computable martingale M succeeds on Z. That is, $M(Z \upharpoonright n)$ is bounded.

While a martingale always bets on the next position, a *non-monotonic betting strategy* can choose some position that has not been visited yet. Z is *Kolmogorov-Loveland random* (KLRand) if not even a non-monotonic betting strategy can succeed on Z. It is easy to verify that

$$\mathsf{MLR} \subseteq \mathsf{KLR} \subset \mathsf{CR},$$

where MLR, KLR and CR denote the classes of Martin-Löf-random, KL-random and computably random sets, respectively. Whether the first inclusion is proper is a major open problem [18] .

Let us discuss the associated lowness notions. Recall that each low for ML-randomness set is a basis for ML-randomness, hence low for ML-randomness implies low for K by Theorem 2.5. First we consider a strengthening of this result, which actually is the version in which it first appeared.

Theorem 2.6: [24]**.** *If* $\mathsf{MLR} \subseteq \mathsf{CR}^A$ *then* A *is low for* K*.*

The converse implication is easy, since low for K implies low for ML-randomness by Schnorr's Theorem, as discussed above.

We sketch the proof of Theorem 2.6.

Proof: Let R be any c.e. open set such that $\mu R < 1$ and $\mathsf{Non\text{-}MLRand} \subseteq R$, for instance, $R = \{Z : \exists n K(Z \upharpoonright n) \leq n - 1\}$. We will define a Turing functional L such that L^A is a martingale. If $\mathsf{MLR} \subseteq \mathrm{CRand}^A$ then $\mathsf{Success}(L^A) \subseteq \mathsf{Non\text{-}MLRand}$, and the following lemma applies to $N = L^A$. It says that there is a basic open cylinder $[v] \not\subseteq R$ such that, for each x extending v, if $N(x)$ is large, then x is not too random as a string because x is in R.

Lemma 2.7: *Let* N *be any martingale such that* $\mathsf{Success}(N) \subseteq \mathsf{Non\text{-}MLR}$*. Then there are* $v \in 2^{<\omega}$ *and* $m \in \mathbb{N}$ *such that* $[v] \not\subseteq R$*, and*

$$\forall x \succeq v \, [N(x) \geq 2^m \Rightarrow x \in R]. \tag{2.2}$$

To prove the Theorem, let us at first assume we know the witnesses v, m in the Lemma. Thus

$$\forall x \succeq v \, [L^A(x) \geq 2^m \Rightarrow x \in R].$$

The proof parallels the proof of the hungry sets theorem. (The present argument actually was given first, in [24].) Once again, when we see a description $\mathbf{U}^\eta(\sigma) = y$ where $\eta \preceq A$, we want a corresponding set C^η_σ such that $\mu C^\eta_\sigma \geq 2^{-|\sigma|-c}$, c some constant (there is no analog of the parameter d here). While $\mu C^\eta_\sigma < 2^{-|\sigma|-c}$, the set is hungry. The sets for different descriptions have to be disjoint. We feed a set C^η_σ in small servings, as follows: at stage s, pick a clopen set D, $\mu D = \epsilon$ of long strings $x \succeq v$, $D \cap R_s = \emptyset$. Here ϵ is an appropriate small quantity. Define $L^\eta(x) \geq 2^m$ for each $x \in D$ and put D into C^η_σ. If $\eta \prec A$, then D will go into R eventually. Once this happens, repeat with a new set, but again of measure ϵ, as long as C^η_σ is hungry. The fact that the old set D has to enter R before we pick a new one enables us to stuff the right sets up to the desired measure, while the ones where $\eta \not\prec A$ only get a small serving outside of R. To make the sets C^η_σ disjoint, we simply ensure the different portions outside R they are fed with are disjoint.

Actually, we do not know the right witnesses v, m. But it is enough to let L be an infinite weighted sum, over all possible witnesses, of the martingales obtained for those witnesses. See [24] for the rather tricky details. $\square$

Next we consider the class of low for KL-random sets If A is low for KLRand, then

$$\mathsf{MLR} \subseteq \mathsf{KLR} = \mathsf{KLR}^A \subseteq \mathsf{CR}^A.$$

Therefore, the following is a consequence of Theorem 2.6.

Corollary 2.8: *Each low for KL-random set is low for K.*

As one would expect, it is an open problem whether the two classes are the same.

Next we show that the only low for computably random sets are the computable ones [24]. An earlier result in this direction was obtained in joint work with Benjamin Bedregal, then at UFRN, Natal, Brazil.

Theorem 2.9: [2]. *Each low for computably random set A is of hyperimmune free degree.*

But also, by Theorem 2.6 each low for computably random set is K-trivial, and hence Δ^0_2. Since the only hyper-immune free Δ^0_2 sets are the computable ones, we have

Theorem 2.10: [24]. *Each low for computably random set A is computable.*

This answers Question 4.8 in Ambos-Spies and Kucera [1] in the negative. It was conjectured this way by R. Downey.

An *order function* is a non-decreasing unbounded computable function. A notion weaker still than computable randomness is the following. Z is *Schnorr random* (SRand) if no computable martingale M succeeds fast on Z, in the sense that there is an order function h (for instance, $h(n) = \lfloor \log n \rfloor$) such that $M(Z \upharpoonright n) \geq h(n)$ for infinitely many n. Equivalently, Z passes each Schnorr test, namely each ML-test $(V_n)_{n\in\mathbb{N}}$ such that $\mu(V_n) = 2^{-n}$. Just as in the case of lowness for ML-randomness, the associated lowness notion can be characterized in a combinatorial way. A is *computably traceable* if the value $f(x)$ of each $f \leq_T A$ is in a small effectively given set $D_{g(x)}$: g is a computable function depending on f, and $|D_{g(x)}| \leq h(x)$ for an order function h not depending on g. Each computably traceable set is hyper-immune free.

A is low for Schnorr tests if for each Schnorr test $(V_n)_{n\in\mathbb{N}}$ relative to A, there is a Schnorr test $(S_n)_{n\in\mathbb{N}}$ such that $\bigcap_n V_n \subseteq \bigcap_m S_m$. Clearly each set that is low for Schnorr tests is low for Schnorr randomness. Terwijn and Zambella [27] proved that A is low for Schnorr *tests* iff A is computably traceable. They asked if this is also the same as being low for Schnorr randomness. In fact a stronger result holds.

Theorem 2.11: [14]. *The following are equivalent.*

(i) Each computably random set is Schnorr random relative to A
(ii) A is computably traceable.

One key ingredient is Theorem 2.9, which persists when one weakens the hypothesis to: each computably random set is Schnorr random relative to A.

3. Far from random

3.1. *Brief introduction to K-triviality.*

For a string y, up to constants, $K(|y|) \leq K(y)$, since one can compute $|y|$ from y (where $|y|$ is represented in binary). A set A is K-trivial if, for some

$b \in \mathbb{N}$

$$\forall n\ K(A \upharpoonright n) \leq K(0^n) + b,$$

namely, the K complexity of all initial segments is minimal up to a constant. This notion is opposite to ML-randomness: Schnorr's Theorem (see [8]) says that Z is ML-random iff $\exists b \forall n\, K(Z \upharpoonright n) \geq n - b$. Thus Z is ML-random if all the complexities $K(Z \upharpoonright n)$ are near the upper bound $n + K(n)$, while Z is K-trivial if they have the minimal possible value $K(n)$ (all within constants). If one defines K-triviality using the plain Kolmogorov complexity C instead of K, then one obtains nothing beyond the computable sets [4]. However, Chaitin still managed to prove that the K-trivial sets are Δ^0_2 [5]. As mentioned in the introduction, Solovay (unpublished, 1975) constructed a non-computable K-trivial set A , which was Δ^0_2, as expected, but not c.e.

3.2. *Constructions*

In [9] a short "definition" of a promptly simple K-trivial set is given, which had been anticipated by various researchers (for instance Kummer and Zambella) and is similar to the earlier construction of a non-computable c.e. low for ML-randomness set [17]. We meet the prompt simplicity requirements

$$S_e\colon\ |W_e| = \infty \Rightarrow \exists s \exists x\ [x \in W_{e,s} - W_{e,s-1}\ \&\ x \in A_s].$$

The key ingredient is the "cost function"

$$c(x,s) = \sum_{x<y\leq s} 2^{-K_s(y)}.$$

The c.e. set A is given by letting $A_0 = \emptyset$ and, for $s > 0$,

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	we haven't met e-th prompt simplicity requirement
$x \in W_{e,s} - W_{e,s-1}$	we can meet it, via x
$x \geq 2e$	to make A co-infinite
$c(x,s) \leq 2^{-e}\}$	to ensure A is K-trivial.

To see that each S_e is met, note that

$$\forall e \exists y \forall s > y\, [c(y,s) < 2^{-e}]. \tag{3.1}$$

So if $x \geq y$ enters W_e at a stage $s > y$ then x can be enumerated into A.

The K-triviality of A is shown by enumerating a KC-set L such that $\langle K(n)+2, A \upharpoonright n\rangle \in L$ for each n. By convention, let $K_0(y) = \infty$ for each y. When $r = K_s(y) < K_{s-1}(y)$ we enumerate a request $\langle r+2, A_s \upharpoonright y\rangle$ into L. The total weight of those requests is $\leq \Omega/4$. When x enters A to meet S_e, then all the initial segments of A from $x+1$ on change. So for each y such that $x < y < s$, we enumerate a request $\langle K_s(y)+2, A_s \upharpoonright y\rangle$. The weight added to L is $c(x,s)/4$. Since each S_e is active at most once, the total weight added in this way is at most $(\sum_e 2^{-e})/4 = 1/2$.

Reverse computability theory. Recall the fragments of Peano arithmetic $I\Sigma_1$ (induction over Σ_1 formulas) and $B\Sigma_1$ (for each Σ_1 function f and each x, $f([0,x])$ is bounded). For each $\mathcal{M} \models I\Sigma_1$, there is a promptly simple K-trivial set in $\mathcal{M}$ (Hirschfeldt and Nies). It suffices to verify (3.1) in $\mathcal{M}$. Suppose it fails for $e \in \mathcal{M}$. Consider the Σ_1 formula $\phi(m,e)$ given by

$$\exists u\, [|u| = m \;\&\; \forall i\, (0 \leq i < m \;\Rightarrow\; c(u_i, u_{i+1}) > 2^{-e})].$$

By $I\Sigma_1$ and the failure of (3.1) for e, $\mathcal{M} \models \forall m\, \phi(m,e)$. Now let $m = 2^e+1$ and $u \in M$ be the witness for m. Then, in $\mathcal{M}$,

$$\Omega \geq \sum_{0 \leq i \leq 2^e} c(u_i, u_{i+1}) \geq (2^e+1)2^{-e} > 1,$$

contradiction.

On the other hand, $B\Sigma_1$ is not sufficient to verify the construction, because of work of Chong and Slaman. Let $\mathcal{M} \models I\Delta_1$. $A \subseteq \mathcal{M}$ is *regular* if for each $n \in \mathcal{M}$, $A \upharpoonright n$ is a string of $\mathcal{M}$ (i.e., encoded by an element of $\mathcal{M}$). Each K-trivial set $A \subseteq \mathcal{M}$ is regular, since $A \upharpoonright n$ has a prefix free description in $\mathcal{M}$, for each n. There is a saturated $\mathcal{M} \models B\Sigma_1$ with a Σ_1 cofinal f whose domain is the standard part. In such an $\mathcal{M}$, each regular c.e. set A is computable.

Necessity of the cost function method, c.e. case. Suppose the c.e. set A is K-trivial via a constant b. Then one can think of A as being built by the cost function construction, when restricting to an appropriate computable set of stages $\{s_i : i \in \mathbb{N}\}$. For each s, one can effectively determine an $f(s) > s$ such that $\forall n < s\; K(A \upharpoonright n) \leq K(n) + b\; [f(s)]$. Let $s(0) = 0$ and

$$s(i+1) = f(s(i)). \tag{3.2}$$

Proposition 3.1: *Let A be c.e. and K-trivial via b. Then*

$$\sum\{c(x, s(i)) : x < s(i) \textit{ is minimal s.t. } A_{s(i)}(x) \neq A_{s(i+1)}(x)\} \leq 2^b.$$

Proof: Let $x_i < s(i)$ be minimal such that $A_{s(i)}(x) \neq A_{s(i+1)}(x)$. For each y, $x_i < y \leq s(i)$, by definition there is at stage $s(i+1)$ a prefix free description of $A_{s(i+1)} \upharpoonright y$ of length $\leq K_{s(i+1)}(y) + b$. Let D_i be the open set generated by such descriptions of a $A_{s(i+1)} \upharpoonright y$, $x_i < y \leq s_i$. Since A is c.e., the strings $A \upharpoonright y$ described at different stages $s(i+1)$ are distinct, so that $D_i \cap D_j = \emptyset$ for $i \neq j$. Hence $\sum_i \mu D_i \leq 1$.

Since

$$c(x, s_i) = \sum_{x_i < y \leq s(i)} 2^{-K_{s(i)}(y)} \leq \sum_{x_i < y \leq s(i)} 2^{-K_{s(i+1)}(y)} \leq 2^b \mu D_i,$$

this shows $\sum_i c(x_i, s(i)) \leq 2^b$, as required. □

Using deeper methods, Proposition 3.1 can be extended to all K-trivial sets. See Subsection 3.4.

We have seen two constructions of a non-computable c.e. K trivial set A. Both are injury free.

(i) Take an ML-random $Z <_T \emptyset'$, and build $A \leq_T Z$ using Kučera's method in Theorem 2.3. Then A is a basis for ML-randomness, hence low for K, and hence K-trivial.

(ii) The cost function construction.

By the extended form of Proposition 3.1, each K-trivial set can be thought of as being obtained via a cost function construction. It is an open question whether each K-trivial set can be obtained via (i):

Question 3.2: *If A is K-trivial, is there a ML-random set $Z <_T \emptyset'$ such that Z is Turing above A?*

In Subsection 3.4. we will see that each K trivial set is Turing below a c.e. K-trivial set. So there is no need to require that the given K-trivial set A is c.e. See [18] for more details.

3.3. *A is K-trivial iff A is low for K*

We will get there in small steps. First we consider the fact that each K-trivial is wtt-incomplete. Showing the downward closure of the K-trivials under $\leq_{wtt}$ is easy: Suppose $B = \Gamma^A$, where Γ is a wtt reduction procedure with a computable bound f on the use. Then, for each n, within constants,

$$K(A \upharpoonright n) \leq K(A \upharpoonright f(n)) \leq K(f(n)) \leq K(n).$$

Now, since the *wtt*-complete set Ω is ML-random and hence not K–trivial, no K–trivial set A satisfies $\emptyset' \leq_{wtt} A$. To introduce some new techniques, we give a direct proof of wtt-incompleteness.

Suppose that $\emptyset' \leq_{wtt} A$ for a K-trivial A. We build an c.e. set B, and by the Recursion Theorem we can assume we are given a total *wtt*-reduction Γ such that $B = \Gamma^A$, whose use is bounded by a computable function g.

We also build a KC-set L. Thus we enumerate requests $\langle r, n\rangle$ and have to ensure the total weight is at most 1. By the Recursion Theorem, we may assume the coding constant d for L is given in advance. Then, putting $\langle r, n\rangle$ into L causes $K(n) \leq r + d$ and hence $K(A \upharpoonright n) \leq r + b + d$, where b is the triviality constant. (In fact we apply the Double Recursion Theorem.)

Let

$$\mathbf{k} = 2^{b+d+1}$$

Let $n = g(\mathbf{k})$ (the use bound). We wait till $\Gamma^A(\mathbf{k})$ converges, and put the single request $\langle r, n\rangle$ into L, where $r = 1$. Our total cost is $1/2$.

Each time the opponent (named Otto here) has a prefix free description of $A \upharpoonright n$ of length $\leq r + b + d$, we force $A \upharpoonright n$ to change, by putting into B the largest number $\leq \mathbf{k}$ which is not yet in B. If we reach $\mathbf{k} + 1$ such changes, then his total cost is

$$(\mathbf{k} + 1)2^{-(b+d+1)} > 1,$$

contradiction.

Turing-incompleteness. Consider the more general result that each K-trivial set is T-incomplete [9]. There is no recursive bound on the use of $\Gamma^A(\mathbf{k})$. The problem now is that Otto might, before giving a description of $A_s \upharpoonright n$, move this use beyond n, thereby depriving us of the possibility to cause further changes of $A \upharpoonright n$. The solution is to carry out many attempts in parallel, based on computations $\Gamma^A(m)$ for different m. Each time the use of such a computation changes, the attempt is cancelled. What we placed in L for this attempt now becomes garbage. We have to ensure that the weight of the garbage does not build up too much, otherwise L is not a KC set.

More details: j-sets The following is a way to keep track of the number of times Otto had to give new descriptions of strings $A_s \upharpoonright n$. We only consider the stages $s(0) < s(1) < s(2) < \ldots$ where A looks K-trivial with constant b, defined as in 3.2. We write *stage* (in italics) when we mean a stage of this type.

At *stage* t, a finite set E is a j-set if for each $n \in E$

- first we put a request $\langle r_n, n\rangle$ into L
- and then j times at *stages* $s < t$ Otto had to give new descriptions of $A_s \restriction n$ of length $r_n + b + d$.

A c.e. set with an enumeration $E = \bigcup E_t$ is a j-set if E_t is a j-set at each *stage* t.

For $E \subseteq \mathbb{N}$, the weight is defined by $wt(E) = \sum\{2^{-r_n} : n \in E\}$. The weight of a $\mathbf{k}$-set is at most $1/2$.

Lemma 3.3: *If the c.e. set E is a $\mathbf{k}$-set, $\mathbf{k} = 2^{b+d+1}$ as defined above, then $wt(E) \leq 1/2$.*

This is so because $\mathbf{k}$ times Otto has to match our description of n, which has length r_n, by a description of a string $A_s \restriction n$ that is at most $b + d$ longer.

Procedures. Assume A is K-trivial and Turing complete. As in the case of wtt-incompleteness, we attempt to build a $\mathbf{k}$-set $F_{\mathbf{k}}$ of weight $> 1/2$ and reach a contradiction.

The procedure P_j $(2 \leq j \leq \mathbf{k})$ enumerates a j-set F_j. The construction begins calling $P_{\mathbf{k}}$, which calls $P_{\mathbf{k}-1}$ many times, and so on down to P_2, which enumerates L (and F_2).

Each procedure P_j is called with rational parameters $q, \beta \in [0, 1]$. The *goal* q is the weight it wants F_j to reach. When the procedure reaches its goal it returns. The *garbage quota* β is how much garbage it is allowed to produce.

Decanter model. We visualize this construction by a machine similar to Lerman's pinball machine. However, since we enumerate rational quantities instead of single objects, we replace the balls there by amounts of a precious liquid, 1955 Biondi-Santi Brunello wine.

Our machine consists of decanters $F_k, F_{k-1}, \ldots, F_0$. At any *stage* F_j is a j set. F_{j-1} can be emptied into F_j.

The procedure $P_j(q, \beta)$, $2 \leq j \leq \mathbf{k}$, wants to add a weight of q to F_j. In the beginning it picks a new number m targeted for B. It fills F_{j-1} up to q and then returns, by emptying it into F_j. All numbers put into F_{j-1} are above $\gamma^A(m)$. The emptying is done by enumerating m into B and hence causing another A-change.

The emptying device for F_{j-1} is the $\gamma^A(m)$-marker. It is depicted as a hook, which besides being used once on purpose may go off finitely often by itself (this is the visualization of a premature A-change). When F_{j-1} is emptied into F_j then $F_{j-2}, \ldots, F_0$ are spilled on the floor.

Though the recursion starts by calling P_k with goal 1, wine is first poured into the highest decanter F_0, and thereby into the left domain of L. We want to ensure that at least half the wine we put into F_0 reaches F_k. Recall that the parameter β is the amount of garbage $P_j(q, \beta)$ allows. If v is 1+the number of times the emptying device $\gamma^A(m)$ has gone off by itself, then P_j lets P_{j-1} fill F_{j-1} in portions of $2^{-v}\beta$ (ie it calls P_{j-1} with goal $2^{-v}\beta$ as often as necessary). Then, when F_{j-1} is emptied into F_j, at most $2^{-v}\beta$ can be lost because of being in higher decanters $F_{j-2}, \ldots, F_0$. Alltogether the garbage due to $P_j(q, \beta)$ is at most $\beta \sum_{v \geq 1} 2^{-v} = \beta$.

Let us stress this key idea: when we have to cancel a run $P_j(q, \beta)$ because of a premature A-change, what becomes garbage is *not* F_{j-1}, but rather what the sub-procedures called by this run were working on. The set F_{j-1} already is a $j-1$-set, so all we need is another A-change, which is provided here by the cancellation itself, as opposed to being caused actively once the run reaches its goal.

Who enumerates L? The bottom procedures $P_2(q, \beta)$, which is where the recursion reaches ground. It puts requests $\langle r_n, n\rangle$ into L and the top decanter F_0, where n is large and $2^{-r_n} = 2^{-v}\beta$ for v as above. Once it sees the corresponding $A \upharpoonright n$ description, it empties F_0 into F_1. However, if the hook $\gamma^A(m)$ belonging to P_2 moves before that, then F_0 is spilled on the floor, while F_1 is emptied into F_2.

So much for the discussion of Turing incompleteness. Next, we improve this to lowness.

Theorem 3.4: *Each K-trivial set is low.*

Proof: Let $J^A(e)$ denote $\Phi_e^A(e)$. A procedure $P_j(q, \beta)$ is started when $J^A(e)$ newly converges. The goal q is $\alpha 2^{-e}$, where α is the garbage quota of the procedure of type P_{j+1} that called it (assuming $j < k$).

For different e they run in parallel, so we now have a tree of decanters. This could be avoided by letting a new convergence $J^A(e')$, $e' < e$, cancel a run for $J^A(e)$.

We cannot change A actively any more (and we are happy if it doesn't). However, this creates a new type of garbage, where $P_j(q, \beta)$ reaches its goal, but no A change happens after that would allow us to empty F_{j-1} into F_j. In this case no more procedures are started because of a new convergence of $J^A(e)$. So the total weight of garbage of this type is $\leq \sum_e 2^{-e}\alpha$, which can be tolerated.

The man with the golden run. The initial procedure $P_{\mathbf{k}}$ never returns, since it has goal 1, while a $\mathbf{k}$-set has weight at most 1/2 by Lemma

3.3. So there must be a golden run of a procedure $P_{j+1}(p, \alpha)$: it doesn't reach its goal, but all the subprocedures it calls either reach their goals or are cancelled by premature A changes. The golden run shows that A is low: When the run of the subprocedure P_j based on a computation $J^A(e)$ returns, then we guess that $J^A(e)$ converges. If A changes below the use of $J^A(e)$, then P_{j+1} receives the fixed quantity $2^{-e}\alpha$. In this case we change the guess back to "divergent". This can only happen r times where $r = 2^e p/\alpha$, else P_{j+1} reaches its goal. (Note that α is chosen of the form 2^{-l}.) □

This proof actually shows that A is super-low: the number of changes in the approximation of A' is computably bounded. The lowness index is not obtained uniformly: we needed to know which run is golden. This non-uniformity is necessary [9, 23].

We are now ready for the full result.

Theorem 3.5: *A is K-trivial iff A is low for K.*

This was obtained joint with Hirschfeldt, via a modification of my result that the K-trivial sets are closed downward under $\leq_T$. It implies lowness, as we have seen an easy proof that each low for K set is GL_1, and each K-trivial set is Δ^0_2.

Proof: A procedure $P_j(q, \beta)$ is started when $U^A(\sigma) = y$ newly converges. The goal q is $\alpha 2^{-|\sigma|}$. Thus the construction is similar to the one in the proof of Theorem 3.4, but it is now necessary to call procedures based on different inputs σ in parallel. So we necessarily have a tree of decanters. At the golden run node $P_{j+1}(p, \alpha)$, we can show that A is low for K, by emulating the cost function construction of a low for K set. When $P_j(q, \beta)$ associated with $U^A(\sigma) = y$ returns, we have the right to put a request $\langle |\sigma| + c, y\rangle$ into a set W (where $c = 1 + \log_2(p/\alpha)$). An A change has a cost, since we put a request for a wrong computation. The fact that P_{j+1} does not reach its goal implies that the cost is bounded. Hence W is a KC-set□

3.4. *Further applications of the golden run method.*

Here are two further applications.

1. The cost function construction is necessary even for K-trivial Δ^0_2 sets A . This can be used to show that there is a c.e. K-trivial set Turing above A [24].

2. A real number r is left-c.e. if $\{q \in \mathbb{Q} : q < r\}$ is c.e. For each K-trivial set A, the relativized Chaitin probability Ω^A is left-c.e. [7]. (The converse

also holds here, in case that A is Δ^0_2: if Ω^A is left-c.e. then it is Turing complete, so A is a basis for ML-randomness, hence K-trivial.)

4. Effective descriptive set theory.

Π^1_1 sets of numbers are a high-level analog of the c.e. sets, where the steps of an effective enumeration are recursive ordinals. For details see [25]. Hjorth and Nies [12] have studied the analogs of K and of ML-randomness based on Π^1_1-sets. The analog of K in the Π^1_1 setting is denoted $\widetilde{K}$. The analogs of the KC-theorem and Schnorr's Theorem hold, but the proofs take considerable extra effort due to the extra complication of limit stages. There is a Π^1_1-set of numbers which is $\widetilde{K}$-trivial and not hyperarithmetical. In contrast,

Theorem 4.1: *If A is low for Π^1_1-ML-randomness, then A is hyperarithmetical.*

So K-trivial and low for ML-randomness differ in the Π^1_1-setting.

Proof: First we show that $\omega_1^A = \omega_1^{CK}$. This is used to prove that A is in fact $\widetilde{K}$-trivial at some $\eta < \omega_1^{CK}$, namely

$$\forall n\ \widetilde{K}_\eta(A \upharpoonright n) \leq \widetilde{K}_\eta(n) + b.$$

Then A is hyperarithmetical, by the same argument Chaitin used to show that K-trivial sets are Δ^0_2: The collection of Z which are $\widetilde{K}$-trivial at η form a hyperarithmetical tree of width $O(2^b)$. $\square$

5. Subclasses.

Next we look at subclasses of the K-trivial sets, downward closed under Turing reducibility, which may be proper. We will mostly restrict ourselves to the c.e. K-trivial sets, which is a minor restriction here since each K-trivial is Turing below a c.e. one, see subsection 3.4.

5.1. *ML-coverable and ML-noncuppable sets*

We have already seen a subclass of the c.e. K-trivials that is downward closed: the c.e. sets A such that there a ML-random set $Z <_T \emptyset'$ Turing above A. Let us call a c.e. set of that kind *ML-coverable*. In Question 3.2 we ask if each (c.e.) K-trivial set is ML-coverable.

A Δ^0_2 set A is *ML-cuppable* if

$$A \oplus Z \equiv_T \emptyset' \text{ for some ML-random } Z <_T \emptyset'.$$

Here is a further subclass: the c.e. sets that are not ML-cuppable (this recent development was initiated by Kučera in 2004).

Many sets *are* ML-cuppable: If A is not K-trivial, then $A \not\leq_T \Omega^A$ by the hungry sets theorem 2.5, and $A' \equiv_T \Omega^A \oplus A \geq_T \emptyset'$. If A is also low, then $Z = \Omega^A <_T \emptyset'$, so A is ML-cuppable. This shows for instance that each c.e. non-K-trivial set B is ML-cuppable, since one can split it into low c.e. sets, $B = A_0 \cup A_1$, and one of them is also not K-trivial. So the ML-noncuppable c.e. sets are K-trivial.

Theorem 5.1: [22]**.** *There is a promptly simple set which is not ML-cuppable.*

The proof combines cost functions with the priority method.

Question 5.2: *Can a K-trivial set be ML-cuppable?*

The cost functions in the proof Theorem 5.1 are much more restricting then the one used to characterize the K-trivial sets. This gives some weak evidence that the question has an affirmative answer. The same applies to the Kučera construction of a simple A below a Δ^0_2 ML-random $Z \upharpoonright e$ related to Question 3.2. Recall here that, if $Z \upharpoonright e$ changes another time at s, then $[0, s)$ becomes taboo for S_e. This can happen as often and as late as the computable approximation of Z determines.

5.2. *A common subclass*

By recent work of Hirschfeldt and Nies, and later Miller, there is natural class $\mathfrak{L}$ which is a subclass of both the ML-coverable and the ML-noncuppable sets. $\mathfrak{L}$ determines an ideal in the c.e. Turing degrees. The following notion is the key. B is *almost complete* if $\emptyset'$ is K-trivial relative to B. That is, there is $c \in \mathbb{N}$ such that

$$\forall n \; K^B(\emptyset' \upharpoonright n) \leq K^B(0^n) + c$$

Such a set is high, in fact $\emptyset'' \leq_{tt} B'$. One can obtain incomplete almost complete sets via Jockusch-Shore pseudojump inversion.

Theorem 5.3: [13]**.** *For each c.e. operator W there is a c.e. set C such that*

$$W^C \oplus C \equiv_T \emptyset'.$$

I observed in [24] that pseudojump inversion, applied to the c.e. operator W given by the cost function construction, yields an incomplete but almost complete c.e. set. A recent result shows that pseudojumps can also be inverted via ML-random sets.

Theorem 5.4: ***Nies 2006, see*** [20]**.** *The conclusion of the Theorem 5.3 also holds for "C ML-random".*

The proof, which was simplified with the help of J. Miller, combines the techniques to prove the Low Basis Theorem with the methods used in the proof of Theorem 5.3.

Corollary 5.5: *There is a ML-random almost complete* Δ^0_2*-set.*

Now let

$$\begin{aligned}\mathfrak{L} = \{A :\ & A \text{ is c.e. } \& \ \forall Z \\ & Z \text{ ML-random, almost complete } \Rightarrow\ A \leq_T Z\}.\end{aligned}$$

Clearly, $\mathfrak{L}$ determines an ideal in the Turing degrees. Hirschfeldt proved that there is a promptly simple set in $\mathfrak{L}$. By the previous corollary, each $A \in \mathfrak{L}$ is ML-coverable, and hence K-trivial.

Surprisingly, $\mathfrak{L}$ also is a subclass of the ML-noncuppable sets. The reason for this inclusion is that each potential Δ^0_2 ML-random cupping partner Z of a K-trivial A is almost complete. The proof, due to Hirschfeldt (Dec. 2005), involves a relativization of the van Lambalgen Theorem: for any sets Z, B and A such that $Z \in \mathsf{MLR}^A$, we have $B \oplus Z \in \mathsf{MLR}^A$ iff $B \in \mathsf{MLR}^{Z \oplus A}$. Now argue as follows. For each set B,

$$\begin{aligned} B \in \mathsf{MLR}^Z &\Rightarrow B \oplus Z \in \mathsf{MLR} \\ &\Rightarrow B \oplus Z \in \mathsf{MLR}^A \\ &\Rightarrow B \in \mathsf{MLR}^{Z \oplus A} \\ &\Rightarrow B \in \mathsf{MLR}^{\emptyset'}. \end{aligned}$$

Thus $\emptyset'$ is low for ML-randomness relative to Z, and hence $\emptyset'$ is K-trivial relative to Z (using that $Z \leq_T \emptyset'$).

Unless $\mathfrak{L}$ coincides with the class of c.e. K-trivial sets, $\mathfrak{L}$ is an ideal of the type we were looking for. The proof that there is a promptly simple set in $\mathfrak{L}$ later was both simplified and generalized.

Theorem 5.6: ***Hirschfeldt, Miller 2006.*** *Let $\mathcal{C}$ be a Σ^0_3 null class. Then there is a promptly simple A such that $A \le_T Z$ for each ML-random $Z \in \mathcal{C}$.*

Proof: The proof uses a cost function argument. First suppose that $\mathcal{C}$ is a Π^0_2-class. Then $\mathcal{C} = \bigcap_x V_x$ for an effective sequence $(V_x)_{x\in\mathbb{N}}$ of Σ^0_1-classes such that $V_x \supseteq V_{x+1}$ for each x. Let $(V_{x,s})_{s\in\mathbb{N}}$ be an effective ascending sequence of clopen sets approximating V_x, and let $c(x,s) = \mu V_{x,s}$. Then (3.1) holds because $\lim_x \mu V_x = 0$. Now run the cost function construction from subsection 3.2, using this new definition of $c(x,s)$, and obtain a promptly simple set A. When x enters A at stage s, enumerate $V_{x,s}$ into a Solovay test $\mathcal{G}$ (that is, put σ into $\mathcal{G}$, for all σ of length s such that $[\sigma] \subseteq V_{x,s}$). The construction ensures that $\mathcal{G}$ is indeed a Solovay test, by the definition of $c(x,s)$. To see that $A \le_T Z$ for any ML-random $Z \in \mathcal{C}$, choose s_0 such that $\sigma \not\preceq Z$ for any σ enumerated into $\mathcal{G}$ after stage s_0. Given an input $x \ge s_0$, using Z, compute $t > x$ such that $Z \in V_{x,t}$. Then $x \in A \;\Leftrightarrow\; x \in A_t$, for if we put x into A at a later stage, this would throw Z out of $V_{x,t}$.

If $\mathcal{C}$ is a Σ^0_3-class, we extend the argument slightly: we have $\mathcal{C} = \bigcup_i \bigcap_x (V^i_x)_{i,x\in\mathbb{N}}$, for an effective double sequence $(V^i_x)_{i,x\in\mathbb{N}}$ of Σ^0_1-classes such that $V^i_x \supseteq V^i_{x+1}$ for each i,x. Run the cost function construction from subsection 3.2, now based on the function

$$c(x,s) = \textstyle\sum_i 2^{-i}\mu V^i_{x,s}.$$

To show (3.1) for this new cost function, note that, given k, there is x_0 such that

$$\forall i \le k+1 \;\forall x \ge x_0 \;\mu V^i_x \le 2^{-k-1}.$$

Then $c(x,s) \le 2^{-k}$ for all $x \ge x_0$ and all s, since the total contribution of terms $2^{-i}\mu V^i_{x,s}$ for $i \ge k+2$ to $c(x,s)$ is bounded by 2^{-k-1}.

For each i, build a Solovay test $\mathcal{G}_i$, by enumerating $V^i_{x,s}$ into $\mathcal{G}_i$ when x enters A at stage s. The sum of all $2^{-|\sigma|}$, for strings σ enumerated into $\mathcal{G}_i$, is bounded by 2^{i+1}. If $Z \in \mathcal{C}$ is ML-random, then choose i such that $Z \in \bigcap_x V^i_x$, and argue as before that $A \le_T Z$ using $\mathcal{G}_i$. □

To obtain a promptly simple set in $\mathcal{L}$, it is now sufficient to observe that the Σ^0_3-class of almost complete sets is a null class (for instance, because the larger class of high degrees has measure 0, but one can also give a direct proof).

If $\mathcal{C} = \{Z\}$, for a ML-random Δ^0_2 set Z. Then $\mathcal{C}$ is a Π^0_2 class: let Z_x be the string of length x approximating Z at stage x, and let $V_{x,s} =$

$\bigcup_{x<y<s}[Z_y \upharpoonright m_y]$, where m_y is least such that $Z_y(m_y) \neq Z_{y-1}(m_y)$, so that $\mathcal{C} = \bigcap_x V_x$. Now the proof turns into the proof of Kučera's Theorem 2.3, for the special case that Z is a ML-random set, discussed in subsection 2.2.

5.3. *Is there a characterization of K-triviality independent of randomness and K?*

Figueira, Nies and Stephan [11] have tried the following strengthening of super-lowness:

For each unbounded nondescending computable function h, A' has an approximation that changes at most $h(x)$ times at x. They build a c.e. noncomputable such set, via a construction that resembles the cost function construction. No relationship to K-triviality is known.

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