Asymptotics of multivariable generating functions

Based on notes for a course at Stanford University, Fall 1999

last updated December 2, 2002

Robin Pemantle

Dept. of Math, Ohio State University

231 W. 18th Avenue

Columbus, OH 43210

www.math.ohio-state.edu/˜pemantle

Contents

without loss of generality“tf@toc=“write3

0 Intro

Begin with an array of numbers indexed by d-tuples of nonnegative integers:

$$\{a_{r_1,\ldots,r_d} : r_1, \ldots, r_d \geq 0\}.$$

These numbers arise from some application in combinatorics or probability theory or queuing theory, etc., and we wish to know something about them. In particular, we want asymptotic information: $a_{\mathbf{r}} = (1 + o(1))H(\mathbf{r})$ as $\mathbf{r} \to \infty$ in some prescribed way, and where H is a function we understand. (The boldface notation is for vectors.) The way in which $\mathbf{r} \to \infty$ will depend on the application, as will the range of $\mathbf{r}$ for which the approximation is uniform or even valid. Sometimes we may need more precise information, such as an asymptotic series, and sometimes we might need less, for example $\lim_{t\to\infty} r^{-1} \log a_{\lfloor t\mathbf{s} \rfloor}$ might suffice.

There are many methods of obtaining asymptotics, but let us focus here on the generating function method. Define

$$F(z_1, \ldots, z_d) := \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} := \sum a_{r_1,\ldots,r_d} z_1^{r_1} \cdots z_d^{r_d}. \tag{0.1}$$

The function F exists as a formal power series, and possibly as a convergent power series in a neighborhood of the origin. In cases where the numbers $a_{\mathbf{r}}$ are determined recursively, it is often far easier to compute the function F than it is to find general formulae for the numbers $a_{\mathbf{r}}$. One must then translate one's information about F into asymptotic information about $\{a_{\mathbf{r}}\}$.

There are as many ways to do this as there are interesting generating functions. Most methods use complex contour integration, together with a somewhat *ad hoc* set of tools for making the integrals tractable. The focus of these lectures is two-fold: first to present some new techniques that work well for certain classes of multivariable generating functions, and second to prove some classification results that change the *ad hoc* art of asymptotics into a science, at least for one nontrivial class of functions.

My personal history with the subject is perhaps relevant here. I began with a concrete problem to which I attempted to apply generating function methods. After doing a fair bit of work to obtain the two-variable generating function, I then went to the literature to "look up" what to do to get out the asymptotics. Surprisingly, I didn't find what I needed. After several iterations of reducing the problem to a toy problem, I started to believe that perhaps it was not known how to extract asymptotics even for the simplest imaginable class, namely rational functions. Part of my surprise at how little was known stemmed from how completely trivial it is to obtain asymptotics for rational functions of one variable.

Working out the asymptotics for this example via contour methods was not too hard and did not much improve on what was known by other means. But soon I had another example for which basic qualitative behavior was not known by any means, and to which the same contour techniques could be applied with a little more work. I became interested in finding a general theory which would guarantee that these methods would always produce the correct leading term asymptotic at least for the class of examples I was interested in, namely meromorphic non-entire functions.

The upshot of all this, to date, is (1) an integration technique useful mainly in cases where the singularities of F are poles (recall we are working over several complex variables, so a pole is a complex analytic variety on which the modulus of F goes to infinity); (2) a theorem proving that this technique always works for two-variable meromorphic functions with nonnegative coefficients; (3) good progress toward such a theorem in any number of variables; and (4) an idea for how to proceed when the coefficients are not assumed nonnegative, and a conjecture that the method still always works.

An outline of the remaining lectures is as follows. The course will begin with a brief introduction to formal power series, together with some detailed examples of multivariable generating functions and how they are obtained. Next will be a review of the main techniques in asymptotics of one variable generating functions. From this it will be apparent that oscillating integrals play a major role in the derivation of asymptotics. The theory of oscillating integrals will first be developed in one variable, following several sources including chiefly Stein (1993). We then go on to outline results in several variables, where the proofs are similar but the intuition is the same. Technical results needed for later will be stated and maybe proved. More details will be provided later but an outline of the method is as follows.

(1) Use the multidimensional Cauchy integral formula to represent $a_{\mathbf{r}}$ as an integral over a d-dimensional torus inside $\mathbf{C}^d$.

(2) Expand the surface of integration across a point $\mathbf{z}$ where F is singular, and use the residue theorem to represent $a_{\mathbf{r}}$ as a $(d-1)$-dimensional integral of one-variable residues. The choice of $\mathbf{z}$ determines the directions in which asymptotics may be computed.

(3) Put this in the form of an integral $\int \exp(\lambda f(\mathbf{z}))\psi(\mathbf{z})\, d\mathbf{z}$ for which the large-λ asymptotics can be read off from the theory of oscillating integrals.

These methods will then be applied to derive asymptotics in various cases of interest, and the results compared with those obtainable by previously known methods. An example of the type of result we will derive is the following.

Theorem ??: *Let $F = G/H$ be a meromorphic function of two variables, not singular at the origin. Define*

$$Q(z,w) := -w^2 H_w^2 z H_z - w H_w z^2 H_z^2 - w^2 z^2 \left(H_w^2 H_{zz} + H_z^2 H_{ww} - 2 H_z H_w H_{zw} \right).$$

Then

$$a_{r,s} \sim \frac{G(z,w)}{\sqrt{2\pi}} z^{-r} w^{-s} \sqrt{\frac{-wH_w}{sQ}}$$

uniformly as (z,w) varies over a compact set of strictly minimal, simple poles of F on which Q and G are nonvanishing, and $(r,s) \in \mathbf{dir}(z,w)$.

After working out a number of examples, I will prove the following classification theorem for two-variable, meromorphic non-entire functions.

Theorem ??: *Let $F = G/H = \sum a_{r,s} z^r w^s$ be the quotient of analytic functions $G, H : \mathbf{C}^2 \to \mathbf{C}$. Suppose that the coefficients $a_{r,s}$ are all nonnegative, and that $F(z, 0)$ and $F(0, w)$ are not entire. Then for every direction $\alpha \in \mathbf{RP}^1$ there is a minimal $\mathbf{z} \in \mathcal{V}$ with $\alpha \in \mathbf{dir}(\mathbf{z})$.*

Extension to more than two variables leads to topological and algebraic-geometric questions, which we will discuss as time permits.

The last part of the course (time permitting again) will be on generating functions that are difficult to obtain, and the various means used to obtain them.

1 Generating functions

Let $\mathbf{C}[[z_1, \ldots, z_d]]$ denote the ring of formal power series in the variables $z_1, \ldots, z_d$. Elements of $\mathbf{C}[[z_1, \ldots, z_d]]$ are parameterized by maps f from d-tuples of nonnegative integers to $\mathbf{C}$ via the correspondence $f \mapsto F := \sum_{\mathbf{r}} f(\mathbf{r})\mathbf{z}^{\mathbf{r}}$. Addition is defined by $(a + b)(\mathbf{r}) = a(\mathbf{r}) + b(\mathbf{r})$ and multiplication is defined by convolution: $a \cdot b(\mathbf{r}) = \sum_{\mathbf{s}} a(\mathbf{s})b(\mathbf{r} - \mathbf{s})$. The sum in this convolution is always finite, so there is no question of convergence. **Exercise: show that $f \in \mathbf{C}[[z_1, \ldots, z_d]]$ is a unit (has a multiplicative inverse) if and only if f has a nonzero constant term. Thus $\mathbf{C}[[z_1, \ldots, z_d]]$ has a unique maximal ideal.**

Let $\mathcal{N}$ be an open polydisk containing the origin in $\mathbf{C}^d$, that is a set $\{\mathbf{z} : |z_i| < t_i, 1 \le i \le d\}$. Suppose that $f, g \in \mathbf{C}[[z_1, \ldots, z_d]]$ are absolutely convergent on $\mathcal{N}$, that is, $\sum_{\mathbf{r}} |f(\mathbf{r})||w_1|^{n_1} \cdots |w_d|^{r_d} < \infty$ when all $|w_i| < t_i$. Then $f + g$ and fg are absolutely convergent on $\mathcal{N}$ as well and the sum and product in the ring of formal power series is the same as in the ring of analytic functions in $\mathcal{N}$. Note that some formal power series fail to converge anywhere (except at the origin) and that for these it will not work to apply analytic methods. One can however make a generating function by letting $F(\mathbf{x}) = \sum_{\mathbf{r}} f(\mathbf{r}\mathbf{z}^{\mathbf{r}}/g(\mathbf{r})$ for a judiciously chosen g. A good choice is often to let $g(\mathbf{r})$ be a product of some or all of the quantities $r_i!$; generating functions normalized by factorials are often called *exponential generating functions.*

The interior of the domain D on which the formal power series F converges is the union of open polydisks. In particular it is the union of tori, and is hence characterized by its intersection $D_{\mathbf{R}}$ with $\mathbf{R}^d$. The set D is in fact pseudoconvex, meaning that the set $D_{\log}$ defined by $(x_1, \ldots, x_d) \in D_{\log}$ iff $(e^{x_1}, \ldots, e^{x_d}) \in D_{\mathbf{R}}$ is a convex order ideal[1]. See Hörmander (1990, Section 2.5) for these and other basic facts about functions of several complex variables.

Some elementary examples will make these formalities clear to those without experience using generating functions.

Example 1 (Fibonacci)

Let $d = 1$ and let a_r be the r^{th} Fibonacci number. Let $F(z) = \sum_{r=0}^{\infty} a_r z^r$ be the generating function for the Fibonacci numbers. We compute $(1-z-z^2)F$. Since $(1-z-z^2)F$ has r^{th} coefficient equal to $a_r - a_{r-1} - a_{r-2}$ (where coefficients with negative indices are zero), it follows that $(1 - z - z^2)F = 1$, whence $F = 1/(1 - z - z^2)$.

In order to pave the way for a more complicated example involving polyominos (Example **??** below), I will complete the asymptotic computation from the generating function. We use a partial fraction expansion to express $F(z)$ as

$$\frac{1}{1 - \lambda_1 z} + \frac{1}{1 - \lambda_2 z}$$

where λ_j are the inverse roots of $H := 1 - z - z^2 = 0$. (The 1's in the numerator happen to be particularly simple.) The n^{th} term is therefore given by summing the n^{th} terms of the series for $1/(1 - \lambda_j z)$, yielding $\lambda_1^n + \lambda_2^n$. Since $\lambda_j = (1 \pm \sqrt{5})/2$, we arrive at

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

the first term giving a simple and very accurate asymptotic.

[1] An order ideal is a set closed under $\leq$ in the coordinatewise partial order on $\mathbf{R}^d$.

This example is a prototype of the way sequences determined by recurrences have easily computed generating functions. As most of you probably know, a sequence satisfying any finite linear recurrence $a_r = \sum_{j=1}^k c_j a_{r-j}$ will have the rational generating function $F(z) = p(z)/(1 - \sum_{j=1}^k c_j z^j)$ where $p(z)$ is a polynomial whose coefficients are determined by the (finitely many) small values r for which $a_r \neq \sum_{j=1}^r c_j a_{r-j}$ (this set necessarily includes the index of the least nonvanishing coefficient, hence $p(z) \neq 0$).

Example 2 (Binomial coefficients)

Let $d = 2$ and let $a_{r,s}$ be the binomial coefficient $a_{r,s} := \binom{r+s}{r,s}$. Since the $\{a_{r,s}\}$ satisfy $a_{r,s} - a_{r-1,s} - a_{r,s-1} = \delta_{0,0}$, we see that $(1 - z - w)F = 1$ and $F = 1/(1 - z - w)$. This converges in the pseudo-convex domain $\{(z, w) : |z| + |w| < 1\}$. Simply identifying the domain of convergence will turn out to give asymptotics for $a_{r,s}$ that are of the correct exponential order.

This example too generalizes to any linear recursion in more than one variable where the term $a_{r_1,\ldots,r_d}$ is expressed as a linear combination of terms $a_{s_1,\ldots,s_d}$ with $\mathbf{s} \preceq \mathbf{r}$. When the linear recursion has no maximal term, we will see the story is quite different.

Rational generating functions come from many places other than linear recursions. Conversely, in higher dimensions, not all linear recursions lead to rational generating functions. The next two examples illustrate these two possibilities.

Example 3 (Coalescing particles)

Suppose we begin with particles at sites $-1, -2, \ldots, -r$. At each unit of time, one of the particles is chosen uniformly at random and it moves a unit to the right. When a particle moves onto another particle, the two coalesce into a single particle. When a particle hits 0, it disappears. How many time units does it take before all particles disappear? This problem, which arises in the analysis of a sorting algorithm, is discussed in Larsen and Lyons (1999).

To solve this we let X_k be the number of moves made by particle k (and any particles it coalesces with that were originally to its left) before it coalesces with a particle to its right. We wish to know the distribution of $\sum_{k=1}^r X_k$. The precise distribution is not known, but we compute its mean. Let $a_{r,k}$ denote the expected number of steps that a particle at site $-r-k$ will make before coalescing with a particle at position $-r$ or disappearing. Then $a_{0,k} = k$, $a_{r,0} = 0$, and for $r, k \geq 1$,

$$a_{r,k} = \frac{1}{2} + \frac{1}{2} a_{r,k-1} + \frac{1}{2} a_{r-1,k+1}.$$

Letting $F(z,w) = \sum_{r,k \geq 0} a_{r,k} z^r w^k$, we see that

$$(w - \frac{1}{2}w^2 - \frac{1}{2}z)F = \frac{w}{2(1-z)(1-w)} + H(z,w) \tag{1.1}$$

where H is the sum of terms $(a_{r,k} - (1/2)a_{r-1,k+1} - (1/2)a_{r,k-1} - \frac{1}{2})z^r w^{k+1}$ with $rk = 0$, i.e., where (**??**) fails. When $r = 0$ and $k > 0$,

$$a_{r,k} - \frac{1}{2}a_{r-1,k+1} - \frac{1}{2}a_{r,k-1} - \frac{1}{2} = k - \frac{k-1}{2} - \frac{1}{2} = \frac{k}{2}.$$

This corresponds to a contribution of $w/(2(1-w)^2)$ to H. Unfortunately, when $k = 0$,

$$a_{r,k} - \frac{1}{2}a_{r-1,k+1} - \frac{1}{2}a_{r,k-1} - \frac{1}{2} = -\frac{1}{2}a_{n-1,1} - \frac{1}{2}$$

and we don't know what this is. Thus we may write

$$(w - \frac{1}{2}w^2 - \frac{1}{2}z)F = \frac{w}{2(1-z)(1-w)} + \frac{w}{2(1-w)^2} + wh(z) \tag{1.2}$$

where h is an unknown function of one variable.

The key now is to use the fact that $a_{r,k}$ is obviously at most $r + k$. In particular, it grows slower than an exponential, so F converges in a neighborhood of $(0,0)$ (in fact in the unit polydisk). Thus the LHS of (**??**) is zero on the set $V := \{(x,y) : w^2 - 2w + z = 0\}$ intersected with the unit polydisk. The RHS must vanish there as well. Near $(0,0)$, V is parameterized by $w = \phi(z) := 1 - \sqrt{1-z}$, where the negative square root is chosen to make w near zero when z is. Thus

$$h(z) = \frac{-1}{2\phi(z)(1-w)^2} - \frac{1}{2(1-z)(1-\phi(z))}$$

and F is gotten by substituting this into (**??**).

At the end of the course is planned a section that generalizes this result by prescribing how to solve arbitrary linear recursions with the sort of boundary conditions given here; these will in general be algebraic functions but not rational functions. Some of the asymptotic methods we develop will be useful for functions other than meromorphic functions, and in particular, asymptotics of the function $F(z, w)$ above are easily obtained from these methods or from older results of Bender (1983). The *tour de force* among derivations of generating functions for linear recursions is, I think, the following set of examples from queuing theory due to Flatto et al, which I will describe only briefly at this point, and examine further if time permits.

Example 4 (Queues)

Suppose a bank has two tellers with separate waiting lines, and each new customer must choose which one to join. Customers arrive at Poisson rate 1, and we assume they join the shorter line. Each teller who is not idle finishes serving a customer at rate $\alpha > 1/2$. What is the joint stationary distribution of the number of customers in each line? Rather than give a tie-breaking algorithm for a customer choosing between equal lines, we project to the unordered pair of line lengths, so the states of the system are parameterized by the length n of the shortest line and the difference k between the two lengths. This problem is considered by Flatto and McKean (1977) and the generating function $F(z, w) := \sum_{r,s} p_{r,s} z^r w^s$ is obtained, where $p_{r,s}$ is the stationary probability of finding lines of length r and $r + s$.

The obvious relation from the forward equation (flow out of state (r, s) equals flow into state (r, s) is:

$$(2\alpha + 1)p_{r,s} = \alpha p_{r,s+1} + \alpha p_{r+1,s-1} + p_{r-1,s+1}.$$

When r or s is 0 or 1, there are boundary effects. The upshot is that

$$[z(z + \alpha) - (1 + 2\alpha)zw + \alpha w^2]F = J_1(z, w)f_1(z) + J_2(z, w)f_2(w)$$

where the polynomials J_1 and J_2 are easy to write down and the functions f_i are unknown. As in the previous example, we do know that the RHS vanishes on $V := \{(z,w) : z(z+\alpha) - (1+2\alpha)zw + \alpha w^2 = 0\}$ in a neighborhood of $(0,0)$. This together with a determination of the genus of V (zero) and of the maps of V to itself that preserve one of the coordinates are enough to determine the poles and zeros of the meromorphic functions f_1 and f_2, hence F. This form of determination is convenient for then deriving asymptotics of $p_{r,s}$ as $r, s \to \infty$ for fixed α and of the mean of $r+s$ as $\alpha \to 1/2$.

Flatto and Hahn (1984) consider another queuing model, this time with each customer served by both queues. The analysis yields again something of the form $\phi(z,w)F(z,w) = J_1 f_1(z) + J_2 f_2(w)$, with $V := \{(z,w) : \phi(z,w) = 0\}$ now having genus 1. An even trickier computation involving elliptic functions ensues. It is the nature of the boundary conditions that determines the course of the computation in these cases. Later we describe a class of boundary conditions which we call *standard*, that include the Larsen-Lyons problem but not the queuing models, which always result in algebraic generating functions.

Example 5 (Polyominos)

Let us now do an example where a rational generating function comes from something other than a simple linear recursion. Define a *horizontally convex polyomino* to be a union of some finite number n of lattice squares with the following property: the squares appear in k consecutive rows, with each row containing a single block of consecutive squares and each block sharing at least one edge with the blocks above and below it (except for the top and bottom block respectively). The number of HCP's with n squares in total, composed of s rows, is denoted $f(n,k)$. It is not obvious, but we will see that

$$F(x,y) := \sum_{n,k} f(n,k) x^n y^k = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2y)}.$$

For the derivation we follow Wilf (1989, page 152) though the original derivation is due to Pólya and the more recent derivations due to Klarner, Gessel, Delest and maybe others. The use of variables x, y rather than z, w keeps Wilf's notation.

Let $f(n,k,t)$ denote the number of HCP's with n squares, k rows, and top row having length t. The trivariate generating function $F(x,y,z) := \sum f(n,k,t)x^n y^k z^t$ is rational and can be derived without too much work beyond the derivation of the bivariate function, but we will skip this and simply use the third variable to get a useful recursion. Given a polyomino with specified values of n, k, t, let r be the number in the second-to-top row. The top and second-to-top rows can be offset in $r + t - 1$ ways, leading to the identity

$$f(n,k,t) = \sum_{r \geq 1} f(n-t, k-1, r)(r+t-1) \tag{1.3}$$

with boundary condition $f(n,1,t) = \delta_{t,n}$. Holding k and t fixed gives a generating function $F_{k,t}(x) := \sum_n f(n,k,t)x^n$ which then clearly satisfies

$$F_{k,t}(x) = x^t \sum_{r \geq 1} (r+t-1) F_{k-1,r}(x) \tag{1.4}$$

when $k \geq 2$, with $F_{1,t} = x^t$.

Recall that we don't care about t. Hence we sum on t, defining $U_k(x) := \sum_{t \geq 1} F_{k,t}(x)$. It will also be convenient to define $V_k(x) := \sum_{t \geq 1} t F_{k,t}(x)$. We have $U_1(x) = x/(1-x)$ and $V_1(x) = x/(1-x)^2$. From (**??**) we get

$$F_{k,t}(x) = x^t \left(V_{k-1}(x) + (t-1) U_{k-1}(x) \right)$$

and summing on t gives

$$U_k(x) = \frac{x}{1-x} V_{k-1}(x) + \frac{x^2}{(1-x)^2} U_{k-1}(x).$$

Multiplying by t before we sum on t yields also

$$V_k(x) = \frac{x}{(1-x)^2} V_{k-1}(x) + \frac{2x^2}{(1-x)^3} U_{k-1}(x).$$

This is a linear recurrence over the field of rational functions of x and can be solved as such. The form of the recurrence is $(u_k, v_k) = (u_{k-1}, v_{k-1}) \binom{a\ b}{c\ d}$, with (u_1, v_1) specified. To determine $\sum u_k y^k$, eliminate $\{v_k\}$ by substituting the first equation into the second twice, yielding

$$u_{k+1} - (a+d)u_k + (ad - bc)u_{k-1} = 0$$

except for $k = 1$. Thus $\sum u_k y^k$ will be of the form

$$\frac{J_1(x)}{1 - \lambda_1 y} + \frac{J_2(x)}{1 - \lambda_2(y)}$$

where J_1 and J_2 are determined by the initial values u_1 and v_1, and λ_i are the eigenvalues of the matrix $\binom{a\ c}{b\ d}$. In our case, $\sum_k U_k(x) y^k$ is exactly what we're looking for, and it comes out to be the rational function displayed above. One can set $y = 1$ to obtain a one-variable generating function for $\sum_{n,k,t} f(n,k,t)x^n$, namely

$$\frac{x(1-x)^3}{1 - 5x + 7x^2 - 4x^3} .$$

While this expression begs a combinatorial proof, the only one apparently known (according to R. Stanley) is an unwritten proof due to D. Hickerson; N.B. is this the same person who haunts the halls of U.C. Davis at night doing computations for Conway's game of Life, and is otherwise on the academic fringe?

There are countless (!) examples in this vein and in the interest of getting on with the analysis of asymptotics, I want to limit the examples discussed to those that make a point. **Exercise: Let $a_{n,k}$ be the number of ways of placing k non-overlapping dominos on a $2 \times n$ grid. Find the generating function for these numbers.** One point worth making is that the methods we will develop do not work for entire functions: singularities, preferably poles, are required. Why should we expect real life generating functions not to be entire? One reason is that counting functions have integer coefficients, and thus cannot converge at the point $(1, \ldots, 1)$. This justification is a bit too facile for the following reason. Often we need to count things that grow very fast, say as the factorial of one of the indices, and the best available generating function is a exponential generating function such as $\sum a_{r,s} z^r w^s / r!$. This is likely to make the function entire, unless $r!$ happens to be the growth rate of $a_{\cdot,s}$ up to an exponential factor, that is $r! = O(|a_{r,s}| e^{\gamma s})$ for some γ. Thus our methods are indeed limited mainly to the ordinary generating function case, though it is worth mentioning that there are plenty of natural examples where the factorial generating function is not entire.

Example 6 (Two factorial GF's)

Two such are the ordered set partitions and the Eulerian numbers $A_r(s)$ counting the number of permutations of $\{1, \ldots, r\}$ having exactly s rises. It is clear why this should have an exactly factorial growth rate, at least for some values of s, and indeed

$$\sum_{r,s} \frac{A_r(s) z^r w^s}{r!} = \frac{w(1-w)}{e^{(w-1)z} - w}.$$

These two examples are discussed in Bender (1973), and the asymptotics given. There will be more to say about this when we compare Bender's results to the ones given here.

At times in our asymptotic analyses, it will be convenient to assume nonnegativity: $a_{\mathbf{r}} \geq 0$ for all $\mathbf{r}$. How much is lost from an applications point of view when we assume this? As you might imagine, most combinatorial applications satisfy this condition, but here is an example that does not.

Example 7 (Tschebysheff polynomials)

The Tschebysheff polynomials $T_n(x)$ are perhaps the oldest known and most useful family of orthogonal polynomials. They are defined by $T_n(x) = \cos(ncos^{-1}x)$. The generating function is not hard to write down (see Comtet 1974, page 50), and it is in fact rational:

$$\sum_{r \geq 0} T_r(z) w^r = \frac{1 - zw}{1 - 2zw + w^2}.$$

The generating function for the Tschebysheff polynomials of the second kind is the same but without the numerator. Interestingly enough, a generating function we are about to encounter in the realm of random tilings has a very similar generating function to that for the Tschebysheff polynomials of the second kind, but with the variable z replaced by a polynomial expression in a way that makes all the resulting coefficients nonnegative.

Example 8 (Random tilings)

The last example I wish to discuss in detail is one that motivated much of my work on this subject. Given a finite connected union of lattice squares in $\mathbf{Z}^2$, let $\mathcal{C}$ be the collection of all tilings of the region by dominos (partitions into a disjoint union of adjacent pairs of squares). Of course $\mathcal{C}$ may be empty, for instance if the number of squares is odd or the number of black and white squares unequal when given a checkerboard coloring, but we assume $\mathcal{C}$ is not empty. Particularly well studied cases are large rectangles (Burton and Pemantle 1993), or the Aztec Diamond. The Diamond is a shape approximating a tilted square, but drawn along lattice lines and having the central row and column doubled. Thus the Diamond of order p has maximum height $2p$; see picture. Let μ_p be the probability measure gotten from choosing a tiling uniformly at random from all tilings of the Aztec Diamond of order p. What are the properties of typical samples from μ_p as $p \to \infty$? In a series of papers Propp, Larsen, Elkies, Kuperberg, Cohn, Kenyon and maybe others give detailed answers to this question. Their methods are varied, including some useful bijections and an analysis of an interacting particle system. Here, however, I will concentrate on what can be gotten from the generating function.

Color the Diamond like a checkerboard, with the left most of the two top row squares colored black, and use this to give one of four types to each domino in a tiling. The domino is called type-1 if the black square is west of the white square, with the types 2, 3, 4 being assigned to the other three possibilities. Let $a_{n,k,p}$ be the proportion of all tilings of the order p Aztec Diamond that have a type-1 domino in position (n, k). More precisely, note that the centers of the squares of the Aztec Diamond are at positions $(n/2, k/2)$ with n and k odd integers such that $|n|+|k| \leq 2p$; a type-1 domino is a horizontal 1×2 rectangle whose midpoint is $(n/2, k/2)$ with n even, k odd, and $n - k \equiv 2p + 1_{\text{mod}4}$. Then the generating function for the proportion of tilings containing the domino with midpoint $(n/2, k/2)$ is given by

$$F(x, y, z) := \sum a_{n,k,p} x^n y^k z^p = \frac{yz/2}{(1 - y^2 z)(1 - 2Tz + z^2)}$$

where $T = (x^2 + x^{-2} + y^2 + y^{-2})/4$ (cf. the generating function for Tschebysheff Polynomials of the second kind); I use x, y, z for three variable generating functions because of the awkwardness of z, w,??.

We may of course simplify this by ignoring the numerator and replacing x^2 and y^2 by x and y, to get

$$\frac{1}{(1-yz)(1-2tz+z^2)}\,; \qquad\qquad t := \frac{x+x^{-1}+y+y^{-1}}{4}\,.$$

When we write this as the quotient of polynomials, the denominator will vanish at the origin, so this is not quite in standard form. By a change of variables $z \mapsto xyz$, which only changes the indexing we can turn it into an honest generating function, but for reasons of preserving the symmetry, it is easiest to work with it in its present form as a Laurent series. A derivation of this generating function appears in Gessel, Ionescu and Propp (1995), making use of a particular construction of a random tiling from IID bits. It is proved in Jockusch, Propp and Shor (1995), by a method involving the analysis of an interacting particle system, that the large scale features of the placement probabilities are as follows. Outside the L^1 ball $|n| + |k| = |p|$, the probabilities are nearly zero, while inside they are not. In Cohn, Elkies and Propp (1996), the saddle point method is used to greatly refine these results. As $p \to \infty$ and $(n/p, k/p) \to (\alpha, \beta)$, there is a phase transition depending on whether $\alpha^2 + \beta^2$ is less or greater than 1/2. Inside the circle of radius 1/2, the values of $a_{n,k,p}$ converge to a quantity $g(\alpha, \beta)$ strictly between zero and 1. Between the circle and the L^1-ball, except in the top region, $\lim_{p\to\infty} p^{-1} \log a_{n,k,p}$ converges to a value $h(\alpha, \beta)$ that is strictly negative; in the top region, the same is true of $1 - a_{n,k,p}$. Thus outside the circle, domino placements are deterministic except for an exponentially small probability.

The analytic methods of Cohn, Elkies and Propp (1996), start from the recurrence responsible for the simple form of the generating function. They carry out an asymptotic analysis of a quantity related to the placement probabilities and then spend a fair amount of effort summing these to get the result. In Cohn and Pemantle (2000) we re-derive the results directly from the final generating function which somehow encodes the summation and subsequent integral approximation. There we also consider a related model, called the Fortress model, for which no other method is known to derive the large scale behavior. The

Fortress of order n is defined to be dual to the usual square-octagon tiling of the plane, and to occupy a region similar in shape to the Aztec diamond of order n. Tiles are defined so that tilings correspond to perfect matchings on a sub-diamond of the square-octagon tiling. The generating function for the probability of a type-1 tile in position (n, k) in an order-p Fortress is given by

$$F(x, y, z) = \text{ similar but messier expression}$$

Pending a rigorous writeup, the results which we believe to hold for the Fortress are qualitatively the same as for the Aztec Diamond, but with the circular boundary (the "arctic circle" of Propp et al) replaced by a boundary given by the zero set of an eighth degree polynomial (the "octic circle"). The analyses of both the Aztec Diamond and the Fortress in Cohn and Pemantle (1999) involve oscillating integrals whose stationary phase points are at singular points of a complex algebraic variety.

Before leaving the subject of obtaining and classifying generating functions, it is worth mentioning one more class of functions. Let $F(z, w) = \sum_{r,s} a_{r,s} z^r w^s$ be a two-variable generating function and let $D(z) = \sum_r a_{r,r} z^r$ denote the generating function for the diagonal. There is a method, which is sometimes reasonably effective, for obtaining D directly from F. The significance of this is two-fold. First, since the form of D depends only on the form of F, one can show that if F is nice, then D is at least somewhat nice. For example, if F is rational then D is algebraic; if F is algebraic, then D is D-finite (meaning that for some k, the functions $D, D', D'', \ldots, D^{(k)}$ are linearly dependent over polynomials. Secondly, the problem of obtaining asymptotics for $a_{r,r}$ can be solved by using one-variable methods on D. Asymptotics in one variable are considerably easier, so this represents an advance. Unfortunately the limitation to the diagonal is severe. By substituting z^a and w^b for z and w, one can get diagonals of any rational slope, but the complexity of the computation increases rapidly and the estimates are not uniform in the slope. This method, due to Doubilet, Rota and Stanley, will be discussed later.

2 Asymptotics in one variable

To set the terminology, I will quickly review the definition of an asymptotic development (or expansion). The notation $f = O(g)$ as $x \to L$ means $\limsup_{x \to L} |f(x)/g(x)| < \infty$. If L is not specified, we take $L = \infty$. Similarly, $f = o(g)$ means $|f/g| \to 0$.

If f is a function on $\mathbf{R}^+$ or $\mathbf{Z}^+$, and L is a number or infinity, then $f \approx \sum_{n=0}^\infty g_n$ means that for any N,

$$f(x) - \sum_{n=0}^N g_n(x) = O(g_{N+1}(x)) \text{ as } x \to L.$$

In this case we call $\sum_{n=0}^\infty g_n$ an asymptotic development of f. The sequence $\{g_n\}$ will always satisfy $g_{n+1} = o(g_n)$, whence a seemingly weaker but actually equivalent formulation is

$$f(x) - \sum_{n=0}^N g_n(x) = o(g_n(x)) \text{ as } x \to L.$$

This does not imply that $f = \sum_{n=0}^\infty g_n$, and in fact the series may be nowhere convergent; the standard example is

$$e^x \int_x^\infty \frac{e^{-t}}{t}\, dt \approx \sum_{n=0}^\infty \frac{(-1)^n n!}{x^{n+1}}$$

as $x \to \infty$. Sometimes we loosen the terminology and call $f(x) \approx \sum_{n=0}^\infty b_n g_n(x)$ an asymptotic development when some values b_n are zero, as long as the b_0 is nonzero and as long as we still have $f(x) - \sum_{n=0}^N b_n g_n(x) = O(g_{n+1}(x))$. Thus for example, the expansion

$$a_n = c_0 n^{-1/2} + c_1 n^{-3/2} + O(ne^{-\alpha n})$$

may be considered a terminating asymptotic expansion in decreasing powers of $n^{-k-1/2}$. We think this way especially when considering simultaneously other functions whose asymptotic expansions in decreasing powers of $n^{-k-1/2}$ do not terminate. Such examples arise in Section **??**.

The most common asymptotic developments are in powers of $x - a$ as $x \to a$ or powers of x^{-1} as $x \to \infty$. Because of this we say that $f(x)$ is *rapidly decreasing* at infinity if

$f(x) = O(x^{-N})$ for all $N \geq 0$. Rapidly decreasing functions are smaller at infinity than any functions with asymptotic developments in powers of x^{-1}. Smaller yet are the functions of *exponential decay*, namely those satisfying $f(x) = O(e^{-\gamma x})$ as $x \to \infty$ for some positive constant γ.

2.1 Rational functions: formal power series solution

It will be instructive first to review the easy and complete description of the coefficients of a rational function of one variable. Let $F(x) = \sum a_n x^n = p(x)/q(x)$ the ratio of two polynomials, with $q(0) = 1$, p and q never vanishing together, and the degree of q denoted by d. Factor $q(x)$ as $\prod_{i=1}^{d}(1 - r_i^{-1}x)$, where $r_1, r_2, \ldots, r_d$ are the roots of q. Assume the roots are distinct. Then

$$\frac{1}{q(x)} = \sum_{i=1}^{d} \frac{c_i}{1 - r_i^{-1}x}$$

where $\{c_i\}$ are nonzero constants which may be computed as rational functions of the roots (we will determine them shortly). The formal power series $1/(1 - r_i^{-1}x)$ is equal to $\sum_{n\geq 0} r_i^{-n} x^n$. Hence the n^{th} coefficient of F is equal to $\sum_{i=1}^{d} c_i r_i^{-n}$. The leading term(s) is of course r_M^{-n}, where M minimizes the modulus of r_M. This approximation is very good unless the next-smallest modulus of a root is close to $|r_M|$. If $p = p_0 + p_1 x + \cdots + p_e x^e$ is some other polynomial, then the i^{th} summand in the previous summation, instead of being $c_i r_i^{-n}$, is

$$c_i p_0 r_i^{-n} + \cdots + c_i p_e r_i^{e-n} = c_i p(r_i) r_i^{-n}.$$

There are no common roots of p and q, so $c_i p(r_i) \neq 0$ and again the leading term is given by the minimum modulus root.

Finally, if the roots are not distinct, then let $r_1, \ldots, r_k$ be the distinct roots with r_i having multiplicity d_i. There are constants $c_{i,j}$ with $1 \leq i \leq k$ and $1 \leq j \leq d_i$ for which $1/q = \sum c_{i,j}/(1 - r_i^{-1}x)^j$. In the same way as before, this leads to

$$a_n = \sum_{i=1}^{k} \sum_{j=1}^{d_i} c_{i,j} p(r_i) \binom{n+j-1}{j-1} r_i^{-n}.$$

The leading term constants c_{i,d_i} are nonzero, hence the leading term asymptotic is given by

$$c_{M,d_M} p(r_M) \binom{n + d_M - 1}{d_M - 1} r_M^{-n}.$$

2.2 Rational functions and more: analytic solution

Let us compare the formal power series solution, which is complete but specialized, to an analytic solution. First note that the radius of convergence of the power series for F is equal to the minimum modulus of a singularity for F, which is r_M. For any power series with radius of convergence R, we have $\limsup n^{-1} \log |a_n| = \log(R^{-1})$. Thus we get the correct exponential rate, at least for the limsup, with no work at all.

Next, use Cauchy's integral formula to write

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{z^{n+1}} F(z)$$

where $\mathcal{C}$ is any contour enclosing the origin and contained in the domain of convergence of F. Let $\mathcal{C}$ be a circle of radius $r < r_M$, and let $\mathcal{C}'$ be a circle of radius $R > r_M$. Assume that r_M is the only root of q of minimum modulus and that the moduli of other roots are greater than R. We then have, by the residue theorem,

$$\int_{\mathcal{C}} \frac{dz}{z^{n+1}} F(z) - \int_{\mathcal{C}'} \frac{dz}{z^{n+1}} F(z) = -2\pi i \text{Res}(z^{-n-1} F(z); r_M).$$

If $F = p/q$ has a simple pole at r_M then the residue is just $r_M^{-n-1} p(r_M)/q'(r_M)$. The integral over $\mathcal{C}'$ is bounded by $2\pi R^{-M} \sup_{|z|=R} |F(z)|$, and is therefore exponentially smaller than the residue. Thus the leading term asymptotic for a_n is

$$a_n = -r_M^{-n-1} p(r_M)/q'(r_M) + O(R^{-n}).$$

In fact we may send $\mathcal{C}'$ to infinity, thus picking up all the terms. This will be a sum of terms $-r_i^{-n-1} p(r_i)/q'(r_i)$. Incidentally, this is an easy way of determining the constants c_i, which are evidently $-1/q'(r_i)$.

If there is more than one root of minimum modulus, simply sum the contributions. If the root r_i appears with multiplicity $d_i > 1$, then the residue at r_i is no longer $r_i^{-n-1}p(r_i)/q'(r_i)$ but is instead equal to $(1/(d_i-1)!$ time the $d_i - 1^{st}$ derivative at r_i of $(z-r_i)^{d_i}z^{-n-1}F$. Writing $\tilde{q} := q/(z-r_i)^d$, the residue becomes

$$\left(\frac{p}{\tilde{q}}z^{-n-1}\right)^{(d-1)}\Bigg|_{z=r_i} = (-n-1)_{d-1}r_i^{-n-d+1}p(r_i)/\tilde{q}(r_i) + O(n^{d-2}).$$

This gives the "polynomial correction" for the case of multiple roots.

The advantage of the analytic solution is that it vastly more general. The formal power series solution requires that $F = p/q$ a quotient of polynomials. Instead, suppose that p and q are only required to be analytic in some disk $B(0,R)$, and that q has a zero, say a, inside the disk. The same computation then gives

$$a_n = \frac{1}{2\pi i}\int_{\mathcal{C}} z^{-n-1}F(z)\,dz = \frac{1}{2\pi i}\int_{\mathcal{C}'} z^{-n-1}F(z)\,dz - \text{Res}(z^{-n-1}F;a) \tag{2.1}$$

where $\mathcal{C}$ is a circle of radius less that $|a|$ and $\mathcal{C}'$ has radius r with $|a| < r < R$ and no other pole of F in $B(0,r)$. Then $|\int_{\mathcal{C}'}| \leq 2\pi r \sup_{z\in\mathcal{C}'}|F(z)|r^{-n}$ and the residue is easily computed as before. For instance, in the case of a simple root of q, the residue is still $p(a)a^{-n-1}/q'(a)$.

What does the analytic method give us for a general function F? If the generating function F is purely formal, i.e., nowhere convergent, then we learn nothing. That's why we use exponential generating functions! If F is entire, we learn very little, though more can be said by means beyond the scope of these lectures. Assume then that the radius of convergence of F is positive and finite. If the minimal modulus singularity of F is a pole (or poles) then the preceding analysis applies. If it is a branchpoint, there are standard modifications of the method which we will see shortly. If it is an isolated singularity, there are good prospects for a successful modification of the method, though it is more of an art than a science; this will be discussed too.

Logically, the worst case is if the domain of convergence of F has an entire circle as its natural boundary (this means F cannot be analytically continued anywhere outside the circle). This can and does happen. In fact a classical theorem of Pólya says that if

F has integer coefficients and radius of convergence 1, then either F is rational or F has the unit circle for its natural boundary (see Pólya 19??). A point of philosophy: we have cited integer coefficients as a motivation for considering non-entire functions. Whenever the growth of the coefficients is not exponential, then, we must be dealing with a rational function. So is all this generality spurious? The answer is no: including sequences with exponential growth and those of combinatorial significance that are not integral, there are quite a lot of meaningful examples. Also, in the multivariable case even rational functions are nontrivial to analyze and we have already seen a number of pertinent examples there. Finally, we note that even when the circle is a natural boundary, analytic methods often give something beyond the determination of the limsup exponential growth rate, depending on the behavior of the function on the boundary circle.

The remaining discussion of one-variable asymptotics will be as follows. A brief discussion of an example with an isolated essential singularity will shed some light on the role of oscillating integrals and the method of stationary phase. We will then look at the classical FPS derivation of asymptotics for branchpoints induced by non-integral powers (following Henrici 1977, Theorem 11.10). These will be compared to the comparison or "transfer" theorems (following Flajolet and Odlyzko 1990) which use analytic methods to obtain the leading term asymptotics for a very wide class of functions, namely whose dominant singularity is a power times a power of log times a power of log log.

2.3 Essential singularities

Since contour integration is an art, it is wise to ask first for the guiding principles behind choosing the contour. To answer this, we recall the Cauchy integral formula:

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}} F(z) \frac{dz}{z^{n+1}}$$

where $\mathcal{C}$ is a contour not containing a singularity of F. Let R be the radius of convergence of F. Then $\mathcal{C}$ may be taken to be the circle of radius r for any $r < R$. What does it mean that all these integrals come out to the same value? The integrand has magnitude on the order of

r^{-n}, whereas we know the integral is only size R^{-n}. Evidently there is a lot of cancellation taking place. The cancellation reduces the integral by an exponential factor $(r/R)^n$. As r expands toward R the oscillation kills at a lower exponential rate, until, right near R, it isn't really killing at all. This leads to the stationary phase principle: find a stretch of the contour where the integrand is not oscillating; this will be the leading contribution to the integral. A related technique is the saddle point method: if there is no point of stationary phase, move the contour until you go through one, and make sure you go through at the right angle. At the moment we do not attempt to prove that this methods are universal, though later as part of our classification we do prove this for some classes of integrals. We turn now to an example.

Example 9 (Isolated essential singularity)

Let $F(x) = \exp(x/(1-x))$. Clearly F has a single essential singularity at the point 1. It is of course possible to compute the coefficients directly from the combinatorial interpretation: this is the exponential generating function for the number of unordered partitions of an n element set into ordered sequences. The analytic approach, however, will allow us to compute these in a way that is more robust with respect to perturbations in the generating functions. Starting as usual from Cauchy's integral formula as in (**??**), we send $\mathcal{C}'$ to infinity and discover that a_n is precisely the residue of $z^{-n-1}F$ at 1. We compute this by integrating over any contour encircling the point $z = 1$, or in the Riemann sphere (since $e^{x/(1-x)}$ is analytic at ∞), any simple closed curve separating 0 and 1. Set $I(z) = \log[z^{-n-1}F(z)]$ and compute

$$I'(z) = \left[\frac{-n-1}{z} + \frac{1}{(1-z)^2}\right].$$

This vanishes at the point $1 - \beta_n$, where

$$\beta_n = -\frac{1}{2(n+1)} + \sqrt{\frac{1}{n+1} + \frac{1}{4(n+1)^2}} = n^{-1/2} + O(n^{-1}).$$

Choosing our contour γ to be the line $\{1 - \beta_n + it : -\infty < t < \infty\}$, we have

$$a_n = \frac{1}{2\pi i}\int_\gamma z^{-n-1}F(z)\,dz.$$

We have chosen β_n so that phase of the integrand is stationary at $1-\beta_n$, that is, the derivative of the (log of the) integrand vanishes. The hope now is that the integral is well approximated by integrating the degree-two Taylor approximation of I, namely we hope that

$$\begin{aligned} & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp(I(1-\beta_n+it))\,(i\,dt) && (2.2)\\ \approx\ & \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[I(1-\beta_n)+\frac{1}{2}I''(1-\beta_n)(it)^2]\,dt && (2.3)\\ =\ & \sqrt{\frac{1}{2\pi I''(1-\beta_n)}}\exp(I(1-\beta_n)). && (2.4)\end{aligned}$$

This hope is easily verified as follows. We compute $I''(1-\beta_n) = (n+1)/(1-\beta_n)^2 + 2/\beta_n^3 = (2+o(1))n^{3/2}$. This tells us that the main contribution to (**??**) should come from the region where $|t|$ is not much larger than $n^{-3/4}$. Accordingly, we pick a cutoff a little greater than that, say $L = 2n^{-3/4}\log n$, and break the integrals (**??**) and (**??**) into two parts, $|t| \le L$ and $|t| > L$. Up to the cutoff the two integrals are close, and past the cutoff they are both small.

The contribution to (**??**) when $|t| > L$ is a Gaussian tail and is $o(\exp(I(1-\beta_n) - \log^2 n))$. The contribution to (**??**) when $|t| > L$ may be bounded in two parts. When $|t| < n^{-1/2}$, use the fact that $|z| \ge 1$ on the line of integration to see that the modulus of the integrand is at most $M(t)\exp(I(1-\beta_n))$ where

$$|M(t)| = \exp \mathrm{Re}\{\frac{x}{1-x}\}\Big|_{1-\beta_n}^{1-\beta_n-it} = \exp\left[\mathrm{Re}\left\{\frac{1}{\beta_n+it} - \frac{1}{\beta_n}\right\}\right].$$

We then compute

$$\begin{aligned}\mathrm{Re}\left\{\frac{1}{\beta_n+it} - \frac{1}{\beta_n}\right\} &= \frac{\beta_n}{\beta_n^2+t^2} - \frac{1}{\beta_n}\\ &= \frac{\beta_n^{-1}}{1+\beta_n^{-2}t^2} - \beta_n^{-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\beta_n^{-3}t^2}{1+\beta_n^{-2}t^2} \\
&\leq \frac{-1+o(1)}{2} n^{3/2} t^2
\end{aligned}
$$

because $\beta_n \sim n^{-1/2}$. Thus again we have part of a Gaussian tail and again the contribution is $o(\exp(I(1-\beta_n)) - \log^2 n)$. When $|t| > n^{-1/2}$ we need to use the z^{-n-1} term as well:

$$
\begin{aligned}
\frac{|\exp(I(1-\beta_n+it))|}{\exp(I(1-\beta_n))} &\leq \frac{|1-\beta_n|^n}{|1-\beta_n+it|^n} \exp(\mathrm{Re}\{\frac{1}{\beta_n+it} - \frac{1}{\beta_n}\}) \\
&\leq (1+t^2)^{-n/2} \exp(\mathrm{Re}\{\frac{1}{\beta_n+in^{-1/2}} - \frac{1}{\beta_n}\}).
\end{aligned}
$$

Integrating from $t = n^{-1/2}$ to ∞ and using our previous computation of the real part of the above difference shows that the contribution is at most the integrand at $t = 0$ times a factor of $\exp(-\sqrt{n}/(2+o(1)))$.

When $|t| \leq L$, we use the Taylor approximation

$$
\left| I(1-\beta_n+it) - I(1-\beta_n) + \frac{1}{2}t^2 I''(t) \right| \leq \frac{1}{6} t^3 \sup_{|s|\leq L} |I'''(s)|.
$$

The RHS is bounded by $(1+o(1))t^3 n^2$, and hence by $8n^{-1/4}\log^3 n$. Since the integrand of (**??**) is everywhere positive, this implies that the difference between the integrals (**??**) and (**??**) on $|t| \leq L$ is at most $\exp(n^{-1/4}\log^3 n) - 1$ times the integral (**??**), as desired.

Having established that (**??**) is the leading term, we finally compute it. Using the formula for $I''(z)$ and the formula

$$
\beta_n = n^{-1/2} - \frac{1}{2}n^{-1} + O(n^{-3/2})
$$

we get

$$
\begin{aligned}
&\sqrt{\frac{1}{2\pi I''(1-\beta_n)}} \exp(I(1-\beta_n)) \\
= \quad &(1+o(1))\sqrt{\frac{1}{4\pi n^{3/2}}} \exp\left(-(n+1)\log(1-\beta_n) - 1 + \frac{1}{\beta_n}\right)
\end{aligned}
$$

$$= \quad (1+o(1))\sqrt{\frac{1}{4\pi n^{3/2}}}\exp\left(-(n+1)(-n^{-1/2}+O(n^{-3/2}))-1+n^{1/2}+\frac{1}{2}+O(n^{-1/2})\right)$$

$$= \quad (1+o(1))\sqrt{\frac{1}{4\pi e}}n^{-3/4}\exp(2\sqrt{n}).$$

Note that computing the full asymptotic development is almost as easy. The cutoff is calibrated so that the remainder after k terms of the Taylor expansion is always small on $|t| \leq L$, so essentially the same computation suffices to derive an asymptotic series.

Example 10 (Non-integral power)

Among singularities that are not isolated, the nicest are branchpoints. Many though not all branchpoints can be written as a non-integral power times an analytic function. Thus we consider the class of functions $(1 - z/A)^{-c}\psi(z)$ analytic on a disk containing A and slit from A to the boundary, where A is any nonzero complex number, c is a complex number that is not a real integer, and ψ is a function analytic in the disk of radius R for some $R > |A|$. Let $F(z) = \sum a_n z^n$ be such a function. Since the coefficients of $(z - A)^c$ are explicitly known, the easiest way to obtain the asymptotics for $\{a_n\}$ is by convolving the formal power series. The only work will be in checking that the resulting series really forms an asymptotic development. We follow Henrici's (1977) exposition of a theorem of Darboux:

Theorem 2.1 *With F, ψ and $\{a_n\}$ as above, let $\sum_{n=0}^{\infty} b_n(z-A)^n$ be the power series for ψ near A. Then for any $k \geq \text{Re}\{c\}^+$,*

$$a_n = (-A)^{-n}\left(\sum_{j=0}^{k}(-A)^j b_j \binom{j-c}{n}\right) + o\left(\left|A^k\binom{k-c}{n}\right|\right)$$

where $\binom{x}{n}$ denotes $(1/n!)\prod_{j=1}^{n}(x-j+1)$.

PROOF: To see where this is going, formally write $F(z) = \sum_{m=0}^{\infty} b_m(-A)^m(1 - z/A)^{-c+m}$. Expand $(1 - z/A)^{-c+m}$ in powers of z using the binomial theorem to get

$$F(z) = \sum_{m=0}^{\infty} b_m(-A)^m \sum_{n=0}^{\infty}(-A)^{-n}\binom{m-c}{n} z^n$$

to see that the n^{th} coefficient ought to be as claimed.

To justify all this, fix k; it will be clear later why we assume k to be greater than the positive part of $\mathrm{Re}\{c\}$. Let r_k be the k^{th} Taylor series remainder for ψ, i.e.,

$$\psi(z) = b_0 + b_1(z - A) + \cdots + b_k(z - A)^k + r_k(z)(z - A)^{k+1}.$$

Then in the domains of convergence of F and ψ,

$$F(z) - \sum_{j=0}^{k}(-A)^j b_j(1 - z/A)^{j-c} = (-A)^{k+1}(1 - z/A)^{k+1-c} r_k(z).$$

The LHS is the sum of finitely many power series converging in a neighborhood of zero, so analytically is equal to the power series whose n^{th} coefficient is

$$a_n - (-A)^{-n}\sum_{j=0}^{k} b_j(-A)^j \binom{j-c}{n}.$$

We need now to estimate the n^{th} coefficient B_n of the RHS, which we call $G(z)$. Note that we have chosen k sufficiently large so that the G is continuous at A. Thus we may use Cauchy's formula

$$B_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z^{n+1}} G(z)$$

for any circle $\mathcal{C}$ of radius less than $|A|$, and take limits to let the radius of $\mathcal{C}$ equal $|A|$. Since the function G is differentiable $\lceil k - c \rceil$ times, integration by parts and the Riemann-Lebesgue lemma (see below for amplification) show that the n^{th} coefficient of the RHS is $o(|A|^{-n} n^{-\lceil k-c \rceil})$. Use Stirling's formula to see that this is $o(|A|^{-n}\binom{k-c}{n})$, proving that we have an asymptotic development.

Riemann Lebesgue lemma: the functions $e^{in\theta}$ converge to zero in the topology determined by integration against bounded continuous functions on the unit circle (and in fact in the dual space to L^1 - see Rudin (1974) "Real and Complex analysis"). Hence $\int_{|z|=A} z^{-n} F\,dz = o(|A|^{-n})$ for any $F \in L^1$. Now suppose G is j times continuously differentiable with $G^{(j)} \in L^1$ on the circle $|z| = A$. Then integrating by parts j times we get

$$\int_{|z|=A} z^{-n} G\,dz = (-1)^j (j!)^{-1} \binom{n-1}{j}^{-1} \int_{|z|=A} z^{j-n} G^{(j)}(z)\,dz$$

with the boundary terms vanishing due to continuity of $G^{(s)}, s \leq j$. By the Riemann-Lebesgue lemma, this is $o(n^{-j})|A|^{-n}$. □

Example 11 (Transfer theorems)

The previous example was in some sense a very special case: we knew the expansion of $(1-z)^{-c}$ explicitly, and were able to show how the series behaved under a perturbation that multiplied by an analytic factor. The next example is in this sense also a special class of functions, but a very wide and hence useful special class. Let $\mathcal{C}$ be the class of functions $G(z) := g(1/(1-z))$ where $g(z) = z^\alpha (\log z)^\gamma (\log\log z)^\delta$ for arbitrary real numbers α, γ and δ. Instead of requiring $F(z) = g(z)\psi(z)$ for ψ analytic, we derive information under the assumption only that $F(z) = O(G(z))$ or $F(z) = o(G(z))$ as $z \to 1$. (Naturally, we can rescale so that the critical point appears somewhere other than 1.) The price we pay is that we require F to be analytic out to a radius $1 + \eta$ for some $\eta > 0$, except on the cone $|\mathrm{Arg}(z-1)| < \xi$ where ξ is a fixed number in $(0, \pi/2)$. We call this domain $\Delta := \{z : |z| \leq 1 + \eta, |\mathrm{Arg}(z-1)| \geq \xi\}$.

The transfer method of Flajolet and Odlyzko (1990) consists of three results. The first is an explicit determination of the coefficients of all functions in $\mathcal{C}$. The second and third are the following theorems.

Theorem 2.2 (Flajolet-Odlyzko O-Theorem) *Let $G(z) = \sum b_n z^n$ be in the class $\mathcal{C}$ and let $F(z) = \sum a_n z^n$ be analytic in Δ. Then the hypothesis $F(z) = O(G(z))$ as $z \to 1$ implies $a_n = O(b_n)$.*

Theorem 2.3 (Flajolet-Odlyzko o-Theorem) *Let* $G(z) = \sum b_n z^n$ *be in the class* $\mathcal{C}$ *and let* $F(z) = \sum a_n z^n$ *be analytic in* Δ*. Then the hypothesis* $F(z) = o(G(z))$ *as* $z \to 1$ *implies* $a_n = o(b_n)$*.*

An immediate corollary of these is a transfer asymptotic development result.

Corollary 2.4 *Suppose* $F(z)$ *is analytic in* Δ *and has asymptotic development* $\sum G_j(z)$ *as* $z \to 1$*, where* $G_j \in \mathcal{C}$ *for all* j *and* $G_{j+1} = o(G_j)$*. Let* $\{a_n\}$ *be the coefficients of* F *and* $\{b_n^{(j)}\}$ *be the coefficients of* G_j*. Then* $\sum_j b_n^{(j)}$ *is an asymptotic development of* a_n*.*

Proof: By hypothesis, for each k,

$$h_k := F - \sum_{j=1}^{k} G_j = o(G_k).$$

All members of $\mathcal{C}$ are analytic in Δ, so the finite sum defining h_k is analytic in Δ. Applying the o-theorem gives that the coefficients of h_k are $o(b_n^{(k)})$. Since the coefficients of h_k are the difference between a_n and $\sum_{j=1}^{k} b_n^{(j)}$, this establishes the result. □

So as not to get too far afield, I will prove only the O-theorem, and only for the restricted class $\mathcal{C}'$ in place of $\mathcal{C}$, where $\mathcal{C}'$ contains all functions $(1-z)^\alpha$. This parallels the exposition in Flajolet and Odlyzko (1990), which is highly recommended if you have not seen this stuff before.

Proof of O-theorem: First note that for $G = (1-z)^\alpha \in \mathcal{C}'$, the n^{th} coefficient is of order $n^{-\alpha-1}$. Next, note that the assumption that $F(z) = O(|1-z|^\alpha)$ near $z = 1$ implies (using only continuity, not analyticity) that for some K, $|F(z)| \leq K|1-z|^\alpha$ everywhere on $\Delta \setminus \{1\}$. We use Cauchy's formula

$$a_n = \frac{1}{2\pi i} \int_\gamma F(z) \frac{dz}{z^{n+1}}$$

where γ is a contour carefully as the union of four elementary contours. Let γ_1 be the circular arc parameterized by $1 + n^{-1} e^{it}$ for $\xi \leq t \leq 2\pi - \xi$. Let γ_2 be the line segment between $1+n^{-1}e^{i\xi}$ and the number β of modulus $1+\eta$ and $\text{Arg}(\beta-1)=\xi$. Let γ_3 be the arc on the circle of radius $1+\eta$ running between β and $\overline{\beta}$ the long way, and let γ_4 be the conjugate of γ_2. We will bound the absolute value of the integral on each segment separately, so we need not worry about the orientations.

On γ_1, the modulus of F is at most $Kn^{-\alpha}$, the modulus of z^{-n-1} is at most $(1-n^{-1})^{-n-1} \le 2e$, and the integral of $|dz|$ is at most $2\pi n^{-1}$, leading to a contribution of size at most $6Kn^{-\alpha-1}$. On γ_3 the z^{-n-1} factor in the integral reduces the modulus to at most $C(\eta)(1+\eta)^{-n}$ which is of course $O(n^{-N})$ for any N. Since the order of the n^{th} coefficient of G is $n^{-\alpha-1}$, we are in good shape so far.

By symmetry, we need now only do the computation for γ_2. Set $\omega = e^{i\xi}$ and parametrize the integral as $z = 1+(\omega/n)t$ for $t=1$ to En for a constant, $E=|\beta-1|$. We have $|F(z)| \le K|z-1|^\alpha \le K(t/n)^\alpha$ and

$$|z^{-n-1}| = \left|1+\frac{\omega t}{n}\right|^{-n-1}$$

so

$$\begin{aligned}\int_{\gamma_2} |F(z)||z^{-n-1}||dz| &\le \int_1^{En} K\left(\frac{t}{n}\right)^\alpha \left|1+\frac{\omega t}{n}\right|^{-n-1} \frac{dt}{n} \\ &\le Kn^{-\alpha-1}\int_1^\infty t^\alpha \left|1+\frac{\omega t}{n}\right|^{-n-1} dt. \qquad (2.5)\end{aligned}$$

We need to see that the integral in (**??**) is bounded above for sufficiently large n. We have $|1+\omega t/n| \ge 1+\text{Re}\{\omega t/n\} = 1+(t/n)\cos(\xi)$. Thus the integral in (**??**) is at most

$$J_n := \int_1^\infty t^\alpha \left(1+\frac{t\cos(\xi)}{n}\right)^{-n} dt.$$

The integrand is monotone decreasing in n, and clearly finite for $n > 1+\alpha^+$, so the decreasing limit is

$$\lim_{n\to\infty} J_n = \int_1^\infty t^\alpha e^{-t\cos(\xi)}\, dt$$

which is finite. We have now bounded all four integrals by multiples of $n^{-\alpha-1}$, so the proof is complete. □

We remark that the o-theorem follows from the O-theorem if you can keep track of the constants. That is, if $F = o(G)$ then $F \leq \epsilon G$ in some neighborhood of 1 for any positive ϵ. Thus if constant in the conclusion of the O-theorem can be made to go to zero as the constant in the hypothesis goes to zero, the o-theorem is proved.

The counterpart of the transfer theorems is the asymptotic determination of coefficients of functions in the class $\mathcal{C}$. We quote Flajolet and Odlyzko's result.

Theorem 2.5 (Flajolet and Odlyzko Theorem 5) *Let* $G(z) = (1/(1-z))^\alpha \log^\gamma(1/(1-z)) \log^\delta \log(1/(1-z)) = \sum b_n z^n$ *be in the class* $\mathcal{C}$. *Suppose that* $\alpha \notin \{0, -1, -2, \ldots\}$. *Then*

$$b_n \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1} (\log n)^\gamma (\log\log n)^\delta.$$

When $\delta = 0$, *there is a full asymptotic development in decreasing powers of* $\log n$.

When α is a nonpositive integer, a complete asymptotic development is also available. In the case $\gamma < 0 = \delta$ the leading term turns out to be

$$b_n = -b\Gamma(1-\alpha) n^{\alpha-1} (\log n)^{\gamma-1}. \tag{2.6}$$

We apply this to an example on branching random walks.

Example 12 (Branching RW)

Begin with a single particle at 1. Each unit of time, each particle not at zero gives birth to two particles, which are displaced from the original particle by ± 1, being independently $+1$ with probability p and -1 with probability $1-p$. A particle at 0 never moves or reproduces. Let $Z \geq 2$ be the total number of particles ever to reach 0. Let $a_n := \mathbf{P}(Z = n)$ and let $\phi(z) = \mathbf{E} z^Z = \sum_{n \geq 2} a_n z^n$ be the generating function. An identity for ϕ may be derived as follows. Start with one particle at 2 and freeze any particle that reaches 1. Then the number of particles to reach 1 will be distributed as Z. Now unfreeze these, so each produces an identically distributed number of particles reaching 0. Thus the generating function for the number to reach 0 starting with a single particle at 2 is $\phi(\phi(z))$. Hence

$$\phi(z) = \left[(1-p)z + p\phi(\phi(z))\right]^2 .$$

We consider the critical case, where $p = (2-\sqrt{3})/4$ is the supremum of values such that $\mathbf{P}(Z = \infty) = 0$. It is shown in Aldous (1999) that $16p(1-p) = 1$ and that for this value of p, the variable Z is almost surely finite and in fact has finite mean. [To see this is the critical value, note that among 2^{2n} branches of a tree to depth $2n$, each of which carrying a sum whose distribution is Binomial with parameters $2n$ and x, the expected number of branches to the right of zero is of exponential order $2^{2n}(4x(1-x))^n$. This goes to zero as $n \to \infty$ exactly when $x < p$].

To apply transfer technology, we will show that

$$\phi(z) = 1 - \frac{1-z}{4p} - (c + O(1))\frac{1-z}{\log(1/(1-z))}, \tag{2.7}$$

where $c = \log(1/(4p))/(4p)$. It follows from this and (**??**) that

$$a_n \sim cn^{-2}(\log n)^{-2},$$

so that Z has a first moment but not a "$1 + \log$" moment.

The derivation of (**??**) is given in Aldous (1999) so we only sketch it here. Fix a $0 < z_0 < 1$ and let $z_n = \phi^{(-n)}(z_0)$ so that $z_n \uparrow 1$. The recursion for ϕ gives

$$z_n = ((1-p)z_{n+1} + pz_{n-1})^2.$$

Changing variables to $y_n = 1 - z_n$ gives

$$\begin{aligned} y_n &= 1 - ((1-p)(1-y_{n+1}) + p(1-y_{n-1}))^2 \\ &= 1 - (1 - ((1-p)y_{n+1} + py_{n-1}))^2. \end{aligned}$$

Solving for y_{n+1} gives

$$y_{n+1} = \frac{1 - \sqrt{1-y_n} - py_{n-1}}{1-p}.$$

Setting $x_n = y_n/(4p)^n$ and using $16p(1-p) = 1$ gives

$$x_{n+1} = 2x_n - x_{n-1} + O(y_n)^2.$$

Verifying first that y_n is small, we then have $x_n \sim An + B$, whence $y_n \sim (4p)^n(An + B)$. We may write this as

$$y_{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{\log y_{n+1}/\log(4p)}.$$

Let $z = 1 - y_{n+1}$ so $\phi(z) = 1 - y_n$. We then have

$$1 - \phi(z) = \frac{1-z}{4p} - (1 + o(1))\frac{1-z}{4p}\frac{\log(4p)}{\log(1-z)},$$

proving (**??**).

3 Oscillating and Laplace-type Integrals in One Variable

To motivate a rather long excursion into oscillating integrals, I am going to do a derivation, just in two dimensions, of the representation of the coefficients of F as an oscillating integral. The material on purely oscillating integrals follows Stein (1993, ch. VIII), which is a very fine source. To arrive at the best results for complex phase, the techniques of Bleistein and Handelsman (1986) and Wong (1989) are added to the mix, and some interpolation is also necessary.

We will need to define a few concepts and quantities associated with F. Let $F = G/H$ the quotient of analytic functions and let $\mathcal{V}$ be the zero set o H.

Definition 1 *Say that a point $(z, w) \in \mathcal{V}$ is a* smooth point *of $\mathcal{V}$ if the partial derivatives of H do not simultaneously vanish there. Although this definition appears overly restrictive, it will be seen to lose no generality. Say that (z, w) is a* strictly minimal *point of $\mathcal{V}$ if there is no other $(z', w') \in \mathcal{V}$ with $|z'| \leq |z|$ and $|w'| \leq |w|$.*

Let (z_0, w_0) be a smooth point of the variety $\mathcal{V}$ where the denominator, H, of F vanishes. By definition, H_z and H_w do not both vanish at (z_0, w_0), and here we assume without loss of generality that H_w is nonvanishing. Define the function g (we reserve this letter globally) to be a parametrization of $\mathcal{V}$ near (z_0, w_0), that is, $(z, g(z)) \in \mathcal{V}$ for z in a neighborhood of z_0, and $g(z_0) = w_0$. The implicit function theorem is effective for series, meaning that the series coefficients for g are easily determined from the series coefficients for H.

Next, define a function ψ by

$$\psi(z) = -\lim_{w \to g(z)} (w - g(z)) \frac{F(z, w)}{w} .$$

This is just $(-1/g(z))$ times the residue at $g(z)$ of F (recall that $F(z, \cdot)$ has a simple pole at $g(z)$). It is convenient to change variables in two ways. We wish to let $z = z_0 e^{i\theta}$ and view ψ and g as functions of θ. So we define

$$\tilde{\psi}(\theta) = \psi(z_0 e^{i\theta}) .$$

For fixed $(r, s) \in \mathbf{RP}^1$, we center $\log g$ by defining

$$\tilde{f}(\theta) = \log \frac{g(z_0 e^{i\theta})}{g(z_0)} + i \frac{r}{s} \theta .$$

We now state the lemma on which all computations for smooth point expansions are based.

Lemma 3.1 (Reduction to oscillating integral) *Let (z_0, w_0) be a strictly minimal simple pole of $F = G/H$. Assume that $wH_w \neq 0$ at (z_0, w_0). On a neighborhood $\mathcal{N}$ of zero, define a quantity*

$$\Xi := (2\pi)^{-1} z_0^{-r} w_0^{-s} \int_{\mathcal{N}} \exp(-s\tilde{f}(\theta)) \tilde{\psi}(\theta) \, d\theta. \tag{3.1}$$

Then the quantity

$$|z_0^r| |w_0^s| \, |a_{r,s} - \Xi|$$

decreases exponentially as $\mathcal{N}$ remains fixed and $(r, s) \to \infty$.

PROOF: For $\epsilon \in (0, |w_0|)$, let T be the torus $\{(z,w) : |z| = |z_0|, |w| = |w_0| - \epsilon\}$. By Cauchy's formula,

$$a_{r,s} = \left(\frac{1}{2\pi i}\right)^2 \int_T z^{-r-1} w^{-s-1} F(z,w)\, dw\, dz \tag{3.2}$$

Write this as an iterated integral

$$a_{r,s} = \left(\frac{1}{2\pi i}\right)^2 \int_{|z|=|z_0|} z^{-r-1} \left[\int_{\mathcal{C}_0} w^{-s} F(z,w) \frac{dw}{w}\right] dz \tag{3.3}$$

where $\mathcal{C}_0 := \{w : |w| = |w_0| - \epsilon\}$. Let $K \subseteq \{z : |z| = |z_0|\}$ be a compact set not containing z_0. For each fixed $z \in K$, by the minimality assumption, the function $F(z, \cdot)$ has radius of convergence greater than $|w_0|$. Hence the inner integral in equation (**??**) is $O(|w_0| + \epsilon)^{-s}$. By continuity of the radius of convergence,we may integrate over K to see that

$$|z_0^r||w_0^s| \int_{K \times \mathcal{C}_0} z^{-r-1} w^{-s-1} F(z,w)\, dw\, dz$$

decreases exponentially. Thus if $\mathcal{N}$ is any neighborhood of z in the torus $\{z : |z| = |z_0|\}$, the quantity

$$|z_0^r||w_0^s| \left| a_{r,s} - \left(\frac{1}{2\pi i}\right)^2 \int_{\mathcal{N}} z^{-r-1} \left[\int_{\mathcal{C}_1} \frac{F(z,w)}{w^s} \frac{dw}{w}\right] dz \right| \tag{3.4}$$

decreases exponentially. Thus we have reduced the problem to an integral over a neighborhood of z.

Near z_0 we have parametrized $\mathcal{V}$ by $w = g(z)$. Let $\mathcal{C}_1$ be the circle of radius $|w| + \epsilon$. When $\mathcal{N}$ is sufficiently small compared to ϵ, the image of $\mathcal{N}$ under g is disjoint from $\mathcal{C}_1$. Fix such a neighborhood. For any $z \in \mathcal{N}$, the function $F(z, \cdot)$ has a single simple pole in the annulus bounded by $\mathcal{C}_0$ and $\mathcal{C}_1$, occurring at $g(z)$. The residue of F at $g(z)$ is equal to

$$R(z) := -\psi(z) g(z)^{-s}.$$

Therefore, for each fixed $z \in \mathcal{N}$,

$$\int_{\mathcal{C}_0} \frac{F(z,w)}{w^{s+1}}\, dw = \int_{\mathcal{C}_1} \frac{F(z,w)}{w_{s+1}}\, dw - 2\pi i R(z).$$

But $|z_0^r||w_0^s|| \int_{\mathcal{C}_1} F(z,w) dw/(z^{r+1}w^{s+1})|$ is bounded by a constant multiple of $(1+\epsilon/|w|)^{-s}$ (the constant depending on the maximum of F on $\mathcal{C}_1$) and hence $|z^r||w^s||a_{r,s} - X|$ is exponentially decreasing, where

$$\begin{aligned} X &= (2\pi i)^{-1} \int_{\mathcal{N}} z^{-r-1} g(z)^{-s} \psi(z)\, dz \qquad (3.5) \\ &= (2\pi i)^{-1} z_0^{-r} w_0^{-s} \int_{\mathcal{N}} \frac{z^{-r}}{z_0^{-r}} \frac{dz}{z} \left(\frac{g(z)}{g(z_0)}\right)^{-s} \psi(z)\,. \end{aligned}$$

Changing variables to $z = z_0 e^{i\theta}$ and $dz = iz d\theta$ turns the quantity X into

$$(2\pi)^{-1} z_0^{-r} w_0^{-s} \int_{\mathcal{N}'} e^{-ir\theta} \tilde{\psi}(\theta) \left(\frac{g(z)}{g(z_0)}\right)^{-s} d\theta$$

and plugging in the definition of $\tilde{f}$ yields

$$(2\pi)^{-1} z_0^{-r} w_0^{-s} \int_{\mathcal{N}'} \exp(-s\tilde{f}(\theta)) \tilde{\psi}(\theta)\, d\theta$$

which is none other than Ξ. □

3.1 Laplace integrals

In this section, we let C_0^∞ denote the class of smooth (C^∞) functions supported on a compact subset of $\mathbf{R}$. Choose $\psi \in C_0^\infty$.

> **Question:** What is $\int_0^\infty e^{i\lambda x^k} \psi(x)\, dx$, and why?

While you think about, I will go on to outline our treatment of oscillating integrals. Our aim is to treat integrals of the form

$$\int e^{\lambda f(x)} \psi(x)\, dx$$

where $f(x)$ may have an imaginary component which causes the integrand to oscillate (rapidly, as $\lambda \to \infty$). To see what is going on, we begin with a real integral. Let $k \geq 2$ and

$l \geq 0$ be integers. Let $t = \lambda x^k$ so that $x = t^{1/k}/\lambda^{1/k}$ and $dx = k^{-1}t^{1/k-1}/\lambda^{1/k}$. Then

$$\begin{aligned} \int_0^\infty e^{-\lambda x^k} x^l \, dx &= \int_0^\infty e^{-t} \frac{t^{l/k}}{\lambda^{l/k}} \frac{t^{1/k-1}}{k\lambda^{1/k}} \, dt \\ &= \frac{1}{k\lambda^{(l+1)/k}} \Gamma\left(\frac{l+1}{k}\right). \end{aligned} \tag{3.6}$$

This leads immediately to

Proposition 3.2 *Let $\psi \in C_0^\infty$ and let $b_j = \psi^{(j)}(0)/j!$. Then as $\lambda \to \infty$, the integral $\int_0^\infty e^{-\lambda x^k} \psi(x) \, dx$ has asymptotic development*

$$\sum_{l=0}^\infty a_+(k,l) b_l \lambda^{-(l+1)/k},$$

where

$$a_+(k,l) := k^{-1}\Gamma\left(\frac{l+1}{k}\right).$$

Remark: When k is even it makes sense to integrate instead from $-\infty$ to ∞. We see by symmetry that the integral in (**??**) will then vanish for odd values of l and double for even values of l. Thus it makes sense to extend our notation to define $a_-(k,l) = (-1)^l a_+(k,l)$ when k is even, and to let $a(k,l) = a_+(k,l) + a_-(k,l)$, also defined only for even k. When k is even, Proposition **??** then holds with a instead of a_+ and the integral going from $-\infty$ to ∞.

Proof: Define $P_N(x) := \sum_{l=0}^N b_l x^l$ and $R_N(x) = x^{-(N+1)}(\psi(x) - P_N(x))$. Choose b greater than any point in the support of ψ and note that on $[0,b]$, $|R_N| \leq C$ where $C := \sup_{0 \leq x \leq b} \psi^{(N+1)}(x)/(N+1)!$. Each function $e^{-\lambda x^k} x^l$ is integrable, hence we may write

$$\int_0^\infty e^{-\lambda x^k} \psi(x) \, dx = \int_0^\infty e^{-\lambda x^k} x^{N+1} R_N(x) \, dx + \sum_{l=0}^N b_l \int_0^\infty e^{-\lambda x^k} x^l \, dx \, .$$

This proves the proposition, if the first integral on the RHS is $O(\lambda^{-(N+2)/k})$. This is easy:

$$\left| \int_0^\infty e^{-\lambda x^k} x^{N+1} R_N(x) \right| dx$$

$$
\begin{aligned}
&\leq \int_0^b \left| e^{-\lambda x^k} x^{N+1} R_N(x) \right| dx + \left| \int_b^\infty e^{-\lambda x^k} P_N(x) \right| dx \\
&\leq C \int_0^\infty \left| e^{-\lambda x^k} x^{N+1} \right| dx + \sum_{l=0}^N |b_l| \int_b^\infty e^{-\lambda x^k} x^l \, dx \\
&\leq C' \lambda^{-(N+2)/k} + C'' e^{-\lambda b^k}, .
\end{aligned}
$$

Thus the RHS is indeed $O(\lambda^{-(N+2)/k})$ and the bound is in terms of b_l for $l \leq N$ and $\sup_{0 \leq x \leq b} \psi^{(N+1)}(x)/(N+1)!$. □

Remark: If the integral is an analytic function of λ, our motivating question is answered by replacing λ with $i\lambda$, introducing a factor of $e^{i\pi(l+1)/(2k)}$ so that the expansion in Proposition **??** now has coefficient $a_+(k,l) b_l e^{i\pi(l+1)/k}$ for the $\lambda^{-(l+1)/k}$ term. The philosophical point here is that oscillation (change of phase) kills the integral just about as fast as amplitude decay (change in the real part of the exponent).

Our outline for generalizing this proposition is as follows. The proof of Proposition **??** is already valid when ψ is complex-valued. Next, we replace $-x^k$ by any $-f \in C_0^\infty$ vanishing to order k at zero, as long as the support of ψ is sufficiently small. Next we allow f to take complex values, as long as the strict minimum of $\text{Re}\{f\}$ occurs at $f(0) = 0$. Finally, the hardest case is when $f = i\phi$ with the real function $\phi \in C_0^\infty$. This is the "purely oscillatory" case. The non-decay of the magnitude of the integrand requires some extra care in handling, though in the end we will see that the computation in the preceding remark can be justified.

Fix a real valued $f \in C_0^\infty$. Denote $c_k := f^{(k)}(0)/k!$, and assume that for some $k \geq 1$, $c_k > 0$ and each c_j vanishes for $j < k$. (This is what I mean by a function vanishing to order k at zero.) Choose b less than the first positive value at which f' vanishes. Then the function $y(x) := [f(x)/c_k]^{1/k}$ is a diffeomorphism of $[0, B]$ to $[0, B^*]$. The derivatives at 0 of the inverse function $x = F(y)$ are easy to compute formally. Since this inversion is computationally necessary, we show the first few terms; in general note that the first $j+1$

coefficients of F depend only on the first j coefficients of F starting at c_k.

$$F(y) = y - \frac{1}{k}\frac{c_{k+1}}{c_k}y^2 + \left(\frac{3+k}{2k^2}\frac{c_{k+1}^2}{c_k^2} - \frac{1}{k}\frac{c_{k+2}}{c_k}\right)y^3 + O(y^4).$$

Let ψ be a smooth complex-valued function with compact support in $[0, B)$. Again let b_j denote $\psi(j)(0)/j!$. Defining $\tilde{\psi} := (\psi \circ F) \cdot F'$, we denote $\tilde{b}_j := \tilde{\psi}^{(j)}(0)/j!$. The first few of these are then given as follows (see the later derivation at (**??**)).

$$\begin{aligned} \tilde{b}_0 &= b_0 \\ \tilde{b}_1 &= \left(b_1 - 2\frac{b_0}{k}\frac{c_{k+1}}{c_k}\right) \\ \tilde{b}_2 &= \left(b_2 - 3\frac{b_1}{k}\frac{c_{k+1}}{k} + 3b_0\left(\frac{3+k}{2k^2}\frac{c_{k+1}^2}{c_k^2} - \frac{1}{k}\frac{c_{k+2}}{c_k}\right)\right). \end{aligned} \tag{3.7}$$

In general the first j depend only on the first j coefficients of f and ψ, and the first nonvanishing $\tilde{b}_l$ is equal to the first nonvanishing b_l. Changing variables gives $\int_0^B e^{-\lambda f(x)}\psi(x)\,dx = \int_0^{B^*} e^{-(c_k\lambda)y^k}\tilde{\psi}(y)\,dy$, giving:

Theorem 3.3 *There is an asymptotic development*

$$\int_0^\infty e^{-\lambda f(x)}\psi(x)\,dx \sim \sum_{l=0}^{\infty} a_+(k,l)\tilde{b}_l c_k^{-(l+1)/k}\lambda^{-(l+1)/k} \tag{3.8}$$

where the coefficients $\tilde{b}_l$ depend only on b_j and c_{k+j} for $j \le l$, and the constant in front of the $\lambda^{-(N+1)/k}$ remainder term may be bounded in terms of the supremum of the first N derivatives of f and ψ on the support of ψ. When k is even, the same result holds with $a(k,l)$ in place of $a_+(k,l)$ and $\int_{-\infty}^{\infty}$ in place of $\int_0^\infty$. □

Remark: The constants c_k and λ may be lumped together: replacing f by f/c_k and λ by $c_k\lambda$ it is clear in advance that the dependence on c_k and λ will be via the product $c_k\lambda$. The reason for separating out the c_k dependence as above is that when f is complex, the phase of the integral will rotate with powers of $c_k^{1/k}$, and subsuming this into the definition of $\tilde{b}_l$ would be less clear.

3.2 Complex phase

It is now almost trivial to extend this to the case where the exponent is complex. Let $f : \mathbf{R}^+ \to \mathcal{C}$ be smooth and let c_j denote $f^{(j)}(0)/j!$. Let k be minimal such that $c_k \neq 0$; we assume $1 \leq k < \infty$. Let m be minimal such that $\mathrm{Re}\{c_m\} \neq 0$. Since we will need $\mathrm{Re}\{f\} > 0$ away from 0, we assume of necessity that $\mathrm{Re}\{c_m\} > 0$.

Theorem 3.4 *Under the above assumptions on f, let ψ be a smooth function supported in $(0, b)$, where $f(0) = 0$, f' has no zeros in $(0, b)$ and f maps $(0, b)$ into $\{z : \mathrm{Re}\{z\} > 0\}$. Let b_j denote $\psi^{(j)}/j!$. Then the asymptotic development (**??**) holds, with the constant in the $O(\lambda^{-(N+1)/m})$ term depending on the derivatives of f and ψ up to $(N+1)m/k - 1$. The numbers $c_k^{l/k}$ must be understood as powers of the principal root $c_k^{1/k}$, namely the root with argument between $-\pi/(2k)$ and $\pi/(2k)$. When k is even and the assumptions of the theorem are extended to an interval $(-b, b)$, with the integral extended to the whole real line, the expansion holds but the terms with odd values of l disappear and the even terms are doubled.*

PROOF: As $x \to 0^+$, the value of $f(x)$ approaches 0 staying in the right half plane. This time we change variables to $z = f^{1/k}$. Define G by $x = G(z)$, and $\tilde{\psi} := (\psi \circ G) \cdot G'$. Since $z = c_k^{1/k} y$, we have $G = F \circ (z \mapsto c_k^{-1/k} z)$, so the j^{th} coefficient of $\tilde{\psi}$ is $c_k^{-(j+1)/k} \tilde{b}_j$. The integral $\int_0^\infty e^{-\lambda f(x)} \psi(x)\, dx$ now becomes

$$\int_\gamma e^{-\lambda y^k} \tilde{\psi}(y)\, dy$$

where γ is a contour from 0 to $b^* := f(b)^{1/k}$. For $0 < N < M$, write $\tilde{\psi}$ as $P_N + P_{N,M} + x_{M+1} R_M$ where $P_{N,M}$ is a series of monomials of degrees $N+1, \ldots, M$. Then the integral becomes the sum of three terms:

$$\int_\gamma e^{-\lambda y^k} P_N(y)\, dy + \int_\gamma e^{-\lambda y^k} P_{N,M}(y)\, dy + \int_\gamma e^{-\lambda y^k} x^{M+1} R_M(y)\, dy$$

The first two of these integrals may be evaluated by moving the contour. Since f stays in the right half plane, the $1/k$ powers may all be chosen to have arguments between $-\pi/(2k)$

and $\pi/(2k)$. Let γ' be the line segment $[0,\epsilon]$ followed by the line segment from ϵ to b^*, where ϵ is small enough so that f and $\tilde{\psi}$ are analytic in the domain bounded by γ and γ' (choosing b as small as necessary as well). Then integrating over $[0,\epsilon]$ gives the series (**??**) out to term $N+M$ (recalling that the l^{th} coefficient of $\tilde{\psi}$ is $c_k^{-(l+1)/k}\tilde{b}_l$). The integral over the segment from ϵ to b^* is exponentially small in λ, since the real part of f is bounded away from zero there.

To bound the third integral, parameterize γ by arc-length. The arc-length to the point $f(t)$ on γ is given by $s(t) \sim c_k t^k$ as $t \to 0$. Since the real part of $f(t)$ is at least a constant times t^m, we see that $\mathrm{Re}\{\gamma(t)^k\} \geq C s^{m/k}$. Thus an upper bound for the absolute value of the third integral is

$$\int_0^\infty e^{-\lambda C t^m} |t^{M+1} R_M(\gamma(t))| \, dt.$$

As before, we see this is bounded by $C'\lambda^{-(M+1)/m}$, where C' depends on the first M derivatives of f and ψ. Choosing $M \geq m(N+1)/k$ we have a remainder term that is $O(\lambda^{-(N+1)/k})$, and we are done. □

When $k \geq 3$ is odd[2], the only way it is possible for f to be defined smoothly on an interval $[-\epsilon,\epsilon]$ and remain in the right half plane is to have c_k purely imaginary and to have m even. In this case we obtain the following corollary.

Corollary 3.5 *Suppose f is a smooth function mapping $(-b,b)$ into the right half plane, strictly except at $f(0)=0$, and ψ is smooth and compactly supported in $(-b,b)$. Define $k, m, \{\tilde{b}_j : j \geq 0\}$ and $\{c_j : j \geq k\}$ as before, assume $k \geq 3$ is odd, and define*

$$a(k,l) := a_+(k,l)\left[e^{i(l+1)\pi/(2k)} + (-1)^l e^{-i(l+1)\pi/(2k)}\right]. \tag{3.9}$$

Then if k is odd, there is an asymptotic development

$$\int_{-\infty}^{\infty} e^{-\lambda f(x)}\psi(x)\,dx \sim \sum_{l=0}^{\infty} a(k,l)\tilde{b}_l \mathrm{sgn}(\arg(c_k))|c_k|^{-(l+1)/k}\lambda^{-(l+1)/k}$$

[2]When $k=1$ one finds that all the coefficients vanish, corresponding to the fact that the two-sided integral is rapidly decreasing at a non-stationary point.

Proof: The integral from 0 to ∞ is given by (**??**). Changing variables to $-x$ instead of x, we see that the integral from $-\infty$ to 0 is the same but with c_k replaced by $\overline{c_k} = -c_k$ and with $f(x)/c_k$ replaced by $f(-x)/(-c_k)$. The term $\tilde{b}_l c_k^{-(l+1)/k}$ is then replaced by a sum

$$\tilde{b}_l c_k^{-(l+1)/k} + (-1)^l \tilde{b}_l \overline{c_k^{-(l+1)/k}}$$

leading to a combined contribution of

$$2\tilde{b}_l c^{-(l+1)/k} \left[e^{i(l+1)\pi/(2k)} + (-1)^l e^{-i(l+1)\pi/(2k)} \right] ,$$

proving the corollary. □

3.3 Purely oscillating integrals

Theorem **??** fails to be sharp in one respect. The magnitude of the $\lambda^{-(l+1)/k}$ error term should still be bounded in terms of the first l derivatives of f and ψ. In other words, our approach begins to be inefficient when the oscillation is of greater magnitude than the amplitude decay. The logical continuation of this is that when there is only oscillation and no amplitude decay at all, a completely new approach is needed.

The prototypical oscillating integral is an integral of the form

$$\int \exp(i\lambda\phi(\mathbf{x}))\psi(\mathbf{x})\,\mathbf{d}\lambda \tag{3.10}$$

where ϕ and ψ are analytic functions, $\psi \in C_0^\infty$, and we desire asymptotics as $\lambda \to \infty$. Integrals of this form are governed by the existence of stationary phase points (points where $\phi' = 0$) and by the behavior of ϕ and ψ near such points. If there are no such points then the integral is typically exponentially small in λ (we have now seen why this should be true "by analytic continuation").

The wrinkle in this case is that the magnitude of the integrand does not decay away from the stationary point, so the step in the proof of Theorem **??** in which the contour is moved to the real axis will have a contribution potentially too large to ignore. We therefore take advantage of the well known partial integration approach to determining the asymptotic development of purely oscillating integrals. Following Stein (1993), we begin with a localization principle.

Lemma 3.6 (localization lemma) *Define*

$$I(\lambda) := \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx \tag{3.11}$$

where ϕ and ψ are given functions in C^∞, with ψ having compact support in (a,b). Suppose $\phi'(x) \neq 0$ for all $x \in (a,b)$. Then $I(\lambda)$ is rapidly decreasing, i.e.,

$$I(\lambda) = O(\lambda^{-N}) \qquad \text{as } \lambda \to \infty$$

for any $N \geq 0$.

Proof: The smooth vanishing of ψ at the endpoints allows us to integrate by parts without introducing boundary terms. Let $dU = i\lambda\phi' e^{i\lambda\phi}\,dx$ and $V = \psi/(i\lambda\phi')$ to get

$$I(\lambda) = -\int_a^b e^{i\lambda\phi(x)} \frac{d}{dx}\left(\frac{\psi}{i\lambda\phi'}\right)(x)\,dx.$$

For any $N \geq 1$ we may repeat this N times to obtain

$$I(\lambda) = -\int_a^b e^{i\lambda\phi(x)}\lambda^{-N}\mathcal{D}^N(\psi)(x)\,dx \tag{3.12}$$

where $\mathcal{D}$ is the differential operator $f \mapsto (d/dx)(f/i\phi')$. Letting

$$A(N,\psi) = \sup_{a\leq x\leq b} |\mathcal{D}^n\psi(x)|,$$

we see that

$$I(\lambda) \leq \lambda^{-N} A(N,\psi)$$

which proves that $I(\lambda)$ is rapidly decreasing. □

We call this the localization lemma for the following reason. Suppose in (**??**) we allow ϕ' to vanish on some finite set of points $x_1,\ldots,x_d \in [a,b]$. Then I claim that the contribution to

$I(\lambda)$ from any closed region not containing some x_i is rapidly decreasing, so the asymptotics for $I(\lambda)$ may be read off as the sum of contributions local to each x_i. Indeed, for each i let $[a_i, b_i]$ be tiny intervals containing x_i, with all intervals disjoint and let $\xi_1, \ldots, \xi_d$ be a partition of unity subordinate to $\{[a_i, b_i] : 1 \leq i \leq d\}$. Once we see how to obtain asymptotics in a neighborhood of x_i containing no other critical points, we can write $\psi = \psi_0 + \sum_{i=1}^d \psi \xi_i$, so that the support of ψ_0 contains no x_i. By the localization lemma, $\int e^{i\lambda\phi(x)} \psi_0(x)\, dx$ is rapidly decreasing. It follows that as long as the integrals $I_i(\lambda) := \int_a^b e^{i\lambda\phi(x)} \psi_i(x)\, dx$ sum to something not rapidly decreasing, the asymptotic development of $I(\lambda)$ is gotten by summing the developments of $I_i(\lambda)$.

In the case of Lemma **??** a simpler proof would be to change variables to $y = \phi(x)$. This reduces the result to the more familiar statement that the Fourier transform of a smooth function $\psi \circ \phi^{-1}$ decreases rapidly. I chose the above proof because the assumption that $\phi' \neq 0$ is about to go out the window.

The assumption of compact support is especially important in the purely oscillatory case. In this case the modulus of the integrand is always order 1, so an integral over an interval near whose endpoints ψ does not vanish will have boundary contributions on the possible order of λ^{-1}. For example,

$$\int_a^b e^{i\lambda x} = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$$

which is $\Theta(\lambda^{-1})$ when λ avoids multiples of $2\pi(b-a)^{-1}$. It should however be noted that integration over a compact manifold without boundary qualifies as compact support. Thus if $\mathcal{C}$ is e.g., the unit circle, the integral $\int_{\mathcal{C}} \exp(i\lambda\phi(x))\psi(x)\, dx$ may be done by parts, with no boundary terms ensuing. If ϕ' is nonvanishing then the integral is rapidly decreasing.

Define some shorthand for several classes of integrals:

$$I(\lambda; \phi, \psi) = \int_{-\infty}^{\infty} e^{i\lambda\phi(x)} \psi(x)\, dx; \tag{3.13}$$

$$I(\lambda, k; \psi) = \int_{-\infty}^{\infty} e^{i\lambda x^k} \psi(x)\, dx; \tag{3.14}$$

$$I(\lambda, k, l, \delta) \quad = \quad \int_{-\infty}^{\infty} e^{i\lambda x^k} e^{-\delta|x|^k} x^l \, dx. \tag{3.15}$$

Let I_+ (resp. I_-) with the same sets of arguments denote integrals from 0 to ∞ (resp. $-\infty$ to 0) of the same integrands. Our final result for one-variable purely oscillating integrals will be the asymptotic development of integrals defined as follows.

Theorem 3.7 *Let ϕ and ψ be smooth real functions with $\psi \in C_0^\infty$. Assume that the order of the first nonvanishing derivative of ϕ at 0 is some number $k \geq 2$ and that 0 is the unique value in the support of ψ at which ϕ' vanishes. Let $b_l := \psi^{(l)}(0)/l!$ denote the Taylor coefficients of ψ and let $\tilde{\psi} := (\psi \circ F) \cdot F'$ where F is the inverse function to $x \mapsto (\phi/c_k)^{1/k}$ where $c_l := \phi^{(l)}(0)/l!$. Then as $\lambda \to \infty$, there is an asymptotic development*

$$I(\lambda; \phi, \psi) = \sum_{l=0}^{\infty} A(k, l) \tilde{b}_l c_k^{-(l+1)/k} \lambda^{-(l+1)/k} .$$

*The coefficient of $\lambda^{-(l+1)/k}$ is a continuous function of the first l derivatives of ϕ and ψ at 0 (for $l \leq 2$ see the explicit equation (**??**)), and the constant in the $O(\lambda^{-(N+1)/k})$ remainder term is bounded by a continuous function of the suprema of the first $N + 1$ derivatives of ϕ and ψ on the support of ψ. An expansion of exactly the same form holds for I_+ and I_-.*

An outline of the proof is as follows. We first restrict attention to the case where $\phi(x) = x^k$. Ideally we would like to solve the case where $\psi(x) = x^l$ as well, and then use the Taylor expansion of ψ to sum these. Since x^l is not compactly supported, this does not make sense, but if we throw in a Gaussian cutoff factor of $e^{-\delta x^2}$ we may then carry out this program. Sending δ to zero, we find that the x^l term of ψ contributes to just one term of the asymptotic expansion, so summing over the Taylor expansion of ψ easily gives the full asymptotic development. More general phase functions ϕ may be handled by a change of variables which reduces to the special case $\phi(x) = x^k$ but with a different function ψ. The effect of this change of variables on the coefficients of the asymptotic development are easily computed by recursion, though we do not present a closed form expression for them, except for the first several terms.

Two lemmas will be useful. The first is an estimate on the magnitude of an oscillating integral when the amplitude at the stationary point vanishes to a given order.

Lemma 3.8 *If $\eta \in C_0^\infty$ and $l \geq 1$ and $k \geq 2$ are integers, then*

$$\left| \int_{-\infty}^{\infty} e^{i\lambda x^k} x^l \eta(x) \, dx \right| \leq C \lambda^{-(l+1)/k} \tag{3.16}$$

for a constant C depending on k, l and the first l derivatives of η. The same estimate holds if the integral is over the positive half-line and the closed support of η is contained in $(0, \infty)$.

An immediate corollary is

Corollary 3.9 *Suppose g is smooth, vanishing in an neighborhood of 0, and that $g(x) = O(|x|^{-N})$ for every N. Then*

$$\int e^{i\lambda x^k} g(x) \, dx = O(\lambda^{-N})$$

for every N.

Proof of lemma: Let α be a smooth function equal to 1 on $|x| \leq 1$ and vanishing on $|x| \geq 2$. Choose an $\epsilon > 0$ and rewrite (**??**) as

$$\int e^{i\lambda x^k} x^l \eta(x) \alpha(x/\epsilon) \, dx + \int e^{i\lambda x^k} x^l \eta(x) [1 - \alpha(x/\epsilon)] \, dx. \tag{3.17}$$

The absolute value of the first integrand is at most $|x|^l ||\eta||^\infty \mathbf{1}_{|x| \leq 2\epsilon}$, yielding an integral of at most $C_1 \epsilon^{l+1}$ where $C_1 = ||\eta||_\infty 2^{l+1}/(l+1)$.

The second integral will be done by parts, and to prepare for this we examine the iteration of the operator $D := (d/dx)(\cdot / x^{k-1})$ applied to the function $x^l \eta(x)(1 - \alpha(x/\epsilon))$. The result will be a sum of monomials, each monomial being a product of a power of x, a derivative of η, a derivative of α and a power of ϵ. In fact if (a, b, c, d) is shorthand for $x^a \eta^{(b)}(x) \alpha^{(c)}(x/\epsilon) \epsilon^{(d)}$, and $a \geq 0$, then

$$D(a,b,c,d) = (a-k+1)(a-k,b,c,d) + (a-k+1,b+1,c,d) + (a-k+1,b,c+1,d-1).$$

By induction, we see that $D^N(a,b,c,d)$ is the sum of terms $C \cdot (r,s,t,u)$ with $r+u \geq a+d-kN$, $s \leq b+N$, $t \leq c+N$, and $C \leq (kN \vee a)!$. In particular, since $\epsilon \leq x$ we may replace positive powers of ϵ by the same power of x to arrive at the upper bound:

$$\left| D^N \left[x^l \eta(x)(1-\alpha(x/\epsilon)) \right] \right| \leq \mathbf{1}_{|x| \geq \epsilon} C |x|^{l-kN} \tag{3.18}$$

where C is the product of $\sup_{j \leq N, |x| \in (1,2)} \eta^{(j)}(x)$ and $\sup_{j \leq N, |x| \in (1,2)} \alpha^{(j)}(x)$.

Now we fix an $N \geq 1$ and integrate the second integrand of (**??**) by parts N times, each time integrating $-ik\lambda x^{k-1} e^{i\lambda x^k}$ and differentiating the rest. The resulting integral is

$$\int e^{i\lambda x^k} (-ik\lambda)^{-N} D^N \left[x^l \eta(x)(1-\alpha(x/\epsilon)) \right] \, dx.$$

By (**??**), the modulus of the integrand is at most $C \mathbf{1}_{|x| \geq \epsilon} |x|^{l-kN} (k\lambda)^{-N}$, which integrates to at most $C_2 \lambda^{-N} \epsilon^{l-kN+1}$. Set $\epsilon = \lambda^{-1/k}$ and add the bounds on the two integrals to obtain an upper bound on $\int e^{i\lambda x^k} g(x) \, dx$ of $(C_1 + C_2) \lambda^{-(l+1)/k}$. We have also shown that C_1 and C_2 depend only on k, l, the first l derivatives of η and the first l derivatives of α. We may take α always to be a fixed, convenient function, and incorporate its derivatives into our constants. This proves the lemma for integration over $\mathbf{R}$. For integration over $\mathbf{R}^+$, the proof is exactly the same, with the absolute value of the first integrand bounded identically, and the second integration by parts done only on the right half-line (no boundary effects occur since 0 is not in the support of $1-\alpha$). □

The second lemma computes the asymptotics when ϕ and ψ are monomials and a damping factor of $e^{-\delta x^k}$ is imposed.

Lemma 3.10 *As $\lambda \to \infty$, there is an asymptotic development*

$$I(\lambda, k, l, \delta) = \lambda^{-(l+1)/k} \sum_{j=0}^{\infty} a(j,k,l,\delta) \lambda^{-j}.$$

The same holds for I_+ and I_-, whose coefficients are denoted respectively by a_+ and a_-. The constants in the N^{th} remainder term remain bounded (in fact go to 0) as $\delta \to 0$.

PROOF: Let $z = (\delta - i\lambda)^{1/k}x$, where we choose the principal branch of $w^{1/k}$ i.e.,

$$|\mathrm{Arg}(w^{1/k})| \leq \pi/(2k).$$

The quantity $I_+(\lambda, k, l, \delta)$ may be written as

$$\int_0^{\infty(\delta - i\lambda)^{1/k}} e^{-z^k}(\delta - i\lambda)^{-l/k} z^k \frac{dz}{(\delta - i\lambda)^{1/k}} \, .$$

The integrand decreases rapidly as $z \to \infty$ with $|\mathrm{Arg}(z)| \in [0, \pi/2 - \epsilon]$, so we may rotate the contour of integration back to the positive real line, obtaining

$$I_+(\lambda, k, l, \delta) = (\delta - i\lambda)^{-(l+1)/k} \int_0^\infty e^{-x^k} x^l \, dx$$

taking the l/k power to be the l power of the principal $1/k$ power. The definite integral has value $a_+(k, l) = k^{-1}\Gamma((l+1)/k)$. Writing $(\delta - i\lambda)^{-(l+1)/k}$ as $(-i\lambda)^{-(l+1)/k}(1 + \delta i/\lambda)^{-(l+1)/k}$, the binomial theorem then gives

$$I_+(\lambda, k, l, \delta) = a_+(k, l) e^{i\pi(l+1)/(2k)} \lambda^{-(l+1)/k} \sum_{j=0}^{\infty} (i\delta)^j \binom{-(l+1)/k}{j} \lambda^{-j}.$$

This proves the lemma with

$$a_+(j, k, l, \delta) = k^{-1}\Gamma\left(\frac{l+1}{k}\right) e^{i\pi(l+1+jk)/(2k)} \binom{-(l+1)/k}{j} \delta^j. \tag{3.19}$$

The proof for I_- is identical, and the result for I follows from $I = I_+ + I_-$. □

Since we are going to send δ to 0, we do not at this point need explicit forms for the coefficients $a_+(j, k, l, \delta)$. Instead, we compute limits as $\delta \to 0$. For $j > 1$, $a_+(j, k, l, \delta) \to 0$ as $\delta \to 0$, and the same with a_- and a. When $j = 0$, we obtain $A_+(k, l)$ easily from (**??**). To obtain $A_-(k, l)$ observe that when k is even, $A_-(k, l) = (-1)^l A_+(k, l)$, while when k is odd, $A_-(k, l) = (-1)^l \overline{A_+(k, l)}$. This leads to

$$A_+(k, l) \ := \ \lim_{\delta \to 0^+} a_+(0, k, l, \delta) = k^{-1}\Gamma\left(\frac{l+1}{k}\right) e^{i\pi(l+1)/(2k)} \ ; \tag{3.20}$$

$$A_-(k, l) \ := \ \lim_{\delta \to 0^+} a_-(0, k, l, \delta) = (-1)^l k^{-1}\Gamma\left(\frac{l+1}{k}\right) e^{(-1)^k i\pi(l+1)/(2k)} \ ; \tag{3.21}$$

$$A(k,l) \;:=\; \lim_{\delta\to 0^+} a(0,k,l,\delta) = \frac{2}{k}\Gamma\left(\frac{l+1}{k}\right) e^{i\pi(l+1)/(2k)} \delta_{l\equiv 0_{(2)}} \quad \text{if } k \text{ is even;} \tag{3.22}$$

$$A(k,l) \;:=\; \lim_{\delta\to 0^+} a(0,k,l,\delta) = \frac{1}{k}\Gamma\left(\frac{l+1}{k}\right) \left[e^{i\pi(l+1)/(2k)} + (-1)^l e^{-i\pi(l+1)/(2k)}\right] \quad \text{if } k \text{ is odd.} \tag{3.23}$$

We are now ready to prove a restricted version of Theorem **??**.

Theorem 3.11 *Let $\psi \in C_0^\infty$ with 0 in the support of ψ. Let $b_j = \psi^{(j)}(0)/j!$. Then $I(\lambda, k; \psi)$ has asymptotic development*

$$I(\lambda, k; \psi) \approx \lambda^{-1/k} \sum_{j=0}^\infty b_j A(k,j) \lambda^{-j/k}$$

and the same with I and A replaced by $I_\pm$ and $A_\pm$. The constant in the $O(\lambda^{-N/k})$ remainder term is bounded in terms of the suprema of the first N derivatives of ψ near 0.

PROOF: We prove the result only for I_+, since the proof for I_- is identical and the result for I follows by summing. Let U be a smooth function that is 1 on the support of ψ and vanishes outside of a compact set. Fix $N \geq 1$ and $\delta > 0$ and define the polynomial $P(x) = P_{N,\delta}(x)$ to be the sum of the Taylor series for $e^{\delta x^k}\psi(x)$ through the x^N term. Let $b_{j,\delta}$ denote the Taylor coefficients. Define the normalized remainder term $R(x) = R_{N,\delta}(x)$ by $e^{\delta x^k}\psi(x) = P(x) + x^{N+1}R(x)$. Now represent $I_+(\lambda, k; \psi)$ as

$$\int_0^\infty e^{i\lambda x^k} \psi(x)\, dx = \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} \left[e^{\delta x^k}\psi(x)\right] U(x)\, dx = A + B + C$$

where

$$\begin{aligned}
A &:= \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} P(x)\, dx \,; \\
B &:= \int_0^\infty e^{i\lambda x^k} x^{N+1} e^{-\delta x^k} R(x) U(x)\, dx \,; \\
C &:= \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} P(x)(U(x) - 1)\, dx \,.
\end{aligned}$$

By Lemma **??**,

$$\begin{aligned} A &= \sum_{l=0}^{N} b_{l,\delta} \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} x^l \, dx \\ &= \sum_{l=0}^{N} b_{l,\delta} I(\lambda, k, l, \delta) \\ &= \sum_{l=0}^{N} b_{l,\delta} \sum_{j=0}^{N} \lambda^{-(l+1)/k} a(j,k,l,\delta) \lambda^{-j} + O(\lambda^{-(l+1)/k - j(N+1)}) \\ &= \sum_{l=0}^{N} b_{l,\delta} a(0,k,l,\delta) \lambda^{-(l+1)/k} + \sum_{j=l+1}^{l+1+j(N+1)} \xi_j \lambda^{-j/k} + O(\lambda^{-(l+1)/k - j(N+1)}) \end{aligned}$$

where the coefficients ξ_j are sums of products of coefficients $b_{l,\delta}$ with coefficients $a(j,k,l,\delta)$ for which $j \geq 1$.

By Lemma **??** with $\eta(x) = e^{-\delta x^k} R(x) U(x)$ and $l = N+1$, we know that the magnitude of B is bounded by $K\lambda^{-(l+2)/k}$. Similarly, by Corollary **??**, we see that C is rapidly decreasing as $\lambda \to \infty$, hence of magnitude at most $K\lambda^{-l}$. Furthermore, in both cases K may be bounded in terms of k, l and the first l derivatives of ψ, the bound being uniform over δ in a neighborhood of 0. It follows that $I_+(\lambda, k; \psi)$ has, up to the $\lambda^{-(l+1)/k}$ term, the same asymptotic development as A. In particular, the asymptotics of A do not depend on δ to this point. As $\delta \to 0$, the coefficients $b_{l,\delta} \to b_l$. Thus we may send $\delta \to 0$, which annihilates all terms with $j \geq 1$ and leaves

$$I_+(\lambda, k; \psi) \approx \sum_{l=0}^{N} b_l A_+(k,l) \lambda^{-(l+1)/k}$$

which is the desired expansion. □

Proof of Theorem **??**: By assumption, $\phi(x) = c_k x^k (1 + \theta(x))$ where $\theta(x) = O(|x|)$. Let $y = x(1+\theta(x))^{1/k}$. This is a diffeomorphism in a neighborhood of 0, and we write $x = F(y)$ to denote its inverse. Then $c_k y^k = \phi(x)$ and so we may change variables to write

$$\int e^{i\lambda\phi(x)} \psi(x) \, dx = \int e^{i\lambda c_k y^k} \tilde{\psi}(y) \, dy \, ,$$

where $\tilde{\psi} = (\psi \circ F) \cdot F'$. Absorbing c_k into λ shows that

$$I(\lambda; \phi, \psi) = c_k^{-(l+1)/k} I(\lambda, k; \tilde{\psi}) \tag{3.24}$$

The derivatives of $\tilde{\psi}$ at zero are easily computed from derivatives at 0 of ϕ and ψ, and depend continuously on them, which complete the proof. $\square$

As promised earlier, we derive expressions for the first few terms of the expansion . We may express $y = (\phi/c_k)^{1/k}$ by

$$\begin{aligned} y &= x\left(1 + \frac{c_{k+1}}{c_k}x + \frac{c_{k+2}}{c_k}x^2 + O(x^3)\right)^{1/k} \\ &= x + \frac{1}{k}\frac{c_{k+1}}{c_k}x^2 + \left(\frac{1}{k}\frac{c_{k+2}}{c_k} + \frac{1}{k}\frac{1-k}{k}\frac{1}{2}\frac{c_{k+1}^2}{c_k^2}\right)x^3 + O(x^4). \end{aligned}$$

Inverting we get $x = F(y)$ where F is expressed as

$$y - \frac{1}{k}\frac{c_{k+1}}{c_k}y^2 + \left(\frac{3+k}{2k^2}\frac{c_{k+1}^2}{c_k^2} - \frac{1}{k}\frac{c_{k+2}}{c_k}\right)y^3 + O(y^4).$$

Then $F'(y) = 1 - (2/k)(c_{k+1}^2/c_k^2)y + 3[(3+k)/(2k^2)(c_{k+1}^2/c_k^2) - (1/k)(c_{k+2}/c_k)]y^2 + O(y^3)$ and $\psi \circ F(y) = b_0 + b_1 y + (b_2 - (b_1/k)(c_{k+1}/c_k))y^2 + O(y^3)$. Recalling that $\{b_j\}$ are the coefficients of ψ, and $\tilde{b}_j$ are the coefficients of $\tilde{\psi}$, we see that

$$\begin{aligned} \tilde{b}_0 &= b_0 \qquad (3.25) \\ \tilde{b}_1 &= b_1 - 2\frac{b_0}{k}\frac{c_{k+1}}{c_k} \\ \tilde{b}_2 &= b_2 - 3\frac{b_1}{k}\frac{c_{k+1}}{k} + 3b_0\left(\frac{3+k}{2k^2}\frac{c_{k+1}^2}{c_k^2} - \frac{1}{k}\frac{c_{k+2}}{c_k}\right). \end{aligned}$$

4 Oscillating integrals in several variables

In one variable, points where the phase function is stationary may be classified by their degree of vanishing: any two functions vanishing to order k are diffeomorphically equivalent.

In more than one variable, there are many possible geometries for the behavior of a function near a critical point. There are as many different oscillating integrals as there are geometries of zero sets of functions. For a wide-ranging treatment (but not all the proofs) see Arnold *et al* (1988). Rather than attempt a comprehensive treatment, I will present two basic theorems and then develop the estimates needed for later asymptotic computations.

4.1 Localization

As before, we let $f : \mathbf{R}^d \to \mathbf{C}$ be a smooth function, let $\psi \in C_0^\infty(\mathbf{R}^d)$, and denote $I(\lambda) := \int e^{-\lambda f(\mathbf{x})} \psi(\mathbf{x})\, d\mathbf{x}$. We assume throughout that $f(\mathbf{0}) = 0$ and that f has a critical point at the origin, i.e., $\bigtriangledown f(\mathbf{0}) = \mathbf{0}$. Letting

$$q_{ij} = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{0}} ,$$

we then have near the origin a Taylor expansion

$$f(\mathbf{x}) = \frac{1}{2} \sum_{i,j} q_{ij} x_i x_j + O(|\mathbf{x}|^3) .$$

Let Q denote the quadratic form $Q(\mathbf{x}) = \sum_{i,j} q_{ij} x_i x_j$ and let $M = M(Q)$ denote the matrix (q_{ij}) representing Q called the *Hessian* of f. The simplest case, and the only one we will need here, is when M has full rank. Accordingly, we define f to have a *nondegenerate stationary point* at the origin if $\bigtriangledown f(\mathbf{0}) = \mathbf{0}$ and if M is non-singular. Intuitively, this means f vanishes to order two but not to any higher order.

We should already be able to guess the expansion of $I(\lambda)$ in this case, via the following steps. (1) localize to a neighborhood of the origin. (2) by a change of variables, f becomes the sum of x_j^2 and ψ becomes $\tilde{\psi}$. (3) expand $\tilde{\psi}$ into monomials. (4) the integral of a monomial against a diagonal quadratic form factors into the product of one-dimensional integrals, of the type we have already done. As before, approximating $\tilde{\psi}$ by a polynomial requires a little work since polynomials are not compactly supported; this is summarized in Lemma **??** below. In addition, step 1 requires a new localization lemma and step 2 requires the Morse Lemma along with some easy linear algebra. We begin with the localization lemmas.

Lemma 4.1 (multivariate localization) *Let* $\psi \in C_0^\infty$ *map* $\mathbf{R}^d$ *to* $\mathcal{C}$ *and* $f \in C^\infty$ *map* $\mathbf{R}_d$ *to* $\mathbf{C}$ *with* $\text{Re}\{f\} \geq 0$ *and* $\bigtriangledown f$ *nonvanishing on the closed support of* ψ*. Then as* $\lambda \to \infty$*, the integral*

$$I(\lambda) := \int_{\mathbf{R}^d} e^{i\lambda f(\mathbf{x})} \psi(\mathbf{x}) \, \mathbf{d}\lambda$$

is $O(\lambda^{-N})$ *for any* $N > 0$*. The estimate is uniform when* $|\bigtriangledown f|$ *is bounded away from zero and depends only on the first* N *derivatives of* f *and* ψ*,and on* $\inf |\bigtriangledown f|$*.*

Proof: For each x in the support of ψ, there is some unit vector ξ_x and a ball B_x around x such that $\xi_x \cdot \bigtriangledown f(y) \geq c > 0$ for all $y \in B_x$. The number of balls necessary to cover and the proximity of c to zero may be bounded in terms of $\inf |\bigtriangledown f|$ and $\sup |\bigtriangledown f|$. We may write $I(\lambda)$ as a finite sum

$$\sum_k e^{i\lambda f(x)} \psi_k(x) \, dx$$

where each ψ_k is supported in some ball B_{x_k}. Fix k and let ξ_{x_k} be the first vector in an orthonormal frame $\xi_1, \ldots, \xi_d$. Write

$$\int e^{i\lambda f(x)} \psi_k(x) \, dx = \int \left(\int e^{i\lambda f(x)} \psi_k(x) \, d\xi_1 \right) \, d\xi_2 \cdots d\xi_d.$$

By the one-variable localization lemma, the inner integral is at most $A(N, \psi_k; f)\lambda^{-N}$, where A depends only on the supremum of the operator $(d/d\xi_1)(f/i(\partial f/\partial \xi_1))$ applied n times to $f := \psi$ [note that equation (**??**) assumes only that $|e^{i\lambda f(\mathbf{x})}| \leq 1$ and not that f is real]. Thus the inner integral is uniformly $O(\lambda^{-N})$ for any N, and by integrating we see that $I(\lambda)$ is as well. □

For later application, we will need a version of the localization lemma that works when the region of integration has a boundary, as long as there is some direction along the boundary in which the phase is changing. To set this up, we discuss a more general boundary condition. The domain of integration, D, will be assumed to be a *nice cell complex*, which we now define. A nice cell complex in d dimensions is built out of manifolds of different dimensions, at least one of which has dimension d. A j-manifold means an orientable manifold without boundary, so it may be diffeomorphic to an open disk in $\mathbf{R}^j$, or to something else such as the torus $(S^1)^j$. We define a nice cell complex in to be a disjoint union of manifolds, called its *cells*. If S and T are j and $j+1$ cells respectively, then we require that either S is disjoint from the closure $\overline{T}$ or else $S \subseteq \overline{T}$,. In the latter case, $S \cup T$ is locally diffeomorphic to a $j+1$ dimensional half-space. It follows that every point $x \in D$ has a well defined dimension, namely the dimension of the unique cell containing x, and that a point x of dimension j has a neighborhood in D diffeomorphic to the orthant $\mathbf{R}^j \times (\mathbf{R}^+)^{d-j}$.

Definition 2 *If D is a nice cell complex and $f : D \to \{z : \text{Re}\{z\} \geq 0\}$ is smooth, we call $\mathbf{x}$ a* stationary point *for f if $\text{Re}\{f\} = 0$ and $\bigtriangledown f(\mathbf{x})$ is orthogonal to the tangent space $T_x(S)$ at $\mathbf{x}$ to the unique cell S containing $\mathbf{x}$.*

Lemma 4.2 (cell-complex localization) *Let D be a nice cell complex and f a smooth function on D with nonnegative real part. Let $\psi \in C^\infty(D)$ and suppose that f has no stationary points in the support of ψ. Then*

$$I(\lambda) := \int_D \exp(-\lambda f(\mathbf{x}))\psi(\mathbf{x})\, d\mathbf{x}$$

is rapidly decreasing.

Sketch of Proof: For each $\mathbf{x}$ in the support of ψ, let $B_{\mathbf{x}}$ be a neighborhood of $\mathbf{x}$ in which either (i) : $\text{Re}\{f\}$ is bounded away from zero or else (ii) : there is a vector ξ in the tangent space to the cell S containing $\mathbf{x}$ such that $\pi_S(\bigtriangledown f) \cdot \xi$ is bounded away from zero; in the latter case, take a sub-neighborhood which is a product of a neighborhood A of $\mathbf{x}$ in S with another set B. By the hypotheses on D and f, these neighborhoods exist for each $\mathbf{x}$. Choose a finite subcover and a partition of unity subordinate to it. Then $I(\lambda)$ is the sum of finitely many contributions, each of one of two types. The contributions of the type (i) are all exponentially decreasing since the modulus of the integrand is exponentially small. The contributions of type (ii) may be written as integrals over B of integrals over A. Each integral over A may further be written as an integral in the ξ direction, then integrated in all other directions. As in the proof of Lemma **??**, the integral in the direction ξ is rapidly decreasing, and uniformity allows us to integrate in the remaining directions to finish the proof. □

4.2 Quadratically non-degenerate stationary points

The next lemma states a simple and well known fact about diagonalizing a quadratic form over the complex numbers. Recall that quadratic forms Q are in one-to-one correspondence with symmetric bilinear forms B by the correspondence $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$ and $B(\mathbf{x}, \mathbf{y}) = (1/4)(Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x} - \mathbf{y}))$ (the polarization identity). These are also in one-to-one correspondence with symmetric matrices once a basis is chosen. Given two bases $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_n\}$, if $(\mathbf{y}) = A(\mathbf{x})$, then the matrices $M_\mathbf{x}$ and $M_\mathbf{y}$ representing Q in the respective bases $(\mathbf{x})$ and $(\mathbf{y})$ are related by $M_\mathbf{y} = A^T M_\mathbf{x} A$. In particular, we see that quadratic (or bilinear) forms have a well defined *rank*, namely the rank of the representing matrix in any basis.

Lemma 4.3 (diagonalization of quadratic forms) *Let Q be a quadratic form of full rank on a d-dimensional complex vector space. Then there is a basis in which Q is represented by the identity matrix. Consequently, any nonsingular symmetric matrix over* $\mathbf{C}$ *may be factored as $A^T A$.*

PROOF: Induct on d, the statement being clear when $d = 1$. When $d > 1$, we observe that there must be an $\mathbf{x}$ with $Q(\mathbf{x}) \neq 0$ or else B and hence M would be singular by the polarization identity. Multiplying by a scalar, we assume that $Q(\mathbf{x}) = 1$. Write $V = \mathbf{x}\mathbf{C} + V'$ where V' is the B-complement of $\mathbf{x}$, namely the set of $\mathbf{y}$ for which $B(\mathbf{x}, \mathbf{y}) = 0$. Since the representing matrix M has full rank on V it has full rank on V', so by induction, there is a basis $\{\mathbf{y}_1, \ldots, \mathbf{y}_{d-1}\}$ for which $Q(\sum_{j=1}^{d-1} a_j y_j) = \sum_{j=1}^{d-1} a_j^2$, and adding $\mathbf{x}$ to the basis completes the induction. □

Continuing with background lemmas, we come to the Morse Lemma, which is the multi-dimensional analogue of the change of variables that straightened out f into a pure power in Theorem **??**. There are many proofs of the Morse Lemma in the literature. The one here is taken appears in Stein (1993) and Milnor (1969), though I have altered the description to a formal power series approach, and the complex case (ours) is a little simpler than the real case (theirs).

Lemma 4.4 (Morse Lemma) *Let f be a smooth complex-valued function on $\mathbf{R}^d$ whose expansion near the origin is given by*

$$f(\mathbf{x}) = \frac{1}{2} Q(\mathbf{x}) + O(|\mathbf{x}|^3)\,.$$

Suppose that the Hessian $M(Q)$ is non-degenerate. Then there are functions $y_1, \ldots, y_d$, analytic on a neighborhood of the origin, for which

$$f(\mathbf{x}) = \frac{1}{2} \sum_{j=1}^{d} y_j(\mathbf{x})^2$$

and Jacobian is a matrix A for which $A^T A = M(Q)$.

PROOF: Assume first that Q is represented by the identity matrix, that is, $Q(\mathbf{x}) = \sum_{j=1}^d x_j^2$. We write $2f(\mathbf{x})$ as $\sum_{i,j} x_i x_j \phi_{ij}$ where $\phi_{ij}(\mathbf{0}) = \delta_{ij}$ and $\phi_{ij} = \phi_{ji}$. It is obvious how to do this as a formal power series, hence if f is analytic at the origin. In the C^∞ category, one first writes $2f = \sum_{j=1}^d x_j g_j$ where $g_j = \int_0^1 \frac{\partial f}{\partial x_j}(t\mathbf{x})\, dt$; then, since g_j vanishes at the origin, it in turn may be written as $\sum_{i=1}^d x_i h_{ij}$; take $\phi_{ij} = (1/2)(h_{ij} + h_{ji})$. Observe that $\phi_{ij}(0) = \delta_{ij}$.

Now assume for induction that f is of the form $(1/2) \sum_{j=1}^{r-1} y_j^2 + \sum_{i,j \geq r} y_i y_j \tilde{\phi}_{ij}$, with $\tilde{\phi}(0) = \delta_{ij}$. Then replacing y_r by

$$y_r' := \tilde{\phi}_{rr}^{1/2} \left(y_r + \sum_{j>r} \frac{y_j \tilde{\phi}_{rj}}{\phi_{rr}} \right)$$

we find that $2f = \sum_{j=1}^{r-1} y_j^2 + (y_r')^2 + \sum_{i,j>r} y_i y_j \overline{\phi}_{ij}$ and $\overline{\phi}(0) = \delta_{ij}$. By induction, we can get to $r = d$ and $2f = \sum_{j=1}^d y_j^2$. The changes of variable from y_r to y_r' all have Jacobian at the origin equal to the identity matrix, so the final result does as well.

Finally, to handle the case of a general nonsingular Q, use the diagonalization lemma to write $Q = A^T A$. Then $f = \tilde{f} \circ A$ where A is the map $\mathbf{x} \mapsto A\mathbf{x}$ and the Hessian of $\tilde{f}$ is the identity. By the previous case, $2\tilde{f}(\mathbf{x}) = \sum_{j=1}^d y_j(\mathbf{x})^2$, whence $2f(\mathbf{x}) = \sum_{j=1}^d (y_j \circ A)^2$; since the Jacobian of $(\mathbf{x}) \mapsto \mathbf{y}$ is the identity, the chain rule shows that the Jacobian of $(\mathbf{x}) \mapsto (\mathbf{y}) \circ A$ is equal to A. $\square$

To make use of polynomial approximation, we need the multivariate analogue of Lemma **??** and Corollary **??**. The proof is the same as in the one variable case, except that we need to integrate by parts r_j times in the x_j-direction for each j.

Lemma 4.5 *Suppose that $f(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^d x_j^2$. Suppose further that ψ is smooth and compactly supported. For a multi-index $\mathbf{r}$ of nonnegative integers, we let*

$$I(\lambda) = \int \exp(-\lambda f(\mathbf{x}))\mathbf{x}^{\mathbf{r}}\psi(\mathbf{x})\, d\mathbf{x}\,.$$

Then

$$\left|\int e^{-\lambda f(\mathbf{x})}\mathbf{x}^{\mathbf{r}}\psi(\mathbf{x})\, d\mathbf{x}\right| = O(\lambda^{-\frac{d+|\mathbf{r}|}{2}})\,.$$

If ψ vanishes in a neighborhood of the origin then the integral is rapidly decreasing. $\square$

One final lemma involves the topology of $S^+ \subset \mathbf{C}^d$ defined as the set $\{(z_1, \ldots, z_d) : \mathrm{Re}\{z_1^2 + \cdots + z_d^2\} \geq 0\}$.

Lemma 4.6 *Let D^* be a the diffeomorphic image of a neighborhood of the origin in $\mathbf{R}^d$ by a map $\Psi : \mathbf{R}^d \to \mathbf{C}^d$ sending $\mathbf{0}$ to $\mathbf{0}$. Suppose $D^* \subseteq S^+$. Then there is a neighborhood $\mathcal{N}$ of the origin in $\mathbf{R}^d$ and a $(d+1)$-manifold (with boundary), Ω, having the following two properties.*

$$\mathbf{0} \notin \partial\Omega \setminus (D^* \cup \mathcal{N})\,; \tag{4.1}$$

$$\Omega \subseteq S^+\,. \tag{4.2}$$

Sketch of proof: Define a homotopy $H : D^* \times [0,1] \to \mathbf{C}^d$ by $H(\mathbf{z}, t) = \mathrm{Re}\{\mathbf{z}\} + (1-t)\mathrm{Im}\{\mathbf{z}\}$. Suppose H is a diffeomorphism and $H(\partial D^* \times [0,1])$ avoids $\mathbf{0}$. Then the first property is immediate for $\Omega = \mathrm{Image}(H)$, and the second property follows from the fact $D^* \in S^+$ together with the closure of S^+ under $x + iy \mapsto x + i\alpha y$ for $\alpha \in [0,1]$. If H is not a diffeomorphism, then either two elements of D^*, $\mathbf{x} + i\mathbf{y}$ and $\mathbf{x} + i\mathbf{y}'$ have the same real (d-dimensional) part, in which case the one with the larger imaginary part will be called the "offending element", or some nonzero "offending" element of D^* is real. In either case, the offending element of D^* is in the interior of S^+, so we may perturb the map H keeping the image inside S^+ and make it a local diffeomorphism. Doing this in a neighborhood of each offending point creates a diffeomorphism, with the two properties required. □

Finally, we put the lemmas together into the following theorem.

Theorem 4.7 *Let D be a nice cell complex in $\mathbf{R}^d$ containing the origin in its interior. Let $f : D \to \mathcal{C}$ be a smooth function and suppose the origin is the only stationary point of f on D, in the sense of Definition* **??** *and suppose that the quadratic approximation Q to f at the origin is nonsingular. Let ψ be another smooth function on D. Then the integral*

$$I(\lambda) := \int_D \exp(-\lambda f(\mathbf{x}))\psi(\mathbf{x})\, d\mathbf{x}$$

has an asymptotic development

$$I(\lambda) \sim \sum_{j=l}^{\infty} C_j \lambda^{-\frac{d+j}{2}}$$

where l is the order of vanishing of ψ at 0, that is, the minimal degree $\sum r_j$ of a nonzero term $a_{\mathbf{r}}\mathbf{x}^{\mathbf{r}}$ in the expansion of ψ at the origin. If $l = 0$ (i.e., $\psi(0) \neq 0$) then

$$I(\lambda) \sim (2\pi/\lambda)^{d/2}\psi(0)\det(M(Q))^{-1/2}$$

where the choice of square root of the determinant of M is the product of the principal square roots of the eigenvalues.

Remark: The factor of 2 is due to the fact that $f \sim \frac{1}{2}Q$; it is easy to confuse Q with $Q/2$ e.g. when $f = \sum x_j^2$ then Q is twice the identity matrix.

PROOF: Let us write $I_D(\lambda; f, \psi)$ to emphasize the dependence of I on D and the functions f and ψ. Let η be a standard bump function, that is, $\eta = 1$ on the unit ball $\{\mathbf{z} : \sum_{j=1}^d |z_j|^2 \leq 1\}$ of $\mathbf{C}^d$, $\eta = 0$ outside the ball of radius 2, and $\eta \in C^\infty(\mathbf{C}^d)$. Write η_r for the function $\eta(\mathbf{x}/r)$. Throughout this proof we will view $\mathbf{R}^d$ as a subset of $\mathbf{C}^d$.

For any $r > 0$, the function $(1 - \eta_r)\psi$ vanishes in a neighborhood of the origin, so f has no stationary points in the support of $(1 - \eta_r)\psi$. Applying Lemma **??** we see that $I_D(\lambda; f, (1 - \eta_r)\psi)$ is rapidly decreasing, and hence

$$I_D(\lambda; f, \psi) \approx I_D(\lambda; f, \eta\psi)$$

modulo a rapidly a decreasing difference. Now apply the Morse Lemma to see that there is a change of variables making the RHS of this equal to

$$I_{D^*}(\lambda; \mathrm{Id}, \psi^*)$$

where $\mathrm{Id}(\mathbf{x}) := \frac{1}{2} \sum_{j=1}^d x_j^2$ has Hessian equal to the identity matrix. Observe that $D^* \subseteq S^+$ since D^* is the image of D when we map f to the sum of squares, and f has nonnegative real part on D. Also, the Jacobian of the change of variables is a matrix A for which $A^T A = M(Q)$. Thus the integrating factor is $|A|^{-1}$ and we have

$$\psi^*(\mathbf{0}) = |M(Q)|^{-1/2} \psi(\mathbf{0}) \tag{4.3}$$

with the choice of square root of the determinant of $M(Q)$ yet to be determined. Approximate ψ^*/η_r to homogeneous degree N by its Taylor polynomial P_N; specifically,

$$P_n(\mathbf{x}) = \sum_{|\mathbf{r}| \leq N} p_{\mathbf{r}} \mathbf{x}^{\mathbf{r}}$$

where $p_{\mathbf{r}}$ are the Taylor coefficients of $\psi\eta$. Note that $p_{\mathbf{0}} = |M(Q)|^{-1/2}\psi(\mathbf{0})$. Then $\psi^* - \eta_r P = O(|\mathbf{x}|^{-N-1})$ near the origin, so by Lemma **??**,

$$I_D(\lambda; f, \psi) = I_{D^*}(\lambda; \mathrm{Id}, \eta P_N) + O(\lambda^{-\frac{d+N+1}{2}}) \tag{4.4}$$

Now we rotate the surface of integration using Lemma **??**. Let ω be the d-form $\exp(-\lambda \mathrm{Id}(\mathbf{z}))\eta(\mathbf{z})P_N(\mathbf{z})\, dz_1 \wedge \cdots \wedge dz_d$. Let Ω be the region given by Lemma **??**. Then

Stokes' Theorem gives

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega.$$

We now choose r less than half the distance from the origin to $\partial\Omega \setminus (D^* \cup \mathcal{N})$. Then η_r vanishes on $\partial\Omega \setminus (D^* \cup \mathcal{N})$, and hence

$$\int_\Omega d\omega = \int_{D^*} \omega + \int_{\mathcal{N}} \omega\,.$$

Since $\int_{D^*} \omega$ is the first term on the RHS of (**??**), we can put this together with (**??**), to get

$$I_D(\lambda; f, \psi) = \pm \int_{\mathcal{N}} \omega \pm \int_\Omega d\omega + O(\lambda^{-\frac{d+N+1}{2}})$$

with the signs determined by the orientations of the boundary pieces. The form $d\omega$ is the differential of a form that is holomorphic on the ball of radius $1/r$, so it vanishes in a neighborhood of the origin. Since $\Omega \subset S^+$, we know that in the region where $d\omega$ does not vanish, the function Id has non-negative real part. Since $d\omega = \exp(-\lambda \mathrm{Id}) P\, d(\eta_r\, dz_1 \wedge \cdots \wedge dz_d)$ we may apply the last part of Lemma **??** to see that $\int_\Omega d\omega$ is rapidly decreasing. Thus

$$I_{\mathcal{N}}(\lambda; f, \psi) = \pm \int_{\mathcal{N}} \omega + O(\lambda^{-\frac{d+N+1}{2}})$$

Finally, we evaluate $\int_{\mathcal{N}} \omega$. The function Id has real part at least $1/r^2$ on the support of $(1 - \eta_r)$, so the difference between $\int_{\mathcal{N}} \omega$ and $\int_{\mathbf{R}^d} \exp(-\lambda \mathrm{Id}) P$ decreases exponentially in λ. But $\int_{\mathbf{R}^d} \exp(-\lambda \mathrm{Id}) P$ may be evaluated term by term to yield

$$\sum_{|\mathbf{r}| \le N} p_{\mathbf{r}} \prod_{j=1}^d a(2, r_j) \left(\frac{\lambda}{2}\right)^{-(r_j+1)/2}$$

which proves the general expansion. When $\mathbf{r} = \mathbf{0}$, this simplifies to $p_{\mathbf{0}}(2\pi/\lambda)^{d/2}$. Substituting $p_{\mathbf{0}} = \psi(\mathbf{0})|M(Q)|^{-1/2}$ recovers the claimed leading term up to sign. To determine the sign of the leading term, we argue as follows.

If $f_n \to f$ in the topology of uniform convergence of the first N derivatives, then since the error bounds depend only on the first N derivatives, the coefficients of the expansions of $I_D(\lambda; f_n, \psi)$ converge to the coefficients of the expansion of $I_D(\lambda; f, \psi)$. In particular, the choice of $|M(Q)|^{-1/2}$ must be continuous in the second partials of f which are the matrix entries of M. Of course there is no global choice of square root of the determinant that is continuous in the matrix entries, but there is if one restricts to the set of matrices representing quadratic forms that map real vectors to complex numbers with nonnegative real parts. Indeed, up to global sign change, the product of the principal square roots of the eigenvalues is the only such function. We know that when $f = \mathrm{Id}$, the positive square root is chosen, which determines the global sign, hence the choice of square root for all matrices. $\Box$

4.3 Stationary manifolds

4.4 Cusps

5 Asymptotics in several variables: isolated smooth points

Recall our previous notation: $F = G/H = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is a function of d complex variables, written as a quotient of functions analytic in some domain $\mathcal{D}$. The analytic variety where H vanishes is denoted $\mathcal{V}$. Writing $a_{\mathbf{r}}$ as an iterated Cauchy integral (as in equation (refeq:cauchy), we see that any function analytic on the closed polydisk $D(\mathbf{z})$ has coefficients satisfying

$$a_{\mathbf{r}} = O(|\mathbf{z}^{-\mathbf{r}}|)\,. \tag{5.1}$$

The following upper bound may be proved directly:

Lemma 5.1 *Let $z_j = x_j e^{i\theta_j}$ for $j = 1, \ldots, d$ and $x_j > 0$. If $\mathbf{z} \in \mathcal{V}$ is on the boundary of $\mathcal{D}$ and the hyperplane through $\mathbf{x}$ normal to $\mathbf{r}$ is not a support hyperplane for* $\log\mathcal{D}$*, then $|\mathbf{z}^{\mathbf{r}} a_{\mathbf{r}}|$ decreases exponentially, the rate being uniform as $\mathbf{r}$ varies over a compact set.*

PROOF: With $\mathbf{x}, \mathbf{z}$ and $\mathbf{r}$ as in the hypotheses, by definition of support hyperplane there is a $\mathbf{y}$ in the interior of $\log\mathcal{D}$ with $\mathbf{y} \cdot \mathbf{r} > \mathbf{x} \cdot \mathbf{r}$. By (**??**), $a_{\mathbf{r}} = O(e^{-\mathbf{y} \cdot \mathbf{r}})$, which is less than

$e^{-\mathbf{x}\cdot\mathbf{r}} = |\mathbf{z}^{-\mathbf{r}}|$ by an exponential factor. □

Evidently, all the work is in proving that this estimate is sharp, and in computing asymptotic expansions in these cases. We proceed from the simplest case to the more intricate ones.

5.1 Two dimensions

Theorem **??** and its two-sided corollary tell us how to do the integral in the reduction lemma (Lemma **??**), provided we verify a couple of hypotheses. We will need to know that $\tilde{f}'(0)$ vanishes and that $\text{Re}\{\tilde{f}\} \geq 0$ with equality only at 0. Indeed, since $(z, g(z)$ is a parametrization of $\mathcal{V}$, the strict minimality of (z, w) implies that $|g(z')| > |g(z)| = |w|$ whenever $|z'| = |z|$, and taking logs shows $\text{Re}\{\tilde{f}\}$ to be strictly positive except at 0. Differentiating the relation $H(z, g(z)) = 0$ gives

$$g'(z) = -\frac{H_z}{H_w} \tag{5.2}$$

and we see that

$$\tilde{f}'(0) = \frac{r}{s} + \frac{d(\log g)}{d(\log z)} = \frac{r}{s} + \frac{zg'}{g} = \frac{r}{s} - \frac{zH_z}{wH_w}\,.$$

To be consistent with future notation, we write $(r, s) \in \mathbf{dir}(z, w)$ to denote the equality of r/s and $zH_z/(wH_w)$. Now recall some quantities defined in the preamble to the reduction lemma. The quantity ψ is defined to be the residue of the function F/w at the smooth point $(z, g(z))$; $\tilde{\psi}$ is ψ after changing variables to $z = z_0 e^{i\theta}$; and $\tilde{f}$ is $\log g$ in terms of θ. We let k be the order of vanishing of $\tilde{f}$ at 0, and $c_k := \tilde{f}^{(k)}(0)/k!$. Let η invert the function $y(x) = (f(x)/c_k)^{1/k}$ and let $\psi^* = (\tilde{\psi} \circ \eta) \cdot \eta'$ be $\tilde{\psi}$ times the integrating factor arising from straightening out $\tilde{f}$; define $b_j^* := (\psi^*)^{(j)}/j!$. The following theorem is simply an application of Corollary **??** the reduction lemma.

Theorem 5.2 *Let $F = G/H = \sum a_{rs} z^r w^s$ have a strictly minimal, simple pole at (z, w). Let l_0 be the order to which G vanishes near (z, w) on $\mathcal{V}$, that is, the largest l such that*

$G(z', w') = O(|z - z'|^l + |w - w'|^l)$ as $(z', w') \to (z, w)$ in $\mathcal{V}$. Then there is a full asymptotic expansion

$$a_{r,s} \sim \frac{1}{2\pi} z^{-r} w^{-s} \sum_{l \geq l_0} a(k, l) b_l^*(c_k s)^{-(l+1)/k} , \tag{5.3}$$

*with $a(k, l)$ given by (**??**) if k is odd, and by $(1 + (-1)^l) a_+(k, l)$ if k is even. The expansion is uniform as (z, w) varies over a compact set with $(r, s) \in \mathbf{dir}(z, w)$ and k and l_0 not changing.* □

We pause to compute one example. The accompanying picture (possibly not drawn yet) should show the graph of $3 - 3x + x^2$ in the positive quadrant, with a minimum at $x = 3/2$ but minimality only when $x \geq 1$.

Example 13 (Cube root asymptotics)

Let $F(z, w) = 1/(3 - 3z - w + z^2)$. The set $\mathcal{V}$ is the set $\{w = z^2 - 3z + 3\}$ and $g(w) = z^2 - 3z + 3$. The point $(1, 1)$ is in $\mathcal{V}$, indicating that the maximal exponential growth rate will be zero. Indeed, for directions above the diagonal, Theorem **??** may be used at the minimal points $\{(z, g(z)) : 0 < z < 1\}$, while each direction below the diagonal corresponds to a pair of complex (finitely) minimal points. The result is that the coefficients decay exponentially at a rate that is uniform over compact subsets of directions not containing the diagonal.

The interesting behavior is near the diagonal. The relevant minimal point is $(1, 1)$, where $z^r w^s \equiv 1$ and the decay is sub-exponential. Computing $\tilde{f}''(0)$ via equation (**??**) below gives

$$\tilde{f}''(z) = -3 \frac{z(z^2 - 4z + 3)}{(z^2 - 3z + 3)^2} .$$

This vanishes when $z = 1$, and computing further, we find that $\tilde{f}$ vanishes to order exactly 3 here, with $c_3 := \tilde{f}'''(0)/3! = i$. Along with $\tilde{\psi}(0) = 1$, this then results in an asymptotic expansion whose leading term is given by

$$a_{n,n} \sim \frac{1}{2\pi} a(3, 0) i^{-1/3} (1 + e^{i\pi/3}) = \frac{\Gamma(1/3)}{6\sqrt{3}\pi} .$$

Later, we discuss the question of computing asymptotics "between the cracks" so as to be able to conclude that $\limsup \log a_{\mathbf{r}} / \log |\mathbf{r}| = -1/3$ or even $\limsup |\mathbf{r}|^{1/3} a_{\mathbf{r}} = \frac{\Gamma(2/3)}{6\sqrt{3}\pi}$.

By far the most usual case is $k = 2$. For this case it is worthwhile computing the expansion (**??**) explicitly in terms of the given data, namely the Taylor coefficients of G and H. This begins with a lemma.

Lemma 5.3 *In a neighborhood of (z, w), ψ and the derivatives of g are as follows.*

$$g'(z) = -\frac{H_z}{H_w} \qquad (??)$$

$$g''(z) = -\frac{1}{H_w}\left[H_{zz} - 2\frac{H_z}{H_w}H_{zw} + \frac{H_z^2}{H_w^2}H_{ww}\right]. \qquad (5.4)$$

$$\psi(z) = \frac{G(z, w)}{-wH_w(z, w)}.$$

PROOF: We have already differentiated the equation $H(z, g(z)) = 0$ once to get (**??**). Differentiate again to get

$$H_{zz} + 2g'H_{zw} + g''H_w + (g')^2 H_{ww} = 0$$

and use (**??**) to eliminate g', giving (**??**). The formula for ψ follows from the definitions of ψ and of the partial derivative. □

We now state the criterion $k = 2$ in terms of the partial derivatives of H. Since the proof of the criterion is subsumed in the computation, we do not give the proof yet. Define

$$Q(z, w) := -w^2 H_w^2 z H_z - wH_w z^2 H_z^2 - w^2 z^2 \left(H_w^2 H_{zz} + H_z^2 H_{ww} - 2H_z H_w H_{zw}\right).$$

We will see that $Q = 0$ if and only if $\tilde{f}''(0)$ vanishes, that is, if and only if $k > 2$. We now state:

Theorem 5.4 *Let $F = G/H$ be a meromorphic function of two variables, not singular at the origin. Then*

$$a_{r,s} \sim \frac{G(z,w)}{\sqrt{2\pi}} z^{-r} w^{-s} \sqrt{\frac{-wH_w}{sQ}}$$

uniformly as (z,w) varies over a compact set of strictly minimal, simple poles of F on which Q and G are nonvanishing, and $(r,s) \in \mathbf{dir}(z,w)$.

Remarks: Usually the expression in the radical will be positive real, as will the coefficients a_{rs}. The result is true in general, though, as long as the square root is taken to be $+1/(wH_w)$ times the principal root of $(-wH_w^3/Q)$. Also note that when $(r,s) \in \mathbf{dir}(z,w)$ then the expression wH_w/s is coordinate-invariant, that is, equal to zH_z/r. Thus the given expression for $a_{r,s}$ has the expected symmetry.

Proof: Recall that $\tilde{f}$ and $\log \tilde{g}$ differ by a linear function of θ. Hence $\tilde{f}'' = (\log \tilde{g})''$. When $Z = ze^{i\theta}$, we have $(d/d\theta) = iZ(d/dZ)$, so

$$\tilde{g}'' = iZ\frac{d}{dZ}\left(iZ\frac{d\log g}{dZ}\right) = -Z\frac{d}{dZ}\left(\frac{Zg'}{g}\right),$$

where $Z = ze^{i\theta}$. Expanding this yields

$$\tilde{f}'' = -Z\frac{g' + Zg''}{g} + \frac{Z^2(g')^2}{g^2}. \tag{5.5}$$

By our assumption, G does not vanish at (z,w), so as long as $\tilde{f}''(0) \neq 0$, we may use Theorem **??** to conclude that the leading term asymptotic for $a_{r,s}$ is the $k = 2, l = 0$ term of (**??**). The term b_0 there is equal to

$$\tilde{\psi}(0)\eta'(0) = \psi(0)\sqrt{2/\tilde{f}''(0)} = \frac{G(z,w)}{-wH_w(z,w)}\sqrt{\frac{2}{\tilde{f}''(0)}}.$$

Thus from Theorem **??**,

$$a_{r,s} \sim \frac{A(2,0)}{2\pi} z^{-r} w^{-s} \frac{G(z,w)}{wH_w(z,w)}\sqrt{\frac{2}{s\tilde{f}''(0)}}.$$

Now evaluate this using the value $A(2,0) = \sqrt{\pi}$ and equation (??) along with (??) and (??) to obtain

$$a_{r,s} \sim \frac{1}{\sqrt{2\pi}} z^{-r} w^{-s} \frac{G(z,w)}{wH_w(z,w)} \sqrt{\frac{(-wH_w(z,w))^3}{sQ}}$$

where

$$Q = (-wH_w(z,w))^3 \tilde{f}''(0) = (-wH_w(z,w))^3 z \frac{-g'(z) - zg''(z)}{g(z)} + \frac{z^2 (g'(z))^2}{(g(z))^2} .$$

With the help of Lemma **??** we see (using $g(z) = w$) that

$$Q = (-wH_w(z,w))^3 \left[-z \frac{H_z}{-wH_w} - z^2 \frac{1}{wH_w} \left(H_{zz} - 2\frac{H_z}{H_w} H_{zw} + \frac{H_z^2}{H_w^2} H_{ww} \right) + \frac{z^2 H_z^2}{w^2 H_w^2} \right] ,$$

which simplifies to the expression in Theorem **??**. We see also that the nonvanishing hypotheses on Q is enough to guarantee $\tilde{f}''(0) \neq 0$, which finishes the proof of Theorem **??**. $\square$

Example 14 (Lattice paths)

Let $a_{r,s}$ be the number of nearest-neighbor paths from the origin to (r,s) moving only north, east and northeast. The generating function is $F(z,w) = 1/(1 - z - w - zw)$. The zero set $\mathcal{V}$ of $H = 1 - z - w - zw$ is given by $w = (1-z)/(1+z)$, and the minimal points of $\mathcal{V}$ are those where $w \in [0,1]$. With the help of relations that hold when $\mathbf{z} \in \mathcal{V}$ we may compute as follows.

$$\begin{aligned} H_z &= -1 - w \\ -zH_z &= 1 - w \\ Q &= (1-z)(1-w)(1-zw) \\ \frac{zH_z}{wH_w} &= \frac{1-w}{1-z} = \frac{1-w^2}{2w} \end{aligned}$$

with H_w and $-wH_w$ given by reversing z and w. As z varies over $[\epsilon, 1-\epsilon]$, the functions Q and $G := 1$ do not vanish. The minimal pair (z,w) that solves $\mathbf{dir}(r,s) \in (z,w)$ is given by $z = (\sqrt{r^2+s^2} - s)/r$ and $w = (\sqrt{r^2+s^2} - r)/s$. Theorem **??** then gives

$$
\begin{aligned}
a_{rs} &\sim \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{1-z}{s}\frac{1}{1-zw}} \\
&= \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{(r+s-\sqrt{r^2+s^2})^2\sqrt{r^2+s^2}}} \, ,
\end{aligned}
$$

uniformly when r/s and s/r remain bounded. In particular, when $r = s = n$, this gives the following formula for the n^{th} diagonal coefficient (which may alternatively be obtained by computing the diagonal generating function $(1-6s+s^2)^{-1/2}$ according to the method given in Stanley (1999, section 6.3):

$$
(\sqrt{2}-1)^{-2n}\sqrt{\frac{1}{2\pi}\frac{2^{-1/4}}{2-\sqrt{2}}} \, .
$$

5.2 Any dimension

5.2.1 The geometry of $\mathcal{V}$

Let $D(\mathbf{z})$ denote the polydisk $\{\mathbf{w} : |w_j| \leq |z_j| \forall j \leq d\}$. Let $T(\mathbf{z})$ denote the distinguished boundary of this polydisk, namely $\{\mathbf{w} : |w_j| = |z_j| \forall j \leq d\}$. As in the one-dimensional case, the points of $\mathcal{V}$ nearest the origin are the most important. Accordingly we define a point $\mathbf{z} \in \mathcal{V}$ to be *minimal* if $\mathcal{V} \cap D(\mathbf{z}) \subseteq T(\mathbf{z})$; we say that $\mathbf{z}$ is *locally minimal* if this holds in a neighborhood of $\mathbf{z}$. Divide the minimal points of $\mathcal{V}$ into three types. Say that $\mathbf{z}$ is *strictly minimal*, *finitely minimal* or *toral*, according to whether the cardinality of $\mathcal{V} \cap D(\mathbf{z})$ is 1, finite, or infinite. When infinite, the intersection must be uncountable. If $\mathbf{z}$ is a minimal point of $\mathcal{V}$ then the interior of $D(\mathbf{z})$ is contained in $\mathcal{D}$, so the assumption that G and H are analytic on a neighborhood of $D(\mathbf{z})$ is just a little stronger than what is true automatically.

It will be convenient to have a notation for the projection of $\mathbf{z}$ onto the first $d-1$ coordinates. We therefore define $\hat{\mathbf{z}} := (z_1, \ldots, z_{d-1})$ and extend the notation so that putting a hat on any d-vector strips off the last coordinate.

A *simple pole* of F is a point $\mathbf{z} \in \mathcal{V}$ where H vanishes to order 1. Equivalently, the gradient $\bigtriangledown H$ does not vanish. Let $\mathbf{z}$ be a simple pole of F and assume for specificity that H_d is nonzero at $\mathbf{z}$. By the implicit function theorem, there is a neighborhood of $\mathbf{z}$ where $\mathcal{V}$ may be parametrized by $z_d = g(\widehat{\mathbf{z}}) = g(z_1, \ldots, z_{d-1})$ for some analytic function g. We will always use g to denote this parametrization.

We will see later (in the proof of Theorem **??**) that under some hypotheses on F, minimal points of $\mathcal{V}$ are always found in the positive real orthant. A relation true in complete generality is the following.

Lemma 5.5 *Let* $\mathbf{z}$ *be a simple pole of* F *and suppose that* $z_d H_d$ *does not vanish there. If* $\mathbf{z}$ *is locally minimal then for all* $j < d$*, the quantity* $z_j H_j / (z_d H_d)$ *is real and nonnegative.*

PROOF: Given θ and j, let $\mathbf{z}^{(\theta)}$ be the result of varying $\mathbf{z}$ by multiplying the j^{th} coordinate by $e^{i\theta}$ and adjusting the last coordinate so as to remain on $\mathcal{V}$ (that is, $z_d^{(\theta)} = g(z_1, \ldots, z_{j-1}, z_j e^{i\theta}, z_{j+1}, \ldots, z_{d-1})$). Differentiating the relation $H(\mathbf{z}^{(\theta)}) = 0$ implicitly with respect to θ at 0 yields

$$i z_j H_j + H_d \frac{d z_d^{(\theta)}}{d\theta} = 0. \tag{5.6}$$

By minimality of $\mathbf{z}$, we know that the modulus of $z_d^{(\theta)}$ has a minimum at $\theta = 0$, hence $(d z_d^{(\theta)} / d\theta) / z_d$ is purely imaginary. Plugging this into (**??**) proves that $z_j H_j / (z_d H_d)$ is real. If $z_j H_j / (z_d H_d) = -\beta < 0$ then $\mathcal{V}$ has a tangent vector at $\mathbf{z}$ in the direction $-z_j e_j - \beta z_d e_d$ which contradicts minimality. Hence $z_j H_j / (z_d H_d) \geq 0$. $\square$

Definition 3 *Define* $\mathbf{dir}(\mathbf{z})$ *to be the equivalence class of scalar multiples of the vector* $(z_1 H_1, \ldots, z_d H_d)$*, defined whenever* $z_j H_j$ *does not vanish for all* j*. By the previous lemma, when* $\mathbf{z}$ *is a minimal pole of* F *with nonzero coordinates,* $\mathbf{dir}(\mathbf{z})$ *is a well defined element of* $\mathbf{RP}^{d-1}$.

The importance of **dir** is that analysis of F near $\mathbf{z}$ yields asymptotic information about $a_{\mathbf{r}}$ with $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. The function **dir** appears in GF-sequence method literature as **m**. When $\mathbf{z} \in \partial\mathcal{D}$ is on the boundary of the domain of convergence, $\mathbf{dir}(\mathbf{z})$ is the normal to the support hyperplane of the convex set $\log\mathcal{D}$ at the point $(\log|z_1|, \ldots, \log|z_d|)$.

We now generalize the definitions of $\psi, \tilde{\psi}$ and $\tilde{f}$ to more than one variable. Again, we will reserve the names of these functions, so as not to burden the notation with subscripts and arguments. If $\mathbf{z}$ is a simple pole of F with $z_d H_d$ not vanishing there, define a function ψ on a neighborhood of $\widehat{\mathbf{z}}$ by

$$\psi(\widehat{\mathbf{w}}) = - \lim_{w \to g(\widehat{\mathbf{w}})} (w - g(\widehat{\mathbf{w}})) \frac{F(\widehat{\mathbf{w}}, w)}{w}. \tag{5.7}$$

Suppose now that $\widehat{\mathbf{w}} \in T(\widehat{\mathbf{z}})$ and write $w_j = z_j e^{i\theta_j}$. For fixed $\mathbf{r}$ with $r_d \neq 0$, define a function f on a neighborhood of $\widehat{\mathbf{z}}$ in $T(\widehat{\mathbf{z}})$ by

$$f(\widehat{\mathbf{w}}) = \log\left(\frac{g(\widehat{\mathbf{w}})}{g(\widehat{\mathbf{z}})}\right) + i \sum_{j=1}^{d-1} \frac{r_j}{r_d} \theta_j. \tag{5.8}$$

We will be parametrizing integrals over $T(\widehat{\mathbf{z}})$ by θ, so we will want the above function expressed in terms of $\widehat{\theta}$. We therefore compose with the map M taking $\widehat{\theta}$ to $\widehat{\mathbf{w}}$ defined by $M(\theta_1, \ldots, \theta_{d-1}) = (z_1 e^{i\theta_1}, \ldots, z_{d-1} e^{i\theta_{d-1}})$, and define the functions $\tilde{g} := g \circ M, \tilde{f} := f \circ M, \tilde{\psi} := \psi \circ M$.

5.2.2 Reduction to an oscillating integral

The smooth point computation in higher dimensions is exactly analogous to the two-dimensional case, so I will state and prove the analogues of Lemma **??** and Theorem **??** without further verbiage. Note though, that we assume that the Hessian is non-degenerate, so we are not proving an analogue of Theorem **??**.

Lemma 5.6 (Multivariate reduction to oscillating integral) *Let $\mathbf{z}$ be a strictly minimal simple pole of $F = G/H$. Assume that $z_d H_d \neq 0$. For a neighborhood $\widetilde{\mathcal{N}}$ of $\mathbf{0}$ in $\mathbf{R}^{d-1}$*

define a quantity

$$\Xi := (2\pi)^{1-d}\mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \exp(-r_d \tilde{f}(\widehat{\theta}))\tilde{\psi}(\widehat{\theta})\, d\widehat{\theta}. \tag{5.9}$$

Then the quantity

$$|\mathbf{z}^{\mathbf{r}}|\, |a_{\mathbf{r}} - \Xi|$$

decreases exponentially as $\widetilde{\mathcal{N}}$ *remains fixed and* $\mathbf{r} \to \infty$.

Proof: For $\epsilon \in (0, |z_d|)$, let T be the torus $T(\mathbf{z})$ shrunk in the last coordinate by ϵ, that is, the set of $\mathbf{x}$ for which $|x_j| = |z_j|$, $j < d$ and $|x_d| = |z_d| - \epsilon$. By Cauchy's formula,

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{w}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{w})\, d\mathbf{w} \tag{5.10}$$

where the multi-exponent $\mathbf{r} - \mathbf{1}$ means $(r_1 - 1, \ldots, r_d - 1)$. Write this as an iterated integral

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{T(\widehat{\mathbf{z}})} \widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \left[\int_{\mathcal{C}_1} w_d^{-r_d} F(\mathbf{w}) \frac{dw_d}{w_d}\right] d\widehat{\mathbf{w}}\,. \tag{5.11}$$

Here $\mathcal{C}_1$ is the circle of radius $|z_d| - \epsilon$. Let $K \subseteq T(\widehat{\mathbf{z}})$ be a compact set not containing $\widehat{\mathbf{z}}$. For each fixed $\widehat{\mathbf{w}} \in K$, the function $F(\widehat{\mathbf{w}}, \cdot)$ has radius of convergence greater than $|z_d|$. Hence the inner integral in equation (**??**) is $O(|z_d| + \epsilon)^{-r_d}$. By continuity of the radius of convergence,we may integrate over K to see that

$$|\mathbf{z}^{\mathbf{r}}| \int_{K \times \mathcal{C}_1} \mathbf{w}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{w})\, d\mathbf{w}$$

decreases exponentially. Thus if $\mathcal{N}$ is any neighborhood of $\widehat{\mathbf{z}}$ in $T(\widehat{\mathbf{z}})$, the quantity

$$|\mathbf{z}^{\mathbf{r}}| \left| a_{\mathbf{r}} - \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{N}} \widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \left[\int_{\mathcal{C}_1} \frac{F(\mathbf{w})}{w_d^{r_d}}\, dw_d\right] d\widehat{\mathbf{w}} \right|$$

decreases exponentially. Thus we have reduced the problem to an integral over a neighborhood of $\widehat{\mathbf{z}}$.

Near $\mathbf{z}$ there is a parametrization $w_d = g(\widehat{\mathbf{w}})$ of $\mathcal{V}$. Let $\mathcal{C}_2$ be the circle of radius $|z_d| + \epsilon$. Then for any sufficiently small $\epsilon > 0$, the image of $\mathcal{N}$ under g is disjoint from $\mathcal{C}_2$. Fix such a neighborhood. For any $\widehat{\mathbf{w}} \in \mathcal{N}$, the function $F(\widehat{\mathbf{w}}, \cdot)$ has a single simple pole in the annulus bounded by $\mathcal{C}_1$ and $\mathcal{C}_2$, occurring at $g(\widehat{\mathbf{w}})$. The residue of F at $g(\widehat{\mathbf{w}})$ is equal to

$$R(\widehat{\mathbf{w}}) := -\psi(\widehat{\mathbf{w}}) g(\widehat{\mathbf{w}})^{-r_d} \tag{5.12}$$

where ψ is defined in (??). Therefore, for each fixed $\widehat{\mathbf{w}} \in \mathcal{N}$,

$$\int_{\mathcal{C}_1} \frac{F(\mathbf{w})}{w_d^{r_d+1}} \, dw_d = \int_{\mathcal{C}_2} \frac{F(\mathbf{w})}{w_d^{r_d+1}} \, dw_d - 2\pi i R(\widehat{\mathbf{w}}).$$

But $|\mathbf{z}^{\mathbf{r}} \int_{\mathcal{C}_2} F(\mathbf{w}) dw_d / \mathbf{w}^{\mathbf{r}+\mathbf{1}}|$ is bounded by a constant multiple of $(1+\epsilon/|z_d|)^{-r_d}$ (the constant depending on the maximum of F on $\mathcal{C}_2$) and hence $|\mathbf{z}^{\mathbf{r}}||a_{\mathbf{r}} - X|$ is exponentially decreasing, where

$$\begin{aligned} X &= (2\pi i)^{1-d} \int_{\mathcal{N}} (\widehat{\mathbf{w}})^{-\widehat{\mathbf{r}}-1} g(\widehat{\mathbf{w}})^{-r_d} \psi(\widehat{\mathbf{w}}) \, d\widehat{\mathbf{w}} \qquad (5.13) \\ &= (2\pi i)^{1-d} \int_N \frac{\widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}}}{\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}}} \frac{d\widehat{\mathbf{w}}}{\prod_{j=1}^{d-1} w_j} \left(\frac{g(\widehat{\mathbf{w}})}{g(z_d)} \right)^{-r_d} \psi(\widehat{\mathbf{w}}) \end{aligned}$$

is exponentially decreasing in r_d. Changing variables to $w_j = z_j e^{i\theta_j}$ and $dw_j = i w_j d\theta_j$ turns the quantity X into

$$(2\pi)^{1-d} \mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \prod_{j=1}^{d-1} e^{-i r_j \theta_j} \tilde{\psi}(\widehat{\theta}) \left(\frac{g(\widehat{\mathbf{w}})}{g(\widehat{\mathbf{z}})} \right)^{-r_d} d\widehat{\theta}$$

and plugging in the definitions of f and $\tilde{f}$ at (??) above yields

$$(2\pi)^{1-d} \mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \exp(-r_d \tilde{f}(\widehat{\theta})) \tilde{\psi}(\widehat{\theta}) \, d\widehat{\theta}$$

which is none other than Ξ. □

5.2.3 Main result

Theorem 5.7 *Let $F = G/H = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have a strictly minimal, smooth point $\mathbf{z} \in \mathcal{V}$. Suppose $z_d H_d$ does not vanish. If the Hessian of $\tilde{f}$ at $\mathbf{z}$ is nonsingular, then there is an expansion*

$$a_{\mathbf{r}} \sim \mathbf{z}^{-\mathbf{r}} \sum_{l \geq l_0} C_l r_d^{(1-d-l)/2}$$

where l_0 is the degree to which G vanishes on $\mathcal{V}$ near the point $\mathbf{z}$. When G does not vanish at $\mathbf{z}$ then $l_0 = 0$ and

$$C_0 = (2\pi)^{(1-d)/2}\mathcal{H}^{-1/2}\frac{G(\mathbf{z})}{z_d H_d}$$

where $\mathcal{H}$ is the determinant of the Hessian at $\mathbf{z}$.

The θ-linear term in the definition of $\tilde{f}$ is designed to make $\tilde{f}$ stationary at $\mathbf{0}$ when $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. Thus we have a lemma similar to the one-dimensional case.

Lemma 5.8 *The quantity $\tilde{f}(\mathbf{0})$ always vanishes. If $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$ then $\bigtriangledown\tilde{f}(\mathbf{0}) = \mathbf{0}$ and the real part of $\tilde{f}$ has a strict minimum at $\mathbf{0}$.*

Proof: The first statement is immediate. To prove the second, let $j \leq d-1$ and see from the definition of f that

$$r_d f_j(\widehat{\mathbf{z}}) = \frac{r_d g_j(\widehat{\mathbf{z}})}{g(\widehat{\mathbf{z}})} + \frac{r_j}{z_j}.$$

By definition of **dir**, the ratio $r_j/(z_j H_j)$ is some constant c independent of j at the point $\mathbf{z} = (\widehat{\mathbf{z}}, z_d)$. Since $g(\widehat{\mathbf{z}}) = z_d$, we find that

$$c^{-1} r_d f_j(\widehat{\mathbf{z}}) = g_j(\widehat{\mathbf{z}}) H_d(\widehat{\mathbf{z}}, z_d) + H_j(\widehat{\mathbf{z}}, z_d).$$

The right hand side of this is the derivative of $H(w_1, \ldots, w_{d-1}, g(\widehat{\mathbf{w}}))$ with respect to w_j at $\widehat{\mathbf{z}}$. By definition of g this vanishes, and hence $f_j(\widehat{\mathbf{z}}) = 0$. But $\tilde{f}_j(\mathbf{0}) = i z_j f_j(\mathbf{z})$, so the gradient of $\tilde{f}$ must vanish at $\mathbf{0}$. Finally, observe that $\mathrm{Re}\{\tilde{f}(\widehat{\theta})\} = -\log|\tilde{g}(\widehat{\theta})/|z_d|$. By strict minimality of $\mathbf{z}$, the modulus of $g(\widehat{\mathbf{w}}) = \tilde{g}(\widehat{\theta})$ is greater than $|z_d|$ for any $\widehat{\mathbf{w}} \in T(\widehat{\mathbf{z}})$. □

Proof of Theorem **??**: We see from Lemma **??** that proving the theorem amounts to evaluating the quantity Ξ in equation (**??**). From Lemma **??** we see that $\mathbf{0}$ is a stationary point for the function $\tilde{f}$ as long as $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. We apply Theorem **??** to get an asymptotic series expansion. The leading term of the integral in (**??**) is $(2\pi)^{(d-1)/2}\tilde{\psi}(\mathbf{0}) r_d^{(1-d)/2}$ divided by the product of the square roots of the eigenvalues of the Hessian. Once we have identified $\tilde{\psi}(\mathbf{0}) = \psi(\mathbf{0})$ as $G(\mathbf{0})/(z_d H_d)$, the theorem follows. □

Exercise: show that when $d = k = 2$ and $l \geq 2$ is even, the leading term asymptotic is given by the following expression.

$$a_{r,s} \sim (z_0 w_0)^l z_0^{-r} w_0^{-s} \sqrt{\frac{wH_w}{sQ}}^{l+1} \sum_{k=0}^{l} (-1)^k \binom{l}{k} H_z^k H_w^{l-k} G_{l-k,k}.$$

5.2.4 Generalizing

One improvement we have already done the work for is when a smooth minimal point is finitely minimal. In this case there are finitely many points $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(n)}$ of $\mathcal{V}$ lying on the torus $T(\mathbf{z}^{(1)})$ with no other points of $\mathcal{V}$ lying in $D(\mathbf{z}^{(1)})$. In this case the integral

$$\int_{\mathcal{C}' \setminus [\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_n]} z^{-r-1} \int_{\mathcal{C}} w^{-s-1} F(z,w)\, dw\, dz\,,$$

which is the analogue of (**??**) but with a neighborhood around each of the n minimal points, will again have modulus at most $O(|\mathbf{z}^{-\mathbf{r}}|)$ where $\mathbf{z}$ is any $\mathbf{z}^{(j)}$. We then obtain n different contributions from the n different minimal points, each computed exactly as in Theorem **??**. The set $S := \{\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(n)}\}$ is called a *complete* set of minimal points if there are no more minimal points on $T(\mathbf{z}^{(1)}) \cap \mathcal{V}$. The one remaining important observation is that the sum

$$\sum_{j=1}^{n} C_j (\mathbf{z}^{(j)})^{\mathbf{r}}$$

cannot always cancel. That is, the sum is not $o(|\mathbf{z}^{(1)}|^{\mathbf{r}})$ for every $\mathbf{r}$ in some neighborhood of $\mathbf{RP}^{d-1}$, and therefore, the leading term of the expansion has been computed. The impossibility of cancellation is due to the linear independence of the arrays $\{\mathbf{z}^{\mathbf{r}} : \mathbf{r} \in (\mathbf{Z}^+)^d\}$. Note though, that one can have cancellation in a single direction, for example one may have $a_{r,2r} \equiv 0$ but $a_{r,2r+1} \equiv 1$.

When defining smooth points as points where the gradient of H does not vanish, we remarked that this restrictive definition of smooth points did not lose any generality. Let's see why. Suppose that (z_0, w_0) is geometrically a smooth point of $\mathcal{V}$ but that both partials vanish. By smoothness, $\mathcal{V}$ has only one branch through (z_0, w_0) and by the vanishing of the partials, H has a double zero at (z_0, w_0), hence along the whole branch. Thus $\mathcal{V}$ has multiplicity greater than 1, locally, and we can write $F = \dfrac{\chi}{(w - g(z))^k}$ for some meromorphic χ analytic in a neighborhood of (z_0, w_0) and some $r \geq 2$. One can now go through the derivation of (**??**) with χ and $(w - g(z))^k$ in place of G and H. At the last step, the residue $\text{Res}(w^{-s-1}F(z,w); w = g(z))$ is computed by multiplying by $(w-g(z))^k$ to remove the singularity, differentiating $k-1$ times with respect to w and dividing by $(k-1)!$. Each time w^{-k-1} is differentiated, a factor of s comes in. If we only want the leading term asymptotic, the term with the most factors of s comes when w^{-s-1} is differentiated $k-1$ times and F is left alone. This gives $\binom{-s-1}{k-1} w^{-s-k} \chi(z_0, w_0)$. With this replacing ψ in (**??**), the computation continues as before. The end result is a leading term s^{k-1} times greater than the order of the asymptotic in the case where H is square-free. The case of only one vanishing partial is left as an exercise for the reader.

Exercise: If one partial of H vanishes but z_0 and w_0 are nonzero, then r or s is zero and the problem reduces to one dimension. If $0 < r/s < \infty$ and one partial, say H_z vanishes, then $w_0 = 0$. In this case, $\mathcal{V}$ is parameterized locally by $w \sim C(z_0 - z)^k$ for some $k \geq 2$, and $\mathbf{dir}(z, w) \asymp 1/w \to \infty$ as $(z, w) \to (z_0, w_0)$ along $\mathcal{V}$. Thus we obtain no new minimal points with one vanishing partial and $0 < r/s < \infty$, and for directions not along any coordinate plane, we may ignore such minimal points.

5.3 A coordinate-free approach

In equation (**??**), we have broken the symmetry between the coordinates, which is reflected in the expressions derived afterwards. In the end, expressions for $a_{n,m}$ such as (**??**), which do not look inherently symmetric, can of course be shown to be so. Theorem **??**, for example, recaptures the symmetry in the expression for $a_{n,m}$. It would be nice, though, to have a more coordinate free way of reducing coefficient extraction to an oscillating integral. This is indeed possible; the idea is as follows.

Let $\mathcal{F}$ denote the form $F(z_1, \ldots, z_d)\, dV$ where $dV := dz_1 \wedge \cdots dz_d$. By Cauchy's integral theorem, we know that $a_{r_1,\ldots,r_d} = (2\pi i)^{-d} \int_T \mathcal{F}$, where T is a torus $|z_1| = c_1, \ldots, |z_d| = c_d$ for which the polydisk $|z_1| \leq c_1, \ldots, |z_d| \leq c_d$ does not intersect the singular set of $\mathcal{F}$ (namely $\mathcal{V}$). Pick T to be an infinitesimal torus, and then expand T until it becomes the torus at infinity. How does $\int_T \mathcal{F}$ change? Letting Ω be a domain bounded by a small torus T_0 and a large torus T_1, we have ought to have

$$\int_{T_0} \mathcal{F} - \int_{T_1} \mathcal{F} = \int_\Omega d\mathcal{F}\,.$$

Then, since normally $\int_{T_1} \mathcal{F} \to 0$ exponentially fast as $T_1 \to \infty$, we will have

$$a_{r_1,\ldots,r_d} \sim \int_\Omega d\mathcal{F}\,.$$

The hitch in making this rigorous is the formalization of $d\mathcal{F}$, which we must interpret as a residue current. Consider by analogy the one variable 1-form $f(z) = dz/z$. Since $1/z$ is holomorphic away from the origin,

$$df = \frac{d}{d\overline{z}}(1/z)\, dz \wedge d\overline{z} = \delta_0 dz \wedge d\overline{z}\,,$$

leading to the residue theorem. We therefore expect $d\mathcal{F}$ to be some kind of a delta function on the singular variety, $\mathcal{V}$. But since it must be integrated on a $d-1$-dimensional set $\Omega \cap \mathcal{V}$, it must be a $d-1$-form.

The following derivation of this $d-1$-form is worked out in Kenyon and Pemantle (2000). They first show that there is a $d-1$-form living on $\mathcal{V}$, that is naturally associated with $\mathcal{F}$.

Lemma 5.9 *Let F be meromorphic with poles but no double poles in a domain $\mathcal{D} \subseteq \mathbf{C}^d$. Let $\mathcal{V}$ denote the pole set of F and let $\iota : \mathcal{V} \to \mathcal{D}$ denote the inclusion map. Then for any representation of $F = G/H$ as a quotient of analytic functions on $\mathcal{D}$ there is a form ω such that $dH \wedge \omega = G\, dV$. The form $\omega_F := \iota^* \omega$ is independent of the particular representation of F as G/H.*

PROOF: The existence of ω is immediate from the fact that dH never vanishes. The kernel of the map ι^* is the same as the kernel of the map $\omega \mapsto dH \wedge \omega$, proving that for fixed G and H, $\iota^*\omega$ is well defined. If ϕ is an analytic function and $dH \wedge \omega = G\,dV$, then $d(\phi H) \wedge \omega = (\phi G)\,dV$, showing independence of the representation $F = G/H$. $\square$

They then establish that $\omega_F \delta_{\mathcal{V}}$ is the working version of the form $d\mathcal{F}$ that we seek.

Theorem 5.10 *Let $\Omega \subseteq \mathbf{C}^d$ be a compact real $d+1$-manifold with boundary. Let F be meromorphic with no double poles on Ω and let $\mathcal{V}$ denote the pole set of F. If $\mathcal{V}$ intersects Ω transversely in a $d-1$-manifold with finite volume and $F\,dV \in L^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} F\,dV = 2\pi i \int_{\Omega\cap\mathcal{V}} \omega_F.$$

PROOF: First suppose $F = G/z_1$ for some analytic function, G. We need to show

$$\int_{\partial\Omega} \frac{G}{z_1}\,dV = \int_{\Omega\cap S} G\,dz_2 \wedge \cdots \wedge dz_d.$$

where S is the $z_1 = 0$ hyperplane. Let S_ϵ denote the set $|z_1| \leq \epsilon$ and let ω_ϵ denote the form $G\,dV/z_1$ outside of S_ϵ and $\epsilon^{-2}\overline{z}_1 G\,dV$ on S_ϵ. Then ω_ϵ is continuous, piecewise smooth and locally in L^1. Its differential is

$$d\omega_\epsilon = \epsilon^{-2}\mathbf{1}_{S_\epsilon} G\,d\overline{z}_1 \wedge dV.$$

Apply Stokes' Theorem to ω_ϵ, yielding

$$\int_{\partial\Omega} \omega_\epsilon = \epsilon^{-2} \int_{\Omega\cap S_\epsilon} G\,d\overline{z}_1 \wedge dV. \tag{5.14}$$

Since $G\,dV/z_1 \in L^1(\partial\Omega)$, the forms ω_ϵ converge to $\omega := G\,dV/z_1$ in $L^1(\partial\Omega)$ and the LHS of (**??**) converges to $\int_{\partial\Omega} \omega$. It therefore suffices to show that the RHS converges to $2\pi i \int_{\Omega\cap S} G\,dz_2 \wedge \cdots \wedge dz_d$.

By transversality, the projection of Ω onto $\mathbf{C} \times (\Omega \cap \mathcal{V})$ is a diffeomorphism. The convergence we are trying to establish is a local property, so we may restrict to a neighborhood where there is a parameterization of Ω by $\mathbf{C} \times (\Omega \cap S)$. Specifically, we may assume there is a diffeomorphic map $f : \mathbf{C} \times (\Omega \cap S) \to S$ such that

$$\Omega = \{(x_1, y_1, \alpha + f(\alpha)) : (x_1, y_1) \in \mathbf{C}, \alpha \in \Omega \cap S\}.$$

Pulling back $d\bar{z}_1 \wedge dV$ by f gives $2i dx_1 \wedge dy_1 \wedge \iota^*(dz_2 \wedge \cdots \wedge dz_d)$ where ι is the inclusion of $\Omega \cap S$ in $\mathbf{C}^d$. Now write (**??**) as an iterated integral:

$$\int_{\alpha \in \Omega \cap S} \left[2i \int_{x_1^2 + y_1^2 \leq \epsilon} \epsilon^{-2} G(x_1, y_1, \alpha + f(x_1, y_1, \alpha))\, dx_1 \wedge dy_1 \right] \iota^*(dz_2 \wedge \cdots \wedge dz_d).$$

The inner integral converges uniformly to $\pi G(0, \alpha)$, so by the finiteness of $\Omega \cap S$, the RHS of (**??**) converges to

$$2\pi i \int_{\Omega \cap S} G \cdot \iota^*(dz_2 \wedge \cdots \wedge dz_d)$$

as desired.

Now remove the assumption $F = G/z_1$. We may always partition Ω into neighborhoods in which there is a bi-analytic map ψ mapping the $\mathcal{V}$ to the set $S := \{z_1 = 0\}$. The function $F \circ \psi^{-1}$ has a simple pole on S so we may represent $F \circ \psi^{-1} = G/z_1$ for some analytic function G. Observe that ψ maps Ω to a manifold intersecting S transversely in a set of finite volume. Pulling back $F\, dV$ by ψ^{-1} gives a form G/z_1, to which we apply the previous analysis and conclude that

$$\int_{\psi \partial \Omega} \frac{G\, dV}{z_1} = 2\pi i \int_{\psi \Omega \cap S} \omega^*$$

where $\omega^* := G\, dz_2 \wedge \cdots \wedge dz_d$. Pulling this equation back by ψ gives

$$\int_{\partial \Omega} F\, dV = \int_{\Omega \cap \mathcal{V}} \psi^* \omega^*.$$

The form ω^* satisfies $dz_1 \wedge \omega^* = G\, dV$; pulling this relation back by ψ yields

$$d(z_1 \circ \psi) \wedge \psi^* \omega^* = G \circ \psi\, dV.$$

Since F may be represented as $\dfrac{G \circ \psi}{H \circ \psi}$, we then conclude that $\psi^* \omega^*$ is the unique form satisfying the definition of ω_F, which finishes the proof. □

When we apply this to the tori T_0 and T_1, we find that

$$a_{\mathbf{r}} \sim \int_E \omega_F$$

where E is the intersection of $\mathcal{V}$ with any homotopy from an infinitesimal torus to an infinite one. The homotopy class must avoid the coordinate planes as well as $\mathcal{V}$, since $\mathbf{z}^{-\mathbf{r}}G$ is not analytic there. Evidently, the possible sets E form a homology class on $\mathcal{V}$ of dimension $d-1$, half the dimension of $\mathcal{V}$. It is always possible to satisfy the transversality condition and to avoid any double or multiple poles of F [since any of these are necessarily isolated, any generic perturbation of the homotopy will miss them].

Choosing E to contain a stationary phase point for ω_F where the modulus of ω_F is maximized leads, as before, to an oscillating integral. Thus for example in the two dimensional case, $\mathcal{F} = (z^{-n-1}w^{-m-1}G)/H \, dz \wedge dw$ and $\omega_F = (z^{-n-1}w^{-m-1}G)(dV/dH)$ on $\mathcal{V}$, where dv/dH is a form that wedges with dH to yield dV. The appearance of $z^{-n}g(z)^{-m}\psi(z)dz$ as the integrand in (**??**) is now explained as a non-coordinate free version of $z^{-n}w^{-m}(dV/dH)$ (recalling that $g(z) = w$ on the set $\mathcal{V}$).

Unfortunately, while we may arrange for the homotopy to avoid singular points of $\mathcal{V}$, these critical points are sometimes the places where stationary phase is obtained for ω_F. Thus we have an open question: **How can we make sense of $\int_E \omega_F$ when E is required to contain a given critical point of $\mathcal{V}$?**

5.4 Furstenburg-Doubilet-Stanley-Rota extraction of the diagonal

Let $F(z,w) = \sum_{r,s} a_{r,s} z^r w^s$ be a function analytic in a neighborhood of 0 in $\mathbf{C}^2$. Define $\xi(y) = \sum_r a_{r,r} y^r$. The function ξ may be determined directly from F. When F is a rational function, it turns out that ξ will always be algebraic; this result in the setting of generating functions is credited to Furstenburg (1967) and appears in Hautus and Klarner (1971). In general, the determination of ξ from F may not be computationally effective, so we stick here to the case where F is rational. Stanley (1999) gives two ways of determining ξ from F. One is a formal power series approach. The other, which I will follow here, is analytic.

When $|y|$ is sufficiently small, the function $F(z, y/z)$ is absolutely convergent for z in some annulus $A(y)$. Treating y as a constant, we view $F(z, y/z)$ as a Laurent series in z inside the annulus $A(y)$; the constant term C of this series is equal to $\xi(y)$. Thus if we are able to evaluate this constant term as a function of y, we will have the function whose power series in a neighborhood of 0 is ξ.

Writing $F = P/Q$, with P and Q polynomial, we observe that the constant term of $F(z, y/z)$ is equal to

$$\frac{1}{2\pi i} \int_{\mathcal{C}} P \frac{(z, y/z)}{zQ(z, y/z)} dz$$

where $\mathcal{C}$ is any circle in the annulus of convergence $A(y)$. By Cauchy's integral formula then,

$$\xi(y) = \sum \text{Res} \left(\frac{P(z, y/z)}{zQ(z, y/z)}; \alpha \right)$$

where the sum is over residues at poles α inside the inner circle of the annulus $A(y)$. For sufficiently small y, the poles may be parametrized by continuous functions of y. As $y \to 0$, the annuli $A(y)$ increase to an annulus with inner radius zero, so any poles $\alpha(y)$ that converge to zero as $y \to 0$ will be included in this sum, and any that do not will be outside $\mathcal{C}$ and will not be included in the sum. The residues are all algebraic functions of y, so we have represented ξ as the sum of algebraic functions of y. We illustrate by re-computing the diagonal for the lattice path generating function (cf. Stanley 1999).

Example ?? continued (lattice paths)

Recall that $F = 1/(1 - z - w - zw)$, so

$$z^{-1} F(z, y/z) = \frac{1}{z - z^2 - y - yz} .$$

The poles of this are at

$$z = \frac{1 - y}{2} \pm \frac{1}{2} \sqrt{1 - 6y + y^2} .$$

Let α_1 denote the root going to zero with y, that is, the one with the minus sign, and let α_2 denote the other root. Since $z^{-1} F(z, y/z) = -1/[(z - \alpha_1)(z - \alpha_2)]$, the residue at $z = \alpha_1$ is just $-1/(\alpha_1 - \alpha_2)$ which is simply $(1-6y+y^2)^{-1/2}$. Thus

$$\xi(y) = \frac{1}{\sqrt{1-6y+y^2}}.$$

To obtain an asymptotic expression for $a_{r,r}$ we write

$$\xi(y) = (1-\beta_+ y)^{-1/2}(1-\beta_- y)^{-1/2}$$

where $\beta_\pm = 3 \pm \sqrt{8}$ are the (reciprocal) roots of $1-6y+y^2=0$. The minimum modulus singularity is at β_-, and as $y \to \beta_-$,

$$\xi(y) \sim (1-\beta_-^2)^{-1/2}\left(1-\frac{y}{\beta_-}\right)^{-1/2}.$$

Applying the Flajolet-Odlyzko Transfer Theorem **??** and Corollary **??**, we see that the n^{th} coefficient, c_n of $\xi(\beta_- y))$ satisfies

$$c_n \sim \frac{(1-\beta_-^2)^{-1/2}}{\Gamma(1/2)} n^{-1/2}$$

and hence that

$$a_{r,r} \sim \frac{(1-\beta_-^2)^{-1/2}}{\sqrt{\pi}} r^{-1/2}\beta_+^r.$$

When F is a rational function of more than two variables, the diagonal need not be algebraic. Extending the hierarchy one more step, we define the class of D-finite functions to be those satisfying a linear differential equation with polynomial coefficients. These are generating functions for P-recursive arrays: arrays satisfying a linear recursion with polynomial coefficients. This class strictly contains the algebraic functions. It was shown in Lipshitz (1988) that the class of D-finite functions is closed under diagonal collapse (taking the diagonal in any two variables) and thus in particular, that the diagonals of any rational or algebraic function are D-finite. Extraction of the polynomial coefficients of the differential equation (equivalently the polynomial coefficients of the recursion) has recently been made effective; see Chyzak and Salvy (1998). One may then effectively obtain asymptotics from these recursions, and indeed for diagonal directions, this is probably the best way to obtain asymptotics. The method may be adapted to give recursions along other fixed rays with rational slopes. Unfortunately, the computation does not admit slope as a parameter, so the method seems ill suited to obtaining any truly multivariate formulae.

5.5 Comparison to GF-sequence results

6 Toral points

Suppose that $\mathcal{V}$ has infinitely many minimal points on a torus $T = T(\alpha_1, \ldots, \alpha_d)$ where $\alpha_j > 0$. What happens when we mimic the derivation of the multivariate reduction to an oscillating integral (Lemma **??** and equation **??**)? Let $\widehat{T}$ denote the torus $T(\widehat{\alpha}) \subseteq \mathbf{C}^{d-1}$ and define the set $E \subseteq \widehat{T}$ by $\widehat{\mathbf{w}} \in E$ if there is a point $\mathbf{w} = (\widehat{\mathbf{w}}, w_d) \in \mathcal{V} \cap T$. In other words, it is the set of projections of $\mathcal{V} \cap T$ onto the first $d-1$ coordinates. Let $\mathcal{N}$ be a neighborhood of E in $\widehat{T}$. As before, we write the Cauchy integral for $a_{\mathbf{r}}$ as an iterated integral over $\widehat{T} \times \mathcal{C}_1$ for a circle $\mathcal{C}_1$ with radius a little smaller than $|z_d|$; then again the contribution from $\mathcal{N}^c \times \mathcal{C}_1$ is $O(R|\mathbf{z}^{-\mathbf{r}}|)$ for some exponentially decreasing function R, so we may localize to $\mathcal{N}$.

Since we have not yet dealt with poles of multiplicity greater than 1, we assume now that $\mathcal{V} \cap T$ contains only simple poles of F. Let Π denote the map from $\mathcal{V} \cap T$ to $\mathbf{C}^{d-1}$ projecting onto the first $d-1$ coordinates. For any $\widehat{\mathbf{z}} \in E$, the set $\Pi^{-1}(\mathbf{z})$ is finite. As $\widehat{\mathbf{z}}$ varies over E each of these inverse images varies analytically except when two inverse images meet. Since $\Pi^{-1}(E)$ contains no double points, two inverse images never meet. Hence $\Pi^{-1}(E)$ is an analytic k-fold cover of E for some $k \geq 1$. It follows that there is a neighborhood $\mathcal{N}_T$ of T and a neighborhood $\mathcal{N}$ of E in $\widehat{T}$ such that for each $\widehat{\mathbf{z}} \in \mathcal{N}$ there are precisely k values of z such that $(\widehat{\mathbf{z}}, z) \in \mathcal{N}_T \cap \mathcal{V}$. Thus $\mathcal{N}_T \cap \mathcal{V}$ is a k-fold cover of $\mathcal{N}$. In other words, for each $(\widehat{\mathbf{z}}, z) \in \mathcal{N}_T \cap \mathcal{V}$ there is a neighborhood of $\widehat{\mathbf{z}}$ in $\widehat{T}$ on which is defined an analytic function g_z with $g_z(\widehat{\mathbf{z}}) = z$ and $(\widehat{\mathbf{z}}, g(\widehat{\mathbf{z}})) \in \mathcal{N}_T \cap \mathcal{V}$. The subscript z is there only to delineate one of the k possible inverse images.

Let g be such a parametrizing function. The residue at $(\widehat{\mathbf{w}}, g(\widehat{\mathbf{w}}))$ of $w^{-r_d-1}F$ is still given by (**??**):

$$R(\widehat{\mathbf{w}}) := -\psi(\widehat{\mathbf{w}})g(\widehat{\mathbf{w}})^{-r_d}\,.$$

Choosing $\mathcal{C}_2$ to be a circle of radius $|z_d| + \epsilon$, and then choosing $\mathcal{N}$ sufficiently small, we arrive once more at (**??**): $|\mathbf{z}^{\mathbf{r}}||a_{\mathbf{r}} - \Xi|$ decays exponentially, where

$$\Xi := (2\pi)^{1-d}\mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \exp(-r_d \tilde{f}(\widehat{\theta}))\tilde{\psi}(\widehat{\theta})\, d\widehat{\theta}.$$

The set E is a j-dimensional sub-manifold of $T(\widehat{\mathbf{z}})$ for some j. Choosing $\mathcal{N}$ smaller if necessary, we may take it to be locally the topological product of E with a closed ball about the origin in $\mathbf{R}^{d-1-j}$. Thus $\mathcal{N}$ is a $(d-1)$-dimensional nice cell complex sitting inside $\widehat{T}$. Since E has no boundary, $\partial\mathcal{N}$ is disjoint from E. For the purposes of the cell complex oscillating integral lemma, any stationary point must have $\text{Re}\{\tilde{f}\} = 0$, which in this case means it must be in E. This means that the stationary points for $\tilde{f}$ in the integral in (**??**) are exactly those places in E where $\bigtriangledown \tilde{f} = \mathbf{0}$. These in turn are exactly projections $\widehat{\mathbf{z}} = \Pi(\mathbf{z})$ of points $\mathbf{z}$ with $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$.

We finally arrive at a result analogous to the main result for strictly minimal simple poles. The following theorem assumes one further nondegeneracy hypothesis (nonsingularity of $\tilde{f}$) in order to apply Theorem **??**.

Theorem 6.1 *Let $F = G/H = \sum a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ have infinitely many minimal points on $T = T(\alpha_1, \ldots, \alpha_d)$. Assume that all the singularities of F on T are simple poles. Let $S(\mathbf{r})$ denote the set of $\mathbf{z} \in T \cap \mathcal{V}$ for which $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. For each $\mathbf{z} \in T \cap \mathcal{V}$ let $l_0(\mathbf{z})$ denote the order of vanishing of G on $\mathcal{V}$ at $\mathbf{z}$. For fixed $\mathbf{z}$ we let g be a local parametrization of $\mathcal{V} = \{(\widehat{\mathbf{w}}, g(\widehat{\mathbf{w}})\}$ near $\widehat{\mathbf{z}}$ and we define $\tilde{f}$ to be $\log g$ centered and expressed in terms of $\widehat{\theta}$ as before. Then whenever $S(\mathbf{r})$ contains only isolated points where the Hessian of $\tilde{f}$ is nonsingular, there is an expansion*

$$a_{\mathbf{r}} \sim \sum_{\mathbf{z}\in S(\mathbf{r})} \mathbf{z}^{-\mathbf{r}} \sum_{l\geq l_0(\mathbf{z})} C_l r_d^{(1-d-l)/2}\,.$$

When G does not vanish at $\mathbf{z}$ then $l_0 = 0$ and

$$C_0 = (2\pi)^{(1-d)/2}\mathcal{H}^{-1/2}\frac{G(\mathbf{z})}{z_d H_d}$$

where $\mathcal{H}$ is the determinant of the Hessian at $\mathbf{z}$*. The expansions are uniform as* $\mathbf{r}$ *varies over compact sets on which l_0 is constant (and $\mathcal{H} \neq 0$).* $\square$

As usual, in dimension two it is easier to handle degeneracy of $\tilde{f}$. As usual for two dimensions, we switch our notation to (z, w) for (z_1, z_2) and we introduce (α, γ) for the radii of the torus, T. Now $E = \widehat{T}$ is just the α-circle (circle of radius α in $\mathcal{C}^1$) and $\Pi^{-1}(E)$ is a disjoint union of circles embedded in the two-torus, $T = T(\alpha, \gamma)$. Since $E = \widehat{T}$, we do not localize but instead must take $\mathcal{N} = E$. Enlarging the inner circle of integration to radius $\gamma + \epsilon$ and comparing gives

$$a_{r,s} = \frac{1}{2\pi i} \int_{|z|=\alpha} z^{-r-1} \sum_{x:|x|=\gamma,(z,x)\in\mathcal{V}} \text{Res}(w^{-s-1}F; (z, x))\, dz + O(\alpha^{-r}(\gamma + \epsilon)^{-s}).$$

The parametrizing functions g always have constant modulus, hence $\tilde{f}' = d(\log g)/d\theta$ will always be purely imaginary. Since $\tilde{f}$ is normalized to be 0 at 0, $\tilde{f}$ is purely imaginary.

We are now in a position to substitute our evaluation of purely oscillating integrals in one variable, Theorem **??**, where before we used the multivariate Theorem **??** and had to assume nondegeneracy of the Hessian of $\tilde{f}$.

Theorem 6.2 *Let $F = G/H = \sum a_{rs} z^r w^s$ and suppose as above that there are $\alpha, \gamma > 0$ such that for each θ there are exactly k values of ϕ for which $(z, w) \in \mathcal{V}$, where $z := \alpha e^{i\theta}$ and $w := \gamma e^{i\phi}$. Suppose further that each such (z, w) is a simple pole of F. Let $\tilde{f}$ be a local parametrization of $i\phi$ in terms of θ. Let $k = k(\theta)$ be the first integer greater than 1 such that the k^{th} derivative of $\tilde{f}(\theta)$ is nonzero. Let $l_0 = l_0(\theta)$ be the vanishing order of G at (z, w). Then there is an expansion*

$$a_{r,s} \sim \alpha^{-r} \gamma^{-s} \sum_{l \geq l_0} C_l s^{-(l+1)/k}, .$$

If $l_0 = 0$, the leading term is given by

$$C_0 = \sum_{(z,w)\in S(r,s)} \frac{A(k, 0)}{2\pi} \frac{G(z, w)}{(wH_w)} e^{-ir\text{Arg}(z) - s\text{Arg}(w)} \left(\frac{\tilde{f}^{(k)}}{k!} \right)^{-1/k}$$

uniformly as $r, s \to \infty$ with $(r, s) \in \mathbf{dir}(\alpha e^{i\theta})$ on compact sets where k and l_0 do not change.

Example 15 (H has degree 1 separately in z and w)

For an example, let us consider generating functions of the form $1/(a+bz+cw+dzw)$. We can factor out a and change variables to bz and cw; thus without loss of generality we take $F(z,w) = 1/(1-z-w+\beta zw)$. When $\beta < 1$, the minimal points of $\mathcal{V}$ are all in the positive quadrant and are all strictly minimal simple poles. The coefficients a_{rs} are all nonnegative. A uniform expansion of central limit type holds and is not hard to compute. The lattice paths are an example of this with $\beta = -1$, as are binomial coefficients ($\beta = 0$). When $\beta > 1$, the coefficients are of mixed sign, and the intersection of $\mathcal{V}$ and the positive quadrant no longer consists entirely of minimal points. Instead, the set of minimal points is describable as follows.

We parametrize $\mathcal{V}$ by

$$w = g(z) := \frac{1-z}{1-\beta z}\,.$$

For $(z, g(z))$ to be locally minimal, recall it is necessary that zg'/g be real. Setting

$$z\frac{g'}{g} = \frac{(\beta-1)z}{(1-z)(1-\beta z)} = r \in \mathbf{R}\,,$$

we get

$$z^2 - 2Cz + \frac{1}{\beta} = 0,$$

where $2C = (1+\beta-(\beta-1)/r)/\beta$ is any real number. When $|C| \geq \beta^{-1/2}$, the solutions to this are real, and when $|C| < \beta^{-1/2}$ the solutions make up the circle $\{z : |z| = \beta^{-1/2}\}$. In other words, the entire circle of radius $\beta^{-1/2}$ is preserved by the linear fractional map g. It is easy to check that the real values of z corresponding to minimal roots are those with $z \in I_1 \cup I_2 := (-\beta^{-1/2}, 0] \cup (\beta^{-1/2}, 1]$.

Computing $zH_z/(wH_w)$ gives $z(\beta w-1)/(w(\beta z-1))$, and on the variety $\mathcal{V}$ this reduces to $(w-1)/(z-1)$. For real values of $w \in I_1$, the ratio $z\phi_z/(w\phi_w)$ increases monotonically from $(\sqrt{\beta}+1)/(\sqrt{\beta}-1)$ to infinity, while for $z \in I_2$, the reciprocal $w\phi_w/(z\phi_z)$ does the same. Hence for $r/s \notin [(\sqrt{\beta}-1)/(\sqrt{\beta}+1), (\sqrt{\beta}+1)/(\sqrt{\beta}-1)]$, Theorem **??** will apply.

Asymptotics when $(\sqrt{\beta}-1)/(\sqrt{\beta}+1) \le r/s \le (\sqrt{\beta}+1)/(\sqrt{\beta}-1)$ are obtained from the torally minimal points of modulus $\beta^{-1/2}$. The geometry is simple since both E and $\mathcal{N}$ are the whole circle $\mathcal{C} := \{z : |z| = \beta^{-1/2}\}$. We have $|z^r||w^s||a_{r,s} - \Xi|$ decreasing exponentially, where since $G = 1$,

$$\Xi = \frac{1}{2\pi} z^{-r} w^{-s} \int_{\mathcal{C}} \exp(s\tilde{f}(\theta))\, d\theta\, .$$

The function $\tilde{f}$ is purely imaginary since E is the entire 1-torus. Direct computation shows that the stationary point of $\tilde{f}$ is of order two except when r/s takes on the extreme values of $(\sqrt{\beta}+1)/(\sqrt{\beta}-1)$ or its inverse; here $\tilde{f}''$ vanishes as well and the stationary point is of order 3. Plugging the computations into Theorem **??** then yields the following result.

Theorem 6.3 *If* $r/s = (\sqrt{\beta}+1)/(\sqrt{\beta}-1)$ *then*

$$(-1)^r \frac{\beta^{(r+s+1)/2}}{6^{2/3}\Gamma(2/3)} (2\sqrt{\beta}(1+\sqrt{\beta}))^{-1/3}\, .$$

If $r/s = (\sqrt{\beta}-1)/(\sqrt{\beta}+1)$ *then the the value is the same with* r *and* s *switched. If* $(\sqrt{\beta}-1)/(\sqrt{\beta}+1) < r/s < (\sqrt{\beta}+1)/(\sqrt{\beta}-1)$, *then*

$$a_{r,s} = 2\beta^{(r+s)/2} \cos(\Psi(r,s)) \sqrt{\frac{\beta}{\pi(\beta-1)}} \left(\frac{4}{\beta-1} rs - (s-r)^2 \right)^{-1/4} + O(s^{-1})$$

where the phase factor is given by

$$\Psi(r,s) = \left(-\frac{\pi}{4} - r\theta_0 - sh(\theta_0) + \arctan \sqrt{\frac{4rs}{(\beta-1)(r+s)^2} - \frac{(r-s)^2}{(r+s)^2}} \right)\, .$$

Remark: If r/s remains fixed, the first and last terms of the phase are constant, so the phase varies linearly along rays from the origin.

Proof: We pick a toral minimal point $(z,w) = (z(\theta), g(z(\theta)))$ and expand. Recall that $z = \beta^{-1/2} e^{i\theta}$ is on the circle of radius $\beta^{-1/2}$, as is $w = g(z) = (1-z)/(1-\beta z)$. Write $\tilde{f}(\theta) = ih(\theta)$ and compute the derivatives of h. Since $h = (1/i)\log g$, and $(d/d\theta) = iz(d/dz)$, we see that

$$\begin{aligned} h'(\theta) &= z(\log g(z))' \\ &= z\left(\frac{-1}{1-z}+\frac{\beta}{1-\beta z}\right) \\ &= 1-\frac{1}{1-z}-\frac{1}{1-\frac{1}{\beta z}}. \end{aligned}$$

Differentiating again,

$$\begin{aligned} h''(\theta) &= iz\frac{d}{dz}h'(\theta) \\ &= \frac{-iz}{(1-z)^2}+\frac{i/(\beta z)}{(1-\frac{1}{\beta z})^2} \\ &= 2\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} \end{aligned}$$

since z and $1/(\beta z)$ are conjugate.

Stationary points are found by solving $h'(\theta)+r/s=0$ which gives

$$\frac{1}{1-\beta z}-\frac{1}{1-z}=-\frac{r}{s}$$

and hence

$$1-\left[\beta+1-\frac{s}{r}(\beta-1)\right]z+\beta z^2=0\,. \tag{6.15}$$

Also, $h''(\theta)$ vanishes simultaneously with $h'(\theta)$ if and only if (**??**) has a double root, which happens when $\beta+1-(s/r)(\beta-1)=\pm 2\sqrt{\beta}$, or when

$$\frac{r}{s}=\frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}$$

or its reciprocal.

Consider first the case when r/s is not one of these two extreme values. There will be two conjugate solutions to (**??**) and we call the one with positive argument (z_0,w_0), where θ_0 is the argument, $z_0=z(\theta_0(r,s))=\beta^{-1/2}e^{i\theta_0}$, and $w_0=g(z_0)=(1-z_0)/(1-\beta z_0)$. The other one is the conjugate, $(z(-\theta_0),g(z(-\theta_0)))$. We now compute the derivatives of h in terms of r and s.

Lemma 6.4 *Suppose z satisfies (??). Then*

$$\frac{1}{1-z} = \sqrt{\frac{\beta r}{(b-1)s}} \exp\left(\pm i \arctan\sqrt{\frac{4rs}{(\beta-1)(r+s)^2} - \frac{(r-s)^2}{(r+s)^2}}\right) .$$

Proof: Begin with the observation that if $1 - bx + ax^2 = 0$, then

$$\frac{1}{1-x} = \frac{b-a-ax}{b-a-1} .$$

This is proved by crossmultiplying. Apply this with $a = \beta$ and $b = \beta + 1 - (s/r)(\beta - 1)$ and use the quadratic formula $z = b/(2a) \pm \sqrt{b^2 - 4a}/(2a)$ to get

$$\frac{1}{1-z} = \frac{(1+(s/r))(\beta-1) \pm i\sqrt{2\frac{s}{r}(\beta+1)(\beta-1) - (1+\frac{s^2}{r^2})(\beta-1)^2}}{2\frac{s}{r}(\beta-1)} .$$

Simplify this to obtain

$$\frac{1}{1-z} = \frac{r+s \pm i\sqrt{\frac{4rs}{\beta-1} - (r-s)^2}}{2s} .$$

The modulus of this is $\sqrt{\beta r/(\beta-1)s}$ and since the real part of $1-z$ is positive for any $|z| < 1$, the argument has absolute value at most $\pi/2$ and is therefore the arctangent of the ratio of imaginary to real parts. □

Lemma 6.5 *Suppose $1 - bz + \beta z^2 = 0$ with $b = \beta + 1 - (s/r)(\beta - 1)$. Then*

$$\begin{aligned} 2\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} &= \pm\frac{r^2}{s^2}\sqrt{\frac{4}{\beta-1}\frac{s}{r} - (\frac{s}{r}-1)^2} \\ &= \pm\frac{r}{s^2}\sqrt{\frac{4}{\beta-1}rs - (r-s)^2} . \end{aligned}$$

Proof: Recall that

$$\begin{aligned} 2\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} &= \frac{-iz}{(1-z)^2} + \frac{(i/\beta z)}{(1-\frac{1}{\beta z})^2} \\ &= -i\frac{(z-\frac{1}{\beta z})(1-\beta^{-1})}{(1-z)^2(1-\frac{1}{\beta z})^2} . \end{aligned}$$

The denominator is the square of $1 - bz + \beta^{-1}$, so this becomes

$$
\begin{aligned}
2\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} &= -i(z - \frac{1}{\beta z})\frac{1-\beta^{-1}}{(\frac{\beta+1-b}{\beta})^2} \\
&= -i(z - \frac{1}{\beta z})\frac{\beta(\beta-1)}{\frac{r^2}{s^2}(\beta-1)^2} \\
&= -i(z - \frac{1}{\beta z})\frac{\beta}{\frac{r^2}{s^2}(\beta-1)} \,.
\end{aligned}
$$

Since z and $1/(\beta z)$ are the two roots of the equation, and since the difference between the two roots is $\pm 2\sqrt{b^2 - 4\beta}/(2\beta)$, we get

$$
\begin{aligned}
2\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} &= \pm\frac{r^2}{s^2(\beta-1)}\sqrt{2\frac{s}{r}(\beta+1)(\beta-1) - \frac{s^2}{r^2}(\beta-1)^2 - (\beta-1)^2} \\
&= \pm\frac{r^2}{s^2}\sqrt{2\frac{s}{r}\frac{\beta+1}{\beta-1} - (\frac{s^2}{r^2}+1)} \\
&= \pm\frac{r^2}{s^2}\sqrt{4\frac{s}{r}\frac{1}{\beta-1} - (\frac{s}{r}-1)^2} \,.
\end{aligned}
$$

□

Now apply Theorem **??** at $\theta = \pm\theta_0$. We see that $a_{r,s} = C_0 \beta^{(r+s)/2} s^{-1/2} + O(s^{-1})$, where

$$
C_0 = \sum_{\theta = \pm\theta_0} e^{-ir\theta - ish(\theta)} \frac{\Gamma(1/2)}{2\pi} e^{\pm i\pi/4} \frac{1}{1 - z(\theta)} \sqrt{\frac{2}{sh''(\theta)}} \,.
$$

The two summands are conjugate, so we rewrite this as

$$
a_{r,s} \sim \sqrt{\frac{1}{2\pi}} \beta^{(r+s)/2} \frac{1}{1-z_0} \sqrt{\frac{2}{sh''(\theta_0)}} \mathrm{Re}\{\exp(-i\pi/4 + i\arg(1-z_0)^{-1} + i(r-s)\theta_0)\} \,.
$$

The computation of $h''(\theta)$ in the previous lemma gives

$$
h''(\theta_0) = \pm\frac{r}{s^2}\sqrt{\frac{4}{\beta-1}rs - (r-s)^2}.
$$

Using Lemma ?? then gives

$$
\begin{aligned}
a_{r,s} \quad \sim \quad & \sqrt{\frac{1}{2\pi}}\beta^{(r+s)/2}\sqrt{\frac{\beta r}{(\beta-1)s}}\sqrt{\frac{2}{\frac{r}{s}\sqrt{4(\beta-1)^{-1}rs-(r-s)^2}}} \\
& \cdot\cos\left(-\frac{\pi}{4}-s\theta_0-rh(\theta_0)+\arctan\sqrt{\frac{4rs}{(\beta-1)(r+s)^2}-\frac{(r-s)^2}{(r+s)^2}}\right)
\end{aligned}
$$

and simplifying yields

$$
\begin{aligned}
a_{m,n} \quad \sim \quad & \beta^{(r+s)/2}\sqrt{\frac{\beta}{\pi(\beta-1)}}\left(\frac{4}{\beta-1}rs-(r-s)^2\right)^{-1/4} \\
& \cdot\cos\left(-\frac{\pi}{4}-s\theta_0-rh(\theta_0)+\arctan\sqrt{\frac{4rs}{(\beta-1)(r+s)^2}-\frac{(r-s)^2}{(r+s)^2}}\right)+O(s^{-1})
\end{aligned}
$$

which finishes the argument in this case.

Finally, we consider the case when r/s is one of the two extreme values. Assume without loss of generality that $r/s=(\sqrt{\beta}+1)/(\sqrt{\beta}-1)$. The first two derivatives of h then vanish at zero and nowhere else. We compute (e.g., with Maple)

$$
h'''(0)=\frac{2\beta^{-1}}{(1-\beta^{-1/2})^3}+\frac{2\beta}{(\beta^{1/2}-1)^3}=\frac{2\sqrt{\beta}(1+\sqrt{\beta})}{(\sqrt{\beta}-1)^3}\neq 0\,,
$$

so the order of vanishing of h at zero is precisely 3. Since $z=-g(z)=\beta^{-1/2}$ when $\theta=0$, $z^{-r}w^{-s}=\beta^{(r+s)/2}$. Theorem ?? then gives

$$
a_{r,s}\sim\beta^{(r+s)/2}\frac{A(k,0)}{2\pi}\frac{1}{1-\beta^{-1/2}}(-1)^r\left(\frac{h'''(0)}{6}\right)^{-1/k}.
$$

Using the identity $A(3,0)/(2\pi)=1/(6\Gamma(2/3))$, we get

$$
a_{r,s}\sim\frac{\beta^{(r+s)/2}}{6\Gamma(2/3)}\frac{1}{1-\beta^{-1/2}}(-1)^r\frac{\sqrt[3]{6}(\sqrt{\beta}-1)}{(2\sqrt{\beta}(1+\sqrt{\beta}))^{1/3}}
$$

which simplifies to

$$
(-1)^r\frac{\beta^{(r+s+1)/2}}{6^{2/3}\Gamma(2/3)}(2\sqrt{\beta}(1+\sqrt{\beta}))^{-1/3}\,.
$$

7 Multiple points

When $\mathbf{z}$ is a minimal, geometrically smooth point of $\mathcal{V}$, it is either strictly minimal, finitely minimal or torally minimal, and is smooth for a square-free denominator in the radical of H. We have obtained asymptotics in all these cases. What remains is to analyze expansions around non-smooth points of $\mathcal{V}$. The simplest geometry in the non-smooth case is an *isolated multiple point*, where locally $\mathcal{V}$ is described as the union of finitely many sheets, $\mathcal{V}_1, \ldots, \mathcal{V}_k$, intersecting only at $\mathbf{z}$. We will assume that each of these sheets projects diffeomorphically onto the first $d-1$ coordinates. In this case, the sheet $\mathcal{V}_j$ may be parametrized by $(\widehat{\mathbf{w}}, u_j(\widehat{\mathbf{w}}))$ for a collection of functions u_j, analytic for $\widehat{\mathbf{w}}$ in a neighborhood of $\widehat{\mathbf{z}}$ and whose graphs intersect only at $\widehat{\mathbf{z}}$. The only generality lost by this assumption is if $\mathcal{V}$ contains a sheet parallel to each coordinate hyperplane, a case of which we know no natural example. A version of the Weierstrass preparation lemma allows us to describe F algebraically near $\mathbf{z}$ as follows.

Definition 4 *Let $v_j(\widehat{\mathbf{w}}) = 1/u_j(\widehat{\mathbf{w}})$ denote the inverses of the zeros of $H(\widehat{\mathbf{w}}, \cdot)$.*

The formulas do not seem particularly simpler in terms of v's than they do in terms of u's, but it will be crucial at a later step to be working with inverse zeros.

Lemma 7.1 (Weierstrass preparation) *There are exponents $n_1, \ldots, n_j$ and a function χ analytic near $\mathbf{z}$, such that $u_j(\widehat{\mathbf{w}}) = w_d$ for each j, $\bigtriangledown u_j \neq \mathbf{0}$ for each j, and such that*

$$F(\mathbf{w}) = \frac{\chi(\mathbf{w})}{\prod_{j=1}^{k}(1 - w_d v_j(\widehat{\mathbf{w}}))^{n_j}} .$$

Proof: For any fixed $\widehat{\mathbf{w}} \neq \widehat{\mathbf{z}}$ in a neighborhood of $\widehat{\mathbf{z}}$, the function $F(\widehat{\mathbf{w}}, \cdot)$ has a pole of some order, n_j at the point $u_j(\widehat{\mathbf{w}})$. The integer n_j must be constant on the sheet $\mathcal{V}_j$. The product $\prod_{j=1}^{k}(1 - w_d v_j(\widehat{\mathbf{w}}))^{n_j} F$ must then have a finite limit as $w_d \to u_j(\widehat{\mathbf{w}})$ for each $\widehat{\mathbf{w}} \neq \widehat{\mathbf{z}}$. A meromorphic function goes to infinity wherever it is not singular, hence the product is analytic everywhere in a neighborhood of $\mathbf{z}$ except possibly at $\mathbf{z}$. By Hartogs' Theorem (see Griffiths and Harris 1994, page 7) there are no isolated singularities in $\mathbf{C}^d$ for $d \geq 2$, hence the product is analytic in a neighborhood of $\mathbf{z}$, which proves the lemma. □

Remark 1: The Weierstrass preparation theorem in its stronger form says that a function whose power series has a nonzero pure w_d term may be multiplied by an analytic function so as to arrive at a function $w_d^l + \sum_{j=1}^l h_j(\widehat{\mathbf{w}}) w_d^{l-j}$, where l is the degree of the minimal pure w_d term. Stated in this way, one sees that coefficients of these power series may be effectively computed.

Remark 2: Immediately following their proof of Hartog's Theorem, Griffiths and Harris give a proof of the Weierstrass preparation lemma, different from the one above, in which they find an analytic function by which to multiply H so that it becomes a polynomial in z_d with coefficients in $\mathbf{C}[[z_1, \ldots, z_{d-1}]]$. They avoid having to argue that n_j is constant on sheets, which is glossed over in the above proof.

Remark 3: The description of the functions u_j, v_j parametrizing the sheets of $\mathcal{V}$ is only as effective as the solution of a general algebraic equation. Thus for example, the set of normal vectors to the sheets is the set of roots of a polynomial equation which may not be solvable by radicals. On the other hand, symmetric functions of the v_j's are effectively represented in terms of G and H, and so are the coefficients of the function χ. In particular,

$$\begin{aligned}
\chi(\mathbf{z}) &= \lim_{\epsilon \to 0} \chi(\widehat{\mathbf{z}}, z_d + \epsilon) \\
&= \lim_{\epsilon \to 0} \prod_{j=1}^k (1 - (z_d + \epsilon) v_j(\widehat{\mathbf{z}}))^{n_j} \frac{G(\mathbf{z})}{H(\widehat{\mathbf{z}}, z_d + \epsilon)} \\
&= \lim_{\epsilon \to 0} G(\mathbf{z}) \frac{\prod_{j=1}^k (-\epsilon v_j(\widehat{\mathbf{z}}))^{n_j}}{\epsilon^D \left((\partial / \partial z_d)^D H(\mathbf{z}) + O(\epsilon) \right) / D!} \\
&= \frac{D! G(\mathbf{z}) (-z_d)^{-D}}{(\partial / \partial z_d)^D H(\mathbf{z})} \qquad (7.16)
\end{aligned}$$

where $D = \sum_{j=1}^k n_j$ is the total degree of multiplicity.

7.1 Exponential rate

Recall that the function **dir** is not defined at a multiple point. We remedy this as follows.

Definition 5 *If* $\mathbf{z} \in \mathcal{V}$ *is a multiple point with sheets* $\mathcal{V}_j$ *parametrized by* u_j *as above, define*

$$\mathbf{dir}_j(\mathbf{z}) = \left(-\frac{z_1}{z_d}\frac{\partial u_j}{\partial z_1}, \ldots, -\frac{z_{d-1}}{z_d}\frac{\partial u_j}{\partial z_{d-1}}, 1\right)$$

in $\mathbf{CP}^{d-1}$. *Since* $\partial u_j / \partial z_l = -H_l^{(j)}/H_d^{(j)}$, *where* $H^{(j)}$ *generates the ideal of* $\mathcal{V}_j$, *we see that* $\mathbf{dir}_j(\mathbf{z})$ *is just the limit of* $\mathbf{dir}(\mathbf{w})$ *as* $\mathbf{w} \to \mathbf{z}$ *along the sheet* $\mathcal{V}_j$. *If* $\mathbf{z}$ *is minimal, then* $\mathbf{z}$ *is minimal for each sheet, and by Lemma* **??** *each* $\mathbf{dir}_j(\mathbf{z})$ *is an element of the nonnegative orthant of* $\mathbf{RP}^{d-1}$. *In this case, we define* $\mathbf{dir}(\mathbf{z})$ *to be the convex hull in* $\mathbf{RP}^{d-1}$ *of the vectors* $\mathbf{dir}_j(\mathbf{z})$ *(a convex hull in the nonnegative orthant of* $\mathbf{RP}^{d-1}$ *can be thought of as a nonnegative cone in* $\mathbf{R}^d$*).*

Remark: If $\mathbf{z}$ is not a minimal point of $\mathcal{V}$, the functions $\mathbf{dir}_j$ are defined but are not necessarily in $\mathbf{RP}^{d-1}$, so the notion of convex hull may not make sense. Assuming that $\mathbf{z}$ is minimal, write $z_j = x_j e^{i\theta_j}$ with $\mathbf{x} \in \log\mathcal{D}$. Then the set of normals to support hyperplanes to $\log\mathcal{D}$ at $\mathbf{x}$ is exactly $\mathbf{dir}(\mathbf{z})$. Thus we may extend the above definition to minimal points (e.g. cone points) that are neither smooth points or multiple points by defining $\mathbf{dir}(\mathbf{z})$ to be the set of normals to support hyperplanes of $\log\mathcal{D}$ at $\mathbf{x}$. Lemma **??** then immediately implies:

Proposition 7.2 *For any minimal point* $\mathbf{z} \in \mathcal{V}$ *and any* $\mathbf{r} \notin \mathbf{dir}(\mathbf{z})$, $|\mathbf{z}^{\mathbf{r}} a_{\mathbf{r}}| \to 0$ *exponentially fast in* $\mathbf{r}$. □

In the remainder of the treatment of multiple points, it will be shown how a point $\mathbf{z}$ determines asymptotics in every direction in $\mathbf{dir}(\mathbf{z})$. There are many variations. Directions that are extreme points of $\mathbf{dir}(\mathbf{z})$ require a separate treatment. As in Section **??**, exponents $n_j > 1$ may be analyzed following the same program but computing the residue by differentiating. Whether at least two of the sheets intersect transversely also affects the analysis. All variations are treated in Pemantle and Wilson (2000b and 2000c), including non-isolated multiple points. After deriving asymptotics in the simplest case (described shortly), none of these variations poses any significant difficulty, so we will be content here to present just the case where each $n_j = 1$ and we impose some transversality conditions on the intersections of the sheets.

The reduction to an oscillating integral begins in a familiar fashion. Just as in the proof of Lemma **??**, when $\mathcal{N}$ is a neighborhood of $\widehat{\mathbf{z}}$ in $T(\widehat{\mathbf{z}})$, the quantity

$$|\mathbf{z}^{\mathbf{r}}| \left| a_{\mathbf{r}} - \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{N}} \widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \left[\int_{\mathcal{C}_1} \frac{F(\mathbf{w})}{w_d^{r_d+1}} \, dw_d \right] d\widehat{\mathbf{w}} \right| \tag{7.17}$$

decreases exponentially. Choosing ϵ small and the $\mathcal{N}$ sufficiently small, we localize the Cauchy integral for $a_{\mathbf{r}}$ to a neighborhood (by (**??**)), write it as an iterated integral, compare to a torus enlarged in the z_d component, observe that the integral over the larger torus is exponentially smaller than $\mathbf{z}^{-\mathbf{r}}$, and hence that $a_{\mathbf{r}}$ is well approximated by the residue term. This time, for each $\widehat{\mathbf{w}}$ in a neighborhood of $\widehat{\mathbf{z}}$, we pick up k different residues. We summarize this in a lemma.

Lemma 7.3 *Let $\mathbf{z}$ be an isolated multiple point of F. Assume all the exponents n_j from the Weierstrass preparation lemma are 1. For a neighborhood $\widetilde{\mathcal{N}}$ of $\mathbf{0}$ in $\mathbf{R}^{d-1}$ define a quantity*

$$\Xi := (2\pi)^{1-d} \widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}} \int_{\widetilde{\mathcal{N}}} R(\widehat{\mathbf{w}}(\widehat{\theta})) \, d\widehat{\theta}, \tag{7.18}$$

where $w_j(\widehat{\theta}) := z_j e^{i\theta_j}$ and

$$R(\widehat{\mathbf{w}}) := \sum_{j=1}^k \operatorname{Res}(w_d^{-r_d-1} F; w_d = u_j(\widehat{\mathbf{w}})) \, .$$

Then the quantity

$$|\mathbf{z}^{\mathbf{r}}| \, |a_{\mathbf{r}} - \Xi|$$

decreases exponentially as $\widetilde{\mathcal{N}}$ remains fixed and $\mathbf{r} \to \infty$. □

We cannot separate the integral into the sum of k integrals of the form in (**??**), since none of the individual residues will be integrable. Instead, to go any further, we need to understand the residue sum, R. It will take some work to put R in a form to which our knowledge of oscillating integrals can be applied. To clarify the exposition, we treat the simplest case first ($d = k = 2$) and then the general case. First though, we use a little algebra to see how much asymptotic information is needed beyond what is available as a corollary of smooth point expansions.

7.2 Dimension of the set of expansions

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be an isolated multiple point with sheets $\mathcal{V}_1, \ldots, \mathcal{V}_k$, with F represented as $\chi / \prod_{j=1}^k (1 - z_d v_j(\widehat{\mathbf{z}}))^{n_k}$. Let $G := \chi$ and $H := \prod_{j=1}^k h_j(\mathbf{z}) := \prod_{j=1}^k (1 - z_d v_j(\widehat{\mathbf{z}}))^{n_k}$. In trying to understand the asymptotics of the coefficients of G/H, when H factors, it is natural to try a partial fraction expansion. Indeed, we understand from the smooth case how to find asymptotics for each $g/(1 - z_j v_j r(\widehat{\mathbf{z}})^l)$ when g is analytic, so if G/H were representable as a sum of such terms for every numerator, G, we would need no further analysis. This is not the case, since whenever g/H that can be written as $\sum g_j/(1 - z_j v_j r(\widehat{\mathbf{z}})^l)$, the function g must vanish at α. We examine in more detail how much the partial fraction expansions fall short of giving us all functions in the numerator.

Let $\Re_\alpha$ be the local ring of power series about α convergent in a neighborhood of α. Let $\mathcal{S}$ be the family of subsets $S \subseteq \{1, \ldots, k\}$ such that α is not an isolated point of $\bigcap_{i \in S} \mathcal{V}_i$, and let $\Im_\alpha$ be the ideal generated by all products $h_S := \prod_{j \in S^c} (1 - z_d v_j(\widehat{\mathbf{z}}))^{n_j}$ for which $S \in \mathcal{S}$. If $\mathcal{A}$ is any variety containing α, let S_A denote the the (possibly empty) set of j for which $\mathcal{A} \subseteq \mathcal{V}_j$. By definition, $S_A \in \mathcal{S}$, hence some generator h_{S_A} of $\Im_\alpha$ does not vanish on A. We conclude that α is an isolated element of $V(\Im_\alpha)$. Hence the radical of $\Im_\alpha$ is $\mathcal{M}$, the (unique maximal) ideal of $\Re_\alpha$ generated by the functions $(z_i - \alpha_i)$. The quotient space $\Re_\alpha / \mathcal{M}$ is naturally isomorphic to $\mathbf{C}$ via $f \mapsto f(\alpha)$. By the Nullstellensatz, $\mathcal{M}$ is finite dimensional over $\Im$, which is the same as $\Re_\alpha / \Im_\alpha$ being finite dimensional over $\Re_\alpha / \mathcal{M} \simeq \mathbf{C}$. By definition of radical, for each j, some power of $z_j - \alpha_j$ is in $\mathcal{M}$, which will be useful later.

The set $\Im_\alpha/H$ is the submodule of $\Re_\alpha/H$ generated by functions h_S/H for $S \in \mathcal{S}$. But $h_S/H = h_{S^c} = 1/\prod_{j\in S}(z_d - v_j(\hat{\mathbf{z}}))^{n_j}$. We have therefore proved the following lemma.

Lemma 7.4 *Every function $g \in \Re_\alpha$ may be represented as a sum*

$$g_0 + \sum_{S\in\mathcal{S}} g_S h_S$$

where g_S is a power series convergent in a neighborhood of α and g_0 is a coset representative of $\Re_\alpha/\Im_\alpha$. For each $j \leq k$ there is an m_j with $(z_j - \alpha_j)^{m_j} f \in \Im_\alpha$. □

We wish to use this partial fraction decomposition to write the coefficients of g as sums of the coefficients of g_0 with coefficients of functions g_S. For the above decomposition to translate to the realm of coefficients, it will need to be a decomposition in the space of functions analytic in a neighborhood of the origin. Consequently, we will need the following strengthening, whose proof will be given at the end of this section. Let $\Re$ be the space of functions analytic in a neighborhood of the closed polydisk $D(\alpha)$. Every function $f \in \Re$ has an image $(f)_\alpha \in \Re_\alpha$ under localization at α.

Theorem 7.5 *There is an ideal $\Im$ of $\Re$, the image of which in $\Re_\alpha$ contains the ideal $\Im_\alpha$, and which has the following property. It is generated by finitely many functions $h_S^{(i)}$, $S \in \mathcal{S}$, $i \geq 1$, whose localizations $(h_S^{(i)})_\alpha$ satisfy*

$$(h_S^{(i)})_\alpha = u h_S$$

for u a unit of $\Im_\alpha$. It follows that there is a vector space $\mathbf{V}'$ of the same dimension as $\Re_\alpha/\Im_\alpha$ such that every $G \in \Re$ has a representation

$$G = g_0^* + \sum g_{S,i} h_S^{(i)}$$

with $g_0 \in \mathbf{V}'$. The space $\mathbf{V}'$ may be chosen to be the span of any basis for $\Re_\alpha/\Im_\alpha$ that lies in $\Re$.

Assuming the theorem for now, we suppose further that the set $\mathbf{dir}(\alpha)$ has nonempty interior. Equivalently, the normals to the hyperplanes of the sheets $\mathcal{V}_1, \ldots, V_k$ span $\mathbf{C}^d$ and hence define a cone with nonempty interior. For each $S \in S$, one may consider the function $F_S := h_S^{(i)}/H$. The pole set of this function is a subset of $\mathcal{V}$ and near $\mathbf{z}$ contains only those sheets $\mathcal{V}_j$ for $j \in S$. With respect to this function, the set $\mathbf{dir}_{F_S}(\alpha)$ is a cone whose extreme points do not span all of $\mathbf{C}^d$ (since all sheets $\mathcal{V}_j$ with $j \in S$ contain a common variety through α of dimension at least 1). Hence the union U of $\mathbf{dir}_{F_S}(\alpha)$ with respect to this function is a union of lower dimensional cones. In particular, it is not all of $\mathbf{dir}(\alpha)$ and is in fact a set of smaller dimension. Let C_1 be any cone with nonempty interior contained in $\mathbf{dir}_{F_S}(\alpha) \setminus U$. We may describe all asymptotics on C_1 of coefficients of functions G/H as follows.

Theorem 7.6 (Polynomial theorem for isolated multiple points) *Suppose the function H has a strictly minimal point at α of multiplicity $k \geq 2$. Let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ denote the sheets of $\mathcal{V}$ near α. Let C_1 be a cone inside $\mathbf{dir}(\alpha) \setminus U$ where $U = \bigcup_{S \in \mathcal{S}, j \geq 1} \mathbf{dir}_{h_S^{(i)}}(\alpha)$, with $\mathcal{S}$ and $\{h_S\}$ as in the previous paragraph. Then there exists a finite dimensional vector space W (over $\mathbf{C}$) of polynomials in a variable $\mathbf{r} \in \mathbf{Z}^d$ such that for every $G \in \Re$, the coefficients of $F := G/H$ are given by*

$$a_{\mathbf{r}} = \alpha^{-\mathbf{r}}(f(\mathbf{r}) + g(\mathbf{r}))$$

where $f \in W$ and g is exponentially decaying on $C_1 \cap \mathbf{Z}^d$.

PROOF: Replacing $F(\mathbf{z})$ with $F(r_1 z_1, \ldots, r_d z_d)$, we move α to $\mathbf{1}$ while multiplying $a_{\mathbf{r}}$ by $\alpha^{\mathbf{r}}$. Thus it suffices to prove the theorem in the case $\alpha = 1$.

Let $\mathbf{V} = \mathbf{V}(d)$ be the vector space of complex valued functions on the positive orthant in $\mathbf{Z}^d$. For $i \leq d$, let σ_i be the i^{th} shift operator on $\mathbf{V}$, defined by $(\sigma_i f)(\mathbf{r}) = f(\mathbf{r} + e_i)$ where e_i is the vector whose j^{th} component is δ_{ij}. Let $E \subseteq \mathbf{V}(d)$ be the subspace of functions decreasing exponentially in $\mathbf{r}$ on C_1. For a function $\phi \in \mathbf{C}[[z_1, \ldots, z_d]]$, let q_ϕ denote the element of $\mathbf{V}$ defined by $q_\phi(\mathbf{r}) = b_\mathbf{r}$, where $b_\mathbf{r}$ are the coefficients of ϕ/H:

$$\frac{\phi(\mathbf{z})}{H(\mathbf{z})} = \sum b_\mathbf{r} \mathbf{z}^\mathbf{r} \, .$$

The function $\mathbf{1}$ is a minimal point of the singular variety of $h_S^{(i)}/H$ for any S, i. Therefore, by Proposition **??**, $h_S^{(i)}/H$ has exponentially decaying coefficients in any direction not in $\mathbf{dir}_{h_S^{(i)}/H}(\mathbf{1})$. By construction, then, each $q_{h_S^{(i)}}$ is in E, and hence the set of functions $q_\Im := \{q_\phi : \phi \in \Im\}$ is a subset of E. The ideal $\mathcal{M}$ contains all functions vanishing at $\mathbf{1}$. Recalling that some power of each $(1 - z_j)$ annihilates $\Re_\alpha/\Im_\alpha$, we let P denote the finite set of $\mathbf{r}$ such that $(1-\mathbf{z})^\mathbf{r} := \prod_{i=1}^k (1-z_i)^{r_i} \notin \Im_\alpha$. The functions $(1-\mathbf{z})^\mathbf{r}$ are generate $\Re_\alpha/\Im_\alpha$, so by Theorem **??** they generate $\Re/\Im$. Since $q_{(1-z_i)\phi} = (I - \sigma_i) q_\phi$, it follows that for $\mathbf{r} \notin P$ and any analytic ϕ,

$$(I - \sigma)^\mathbf{r} q_\phi := \left[\prod_{i=1}^k (I - \sigma_i)^{r_i} \right] q_\phi \in q_\Im \subseteq E \, .$$

We have found a finite set of coset representatives, $\{\phi_1, \ldots, \phi_l\}$ of $\Re/\Im$. Let $\mathbf{V}'$ be their span over $\mathbf{C}$. By Theorem **??** and the fact that $q_\Im \subseteq E$, functions in $\mathbf{V}'$ approximate the coefficients of any function G/H up to a difference that decays exponentially on C_1. We have seen, for each $\mathbf{r} \notin P$, that $(I - \sigma)^\mathbf{r} X \subseteq E$. The theorem then follows from the following lemma with $X = \mathbf{V}'$:

Lemma 7.7 *Let $X \subseteq \mathbf{V}(d)$ be a finite dimensional subspace such that there is a finite set P for which $(I - \sigma)^\mathbf{r} X \subseteq E$ whenever $\mathbf{r} \notin P$. Then for each $f \in X$ there exists a polynomial g whose monomials have multi-degrees in P, and for which $f - g \in E$.*

Proof: Proceed by induction on $|P|$. If $|P| = 1$ then $P = \{\mathbf{0}\}$. In this case, for each $f \in X$ and $i \le k$, the function $\mathcal{E}_i := (I - \sigma_i) f$ is in E. The cone C_1 has nonempty interior, which implies that $C_1 \cap \mathbf{Z}^d$ has a co-finite subset C' which is a connected subgraph of the integer lattice. For any $\mathbf{r} \le \mathbf{s} \in C_1$, there is an oriented path $\gamma_0, \gamma_1, \ldots, \gamma_l$ connecting $\mathbf{r}$ to $\mathbf{s}$ in C', where $l = \sum_{i=1}^k (s_i - r_i)$. (An oriented path takes steps only in the increasing coordinate directions.) Then

$$f(\mathbf{s}) - f(\mathbf{r}) = \sum_{j=1}^{l} f(\gamma_j) - f(\gamma_{j-1}) = \sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$$

where $m(j) = i$ if γ_{j-1} and γ_j differ by e_i. Sending $\mathbf{s}$ to infinity, we see that

$$f(\mathbf{r}) = \lim_{s\to\infty} f(\mathbf{s}) + \sum_{j=1}^{\infty} f(\gamma_j) - f(\gamma_{j-1}) = \sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$$

where γ connects $\mathbf{r}$ to infinity. Thus on C', f is a constant plus a tail sum of functions in E, and the conclusion is true with $g = \lim_{s\to\infty} f(\mathbf{s})$, the constant polynomial.

The induction step is similar. Let $P_i = \{\mathbf{r} : \mathbf{r}+e_i \in P\}$. Fix $f \in X$. The space $(I-\sigma_i)X$ satisfies the hypotheses of the lemma with P_i in place of P. Since $|P_i| < |P|$, we may apply the induction hypothesis to conclude that $(I-\sigma_i)f = g_i + \mathcal{E}_i$ where g_i is a polynomial with multi-degrees in P_i and $\mathcal{E}_i \in E$. For any $\mathbf{r} \le \mathbf{s} \in (\mathbf{Z}^+)^d$, and any oriented path γ from $\mathbf{r}$ to $\mathbf{s}$, we have

$$\begin{aligned} f(\mathbf{s}) - f(\mathbf{r}) &= \sum_{j=1}^{l} f(\gamma_j) - f(\gamma_{j-1}) \\ &= \sum_{j=1}^{l} g_{m(j)}(\gamma_{j-1}) + \sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1}) \,. \end{aligned}$$

If $\mathbf{r}, \mathbf{s} \in C'$ then we have already seen that, as a function of $\mathbf{s}$, the last contribution $\sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$ is equal to a constant $C(\mathbf{r})$ plus a term decaying exponentially in $\mathbf{s}$. Fixing $\mathbf{r} \in C'$ so that the set of $\mathbf{s} \in C_1$ not greater than or equal to $\mathbf{r}$ is finite, it remains to show that

$$p(\mathbf{s}) := f(\mathbf{r}) + \sum_{j=1}^{l} g_{m(j)}(\gamma_{j-1})$$

defines a polynomial in $\mathbf{s}$ with multi-degrees in P.

We know that $(I-\sigma_j)g_i = (I-\sigma_i)g_j$ since the difference is a polynomial in E, hence zero. Thus for $\mathbf{x} \in \mathbf{Z}^d$, we see that $g_i(\mathbf{x}) + g_j(\mathbf{x}+e_i) = g_j(\mathbf{x}) + g_i(\mathbf{x}+e_j)$. It follows that the sum defining p is invariant under switching the order of two steps in the path γ, and hence is independent of the choice of γ. Choosing γ to take first $s_1 - r_1$ steps in direction e_1, then $s_2 - r_2$ steps in direction e_2 and so on, we may write

$$p(\mathbf{s}) = f(\mathbf{r}) + \sum_{j=1}^{k} \sum_{t=1}^{s_i - r_i} g_j(s_1, \ldots, s_{j-1}, r_i + t - 1, r_{j+1}, \ldots, r_k) \, .$$

Each of the inner sums is the sum to $s_i - r_i$ of a polynomial with multi-degrees in P_i, which is well known to be a polynomial with multi-degrees in P. Hence $p(\mathbf{s})$ is a polynomial with multi-degrees in P and the proof is done. □

For an isolated multiple point, we always have $k \geq d$. In the case $k = d$ and each $n_j = 1$, the conclusion of the polynomial theorem is strongest. We state it as a corollary.

Corollary 7.8 (plateau result) *Let α be an multiple point of the zero set $\mathcal{V}$ of H and suppose the number of sheets k of $\mathcal{V}$ near α is equal to the dimension, d, each sheet $\mathcal{V}_j$ appearing with multiplicity $n_j = 1$, and the common intersection of the sheets being the singleton $\{\alpha\}$. Then for each analytic function G, the coefficients $a_{\mathbf{r}}$ of G/H may be written as $C\alpha^{-\mathbf{r}} + g(\mathbf{r})$ with g decaying exponentially on compact subcones of $\mathbf{dir}(\alpha)$. Specializing further to $\alpha = \mathbf{1}$, the coefficients in $\mathbf{dir}(\mathbf{1})$ are constant up to a term of exponential decay.*

Proof: Any proper subset of the sheets intersects in variety of positive dimension through α, hence $\mathcal{S}$ contains each subset of sheets of cardinality $d - 1$. Thus $\Im$ contains each sheet function $z_d - v_j(\hat{\mathbf{z}})$ and is therefore equal to the unique maximal ideal, $\mathcal{M}$, from which it follows that $P = \{\mathbf{0}\}$. The set U is just the boundary of $\mathbf{dir}(\alpha)$, whence the cone C_1 in the polynomial theorem may be chosen to be any subcone of the interior of $\mathbf{dir}(\alpha)$. Applying the polynomial theorem now proves the corollary. □

As we increase d, $k - d$ and the n_j's, there number of cases grows quickly. The structure of the space of asymptotics in each case may be worked from Theorem **??**. One more example, the next simplest, will serve to illustrate how this is done.

Example 16 **(**$k = 3$**,** $d = 2$**,** $n_j \equiv 1$**)**

We suppose $F = G/H$ where $H = (1-wv_1(z))(1-wv_2(z))(1-v_3(z))$ and $v_j(z_0) = 1/w_0$ for $j = 1, 2, 3$. We suppose that (z_0, w_0) is minimal and that the values $\mathbf{dir}_j(z_0, w_0)$ are distinct and listed in increasing order of r/s. Let $H_j(z, w)$ denote $1 - wv_j(z)$. Each pair of sheets meets locally only at (z_0, w_0), so $\mathcal{S}$ contains only the singleton sets $\{1\}, \{2\}$ and $\{3\}$. The ideal $\Im$ is generated by H_1H_2, H_1H_3 and H_2H_3. It is not hard to see that $\Im$ contains all analytic functions vanishing to homogeneous degree two near (z_0, w_0), hence contains $(1-z)^r(1-w)^s$ exactly when $r + s \geq 2$. Thus $P = \{(0, 0), (1, 0), (0, 1)\}$. Initially this tells us that

$$a_{rs} = (A + Br + Cs + \mathcal{E})z_0^{-r}w_0^{-s} \tag{7.19}$$

on any compact subcone of $\mathbf{dir}(z_0, w_0) \setminus U$, where U is the set $\{\mathbf{dir}_j(z_0, w_0) : 1 \leq j \leq 3\}$ and $\mathcal{E}$ decays exponentially on the subcone.

Now U is just the boundary of the cone $\mathbf{dir}(z_0, w_0) = [\mathbf{dir}_1(z_0, w_0), \mathbf{dir}_3(z_0, w_0)]$ together with the ray $\mathbf{dir}_2(z_0, w_0)$, so $\mathbf{dir}(z_0, w_0) \backslash U$ is the union of two open intervals in $\mathbf{RP}^1$, namely $C_1 := (direc_1(z_0, w_0), direc_2(z_0, w_0))$ and $C_2 := (direc_2(z_0, w_0), direc_3(z_0, w_0))$. Instead of using Lemma **??** and the ideal $\Im$, we may analyze the situation by going directly to $\mathcal{M}$. This is generated by $\{H_j : 1 \leq j \leq 3\}$. Thus F may be written as a linear combination of functions H_j/H plus the function $1/H$. By the plateau corollary, we know that the coefficients of $H_1/H = 1/(H_2H_3)$ are a constant multiple of $z_0^{-r}w_0^{-s}$ on C_2, while by Proposition **??** they are exponentially smaller elsewhere. Similarly, up to exponentially smaller terms, the coefficients of H_3/H are a constant multiple of $z_0^{-r}w_0^{-s}$ on C_1. Any $G \in \Re$ is a constant plus an element of $\mathcal{M}$, hence asymptotics of any q_G are given by

$$z_0^{-r}w_0^{-s}\left(A_0 l + A_1 \mathbf{1}_{C_1} + A_2 \mathbf{1}_{C_2} + \mathcal{E}\right),$$

where $\mathcal{E}$ decays exponentially, $\mathbf{1}$ is an indicator function, and $z_0^{-r}w_0^{-s}l$ is the coefficient array of $1/H$. We know from (**??**) that q_1 is affine on each C_i, and by altering A_1 and A_2 we can make it linear on each C_i. We will see later that it is continuous and that it vanishes on the boundary of $\mathbf{dir}(z_0, w_0)$, and therefore that it is the "tent function" which grows linearly on the ray $\mathbf{dir}_2(z_0, w_0)$, vanishes on $\mathbf{dir}_1(z_0, w_0)$ and $\mathbf{dir}_3(z_0, w_0)$, and interpolates linearly on each of C_1 and C_2.

We will not go any further into the structure of these polynomial spaces, save for one remark. Whenever $n_j \equiv 1$ and the sheets meet as transversely as possible, the functions H_j generate $\mathcal{M}$, and hence any analytic function is a linear combination of 1 and functions H_j. Recursively, we may assume we have analyzed the case of each $H_j/H = 1/\prod_{i \neq j} H_i$, so we need add only one dimension to the subspace of $\mathbf{V}(d)/E$ that these generate, namely the coefficient array for $1/H$.

It remains to prove Theorem **??**. the first step is to establish:

Lemma 7.9 *For each $\mathbf{x} \in D(\mathbf{z}) \setminus \{\mathbf{z}\}$ there are functions $h_j^{\mathbf{x}}$ for which the following hold:*

(1) each $h_j^{\mathbf{x}}$ is analytic on a neighborhood Ω of $D(\mathbf{z})$;

(2) $h_j^{\mathbf{x}} = u \cdot h_j$ with u a unit in $\Re_\alpha$;

(3) $h_j^{\mathbf{x}}(\mathbf{x}) \neq 0$.

Proof: Let ω be a neighborhood of $\mathbf{z}$ in which the factors h_j are analytic, and in which $\bigcap_{j=1}^k \mathcal{V}_j = \{\mathbf{z}\}$. Since $\mathcal{V}$ does not intersect the interior of $D(\mathbf{z})$, the intersection of $\mathcal{V}_j$ with $\partial\omega$ is disjoint from $D(\mathbf{z})$ and it follows that we may choose a neighborhood Ω of $D(\mathbf{z})$ containing no such intersection point. Let $\mathcal{F}_j$ be the sheaf over Ω of ideals $< h_j >$. That is, when $\mathbf{w} \in \omega$ and $h_j(\mathbf{w}) = 0$, then $(\mathcal{F}_j)_\mathbf{w}$ is the germs of functions divisible by h_j at $\mathbf{w}$, while when $\mathbf{w} \notin \omega$ or $\mathbf{w} \in \omega$ with $h_j(\mathbf{w}) \neq 0$, then $(\mathcal{F}_j)_\mathbf{w}$ is all analytic germs at $\mathbf{w}$. The definition of $(\mathcal{F}_j)^\mathbf{w}$ is potentially ambiguous when $\mathbf{w} \in \partial\omega$ is in the interior of Ω, but since h_j is nonzero here, there is no problem.

The sheaf $\mathcal{F}_j$ is a subsheaf of the structure sheaf $\mathcal{O}$, hence coherent, so by Cartan's Theorem A (see Grauert and Remmert 1979) there is a map ψ from some $\mathcal{O}^l$ onto $\mathcal{F}_j$, where $\mathcal{O}$ is the sheaf of germs of analytic functions (the structure sheaf) on Ω and $l \geq 1$.

Denote the l generators of $\mathcal{O}^l$ by $\mathbf{1}_i, i \le l$. The fact that ψ is surjective at $\mathbf{z}$ means that h_j is in the image of ψ: for some functions u_i in a neighborhood of $\mathbf{z}$,

$$\psi(\sum_i u_i \mathbf{1}_i) = \sum_i u_i \psi(\mathbf{1}_i) = h_j \,. \tag{7.20}$$

By definition of ψ, each $\psi(\mathbf{1}_i)$ may be written as $u_{ij} h_j$ in $\Re_\alpha$. If each $u_{ij} \in \mathcal{M}$, then each $\psi(\mathbf{1}_i) \in \mathcal{M}\cdot < h_j >$ contradicting (**??**); thus at least one $u_{ij} \notin \mathcal{M}\cdot < h_j >$.

If there is an i with $\psi(\mathbf{1}_i) \notin \mathcal{M}\cdot < h_j >$ and $\psi(\mathbf{1}_i)(\mathbf{x}) \ne 0$, then the lemma is proved with $h_j^{\mathbf{x}} := \psi(\mathbf{1}_i)$ and $u = u_{ij}$. If not, then choose i and i' so that $\psi(\mathbf{1}_i) \notin \mathcal{M}\cdot < h_j >$ and $\psi(\mathbf{1}_{i'})(\mathbf{x}) \ne 0$. Since it was not possible to choose $i = i'$, we know that $\psi(\mathbf{1}_i)(\mathbf{x}) = 0$ and $u_{i'j} \in \mathcal{M}$. It follows that $u_{ij} + u_{i'j} \notin \mathcal{M}$ and the lemma is proved with $h_j^{\mathbf{x}} := \psi(\mathbf{1}_{i'}) + \psi(\mathbf{1}_i)$. □

Corollary 7.10 *There is a finite collection $\{h_\alpha : \alpha \in A\}$ analytic on a neighborhood of $D(\mathbf{z})$ such that for each $S \in \mathcal{S}$ and each $\mathbf{w} \in D(\mathbf{z}) \setminus \{\mathbf{z}\}$ there is an $\alpha \in A$ with*

$$h_\alpha(\mathbf{w}) \ne 0 \text{ and } h_\alpha = h_S u \tag{7.21}$$

with u a unit of $\Re_\alpha$.

Proof: Fix $S \in \mathcal{S}$. The function $h_S^{\mathbf{x}} := \prod_{j \in S^c} (h_j^{\mathbf{x}})^{n_j}$ satisfies (**??**) for all $\mathbf{w}$ in some neighborhood $\mathcal{N}_{\mathbf{x}}$ of $\mathbf{x}$. It also satisfies (**??**) for all $\mathbf{w}$ in some neighborhood $\mathcal{N}$ of $\mathbf{z}$. By compactness of $D(\mathbf{z})$, we may choose finitely many $\mathbf{x}$ for which the collection of sets $\mathcal{N}_{\mathbf{x}}$ covers $D(\mathbf{z}) \setminus \mathcal{N}$. Taking the union of such collections over $S \in \mathcal{S}$ proves the corollary. □

Lemma 7.11 *Let Ω be a polydisk and let $\{h_\alpha : \alpha \in A\}$ be a finite collection of functions analytic in Ω. Suppose an analytic function g on Ω is represented in a neighborhood of each $\mathbf{x}$ as $\sum_\alpha g_\alpha^{\mathbf{x}} h_\alpha$ with $g_\alpha^{\mathbf{x}}$ analytic in a neighborhood of $\mathbf{x}$. Then*

$$g = \sum_\alpha g_\alpha^* h_\alpha$$

with g_α^* analytic in Ω.

Proof: Define sheaves over Ω by $\mathcal{F} = \mathcal{O}^{|A|}$ and $\mathcal{G} =< h_\alpha : \alpha \in A >$. The map $\eta : \mathcal{F} \to \mathcal{G}$ defined by $\eta(f_\alpha : \alpha \in A) = \sum_\alpha f_\alpha h_\alpha$ is a surjection of sheaves. Thus we have a short exact sequences of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{E}$ is the kernel of η. The long exact cohomology sequence then gives us an exact sequence

$$H^0(\Omega, \mathcal{F}) \to H^0(\Omega, \mathcal{G}) \to H^1(\Omega, \mathcal{E})$$

Since H^0 is the set of global sections, we see that the natural map of global sections from $\mathcal{F}$ to $\mathcal{G}$ induced by η is surjective if and only if the first cohomology of $\mathcal{E}$ vanishes.

It is well known that the cohomology of $\mathcal{E}$ vanishes. This is essentially Cartan's Theorem B, a short development of which is as follows. The noetherian property of each stalk $\mathcal{O}_{\mathbf{x}}$ translates by Oka's Lemma into a noetherian property for the sheaf $\mathcal{O}$. This guarantees that $\mathcal{E}$ is finitely generated. A set of generators yields a map from some $\mathcal{O}^p$ to $\mathcal{E}$, whose kernel must also be finitely generated. Iterating, we get a presentation of $\mathcal{E}$ which must terminate in at most $|\mathcal{S}|$ steps by the Sysygy Theorem (Griffiths and Harris 1994, p. 694). The vanishing of $H^q(\Omega, \mathcal{O}^p)$ for any p and any $q > 0$ follows from the contractibility of Ω. It then follows by induction that any finitely presented sheaf over Ω has $H^q = 0$ for $q > 0$. Hence any $g \in \mathcal{G}$ is the image of a global section of $\mathcal{F}$, which proves the lemma. $\Box$

Proof of Theorem **??**: Choose coset representatives for a basis of $\Re_\alpha / \Im_\alpha$; let $\mathbf{V}'$ be their span. By construction, if $G \in \Re$, then G may be written as $g_0^* + g$ with $g_0^* \in \mathbf{V}'$ and the germ $(g)_{\mathbf{z}}$ in $\Im_\alpha$. Evidently, the dimension of $\mathbf{V}'$ is equal to the dimension of $\Re_\alpha / \Im_\alpha$, which is finite by Lemma **??**.

We now verify the hypotheses of Lemma **??**. In a neighborhood of $\mathbf{z}$, we know from (**??**) that the functions $\{h_S^{(i)}\}$ generate $\Im_\alpha$. Hence there is a representation $g = \sum g_\alpha^{\mathbf{z}} h_\alpha$. In a neighborhood of any other $\mathbf{x} \in D(\mathbf{z})$ some h_α is nonzero, so there is trivially a representation $g = \sum g_\alpha^{\mathbf{x}} h_\alpha$. Applying Lemma **??**, it follows that $g \in \Im$. $\Box$

7.3 Two intersecting curves in $\mathbf{C}^2$

To see where we are heading, examine the special case $k = d = 2$ near the point (z_0, w_0), with the nondegeneracy assumption of a transverse (non-tangential) intersection of the two sheets $\mathcal{V}_1$ and $\mathcal{V}_2$. The residue sum is given by

$$R = \sum_{j=1}^{2} \text{Res}\left(\frac{w^{-s-1}\chi(z,w)}{(1-wv_1(z))(1-wv_2(z))} ; w = u_j(z) \right) .$$

Observe that $1 - wv_j(z) = -v_j(z)(w - u_j(z))$. Then it is easy to compute the residues, and we find that

$$\begin{aligned} R &= \frac{v_1(z)^{s+1}\chi(z,u_1(z))}{v_1(z)(1-u_1(z)v_2(z))} + \frac{v_2(z)^{s+1}\chi(z,u_2(z))}{v_2(z)(1-u_2(z)v_1(z))} \\ &= \frac{-v_1(z)^{s+1}\chi(z,u_1(z)) + v_2(z)^{s+1}\chi(z,u_2(z))}{v_1(z)-v_2(z)} . \end{aligned}$$

This quantity looks like a difference quotient. Indeed, letting

$$h(x) := h(z,x) := x^{s+1}\chi(z,1/x) ,$$

we may write

$$R = \int_{-1}^{1} h'(v_t)\, \frac{dt}{2}$$

where $v_t := (v_1+v_2)/2 + t(v_1-v_2)/2$ linearly interpolates between v_1 and v_2 [proof: change variables via $dt/2 = dv_t/(v_1 - v_2)$].

We want to apply Theorem **??** to Ξ which is now an integral over $\mathcal{N}$ of an integral over $[-1,1]$ of $h'(v_t)$. With $z = z_0 e^{i\theta}$, we have v_t written as a smooth function of (θ, t). Thus $h'(v_t(\theta,t))$ is a smooth function times $(s+1)v_t^s$. Once we observe that v_t has modulus bounded above by $1/w_0$ [by concavity of the modulus and the fact that $|v_j(z)| \leq |1/w_0|$ for $j = 1, 2$,which is a consequence of minimality], we are in position to apply the oscillating integral technology. We have $|z^r w^s||a_{rs} - \Xi|$ decreasing exponentially, where the quantity Ξ of (**??**) may be written (with $z = z_0 e^{i\theta}$) as

$$\Xi \;=\; \frac{1}{2\pi} z_0^{-r} \int_{\mathcal{N}} \int_{-1}^{1} h'(v_t(z)) \frac{dt}{2}\, d\theta \qquad (7.22)$$

$$= \frac{1}{4\pi} z_0^{-r} \int_{\mathcal{N}} \int_{-1}^{1} v_t(z)^s \left[(s+1)\chi(z, \frac{1}{v_t(z)}) - v_t(z)^{-2} \frac{\partial}{\partial w} \chi(z, 1/v_t(z)) \right] dt\, d\theta\,.$$

Taking the log of $v_t(z)^s$ we now recognize the integrand as a cell complex oscillating integral:

$$\Xi = \frac{1}{4\pi} z_0^{-r} w_0^{-s} \int_{\mathcal{N}} \int_{-1}^{1} (s+1) \exp(-sf(\theta, t)) \psi(\theta, t)\, dt\, d\theta \qquad (7.23)$$

where

$$f(\theta, t) = -\log[v_t(z_0 e^{i\theta}) w_0] + i \frac{r}{s} \theta$$

and

$$\psi(\theta, t) = \chi(z, 1/v_t(z)) - \frac{1}{s+1} v_t(z)^{-2} \frac{\partial}{\partial w} \chi(z, 1/v_t(z)).$$

We remark that here it is essential to have changed from u_j to v_j: the quantity u_t^{-s-1} does NOT necessarily have modulus bounded above by $1/w_0$; taking convex combinations of u_1 and u_2 before inverting ruins the convexity argument.

Theorem 7.12 *Let $F = G/H$ be meromorphic in a neighborhood of $D(z, w)$ and suppose that (z_0, w_0) is an isolated multiple point of $\mathcal{V}$ of multiplicity 2. Assume further that the curves $\mathcal{V}_1$ and $\mathcal{V}_2$ intersect transversely, that $\mathbf{r}$ is in the interior of $\mathbf{dir(z)}$, and that $\mathcal{V}_1$ and $\mathcal{V}_2$ each have multiplicity 1. Let $\mathcal{N}$ be a neighborhood of the origin in $\mathbf{R}$. Then if $G(z_0, w_0) \neq 0$,*

$$a_{r,s} = z_0^{-r} w_0^{-s} \frac{G(z_0, w_0)}{\sqrt{z_0^2 w_0^2 (H_{zw}^2 - H_{zz} H_{ww})}} + Y$$

where $|z^r w^s Y|$ is exponentially decreasing.

PROOF: Observe first that $G(z_0, w_0) \neq 0$ is equivalent to $\psi(0, t) \neq 0$ for sufficiently large s and in fact that as $s \to \infty$, from (**??**) we have

$$\psi(0, t) \to \chi(z_0, w_0) = \frac{2}{w_0^2} \frac{G(z_0, w_0)}{H_{ww}(z_0, w_0)}\,. \qquad (7.24)$$

Next we check that f has a unique stationary point in $\mathcal{N} \times [-1, 1]$. The partial derivative of f with respect to t is equal to $\frac{-1}{v_t(z)}(d/dt) v_t(z)$ where $z = z_0 e^{i\theta}$. Since $(d/dt) v_t(z) =$

$(v_1(z) - v_2(z))/2$ and v_1 and v_2 are distinct except at z_0, this means f cannot be stationary except when $\theta = 0$. When $\theta = 0$, the θ-partial derivative of f is given by (using $v_t(z_0) = 1/w_0$ for all t in the second equality)

$$\begin{aligned} \frac{\partial f}{\partial \theta}(0,t) &= i\frac{r}{s} - \frac{\partial}{\partial \theta}\bigg|_{(0,t)} \log\left[\frac{v_1(z)+v_2(z)}{2} + t\frac{v_1(z)-v_2(z)}{2}\right] \qquad (7.25) \\ &= i\frac{r}{s} - \left[iz_0\frac{v_1'(z_0)+v_2'(z_0)}{2} + iz_0 t\frac{v_1'(z_0)-v_2'(z_0)}{2}\right] / \frac{1}{w_0} \\ &= i\frac{r}{s} - iz_0 w_0\left(\frac{1+t}{2}v_1'(z_0) + \frac{1-t}{2}v_2'(z_0)\right). \end{aligned}$$

We have assumed transverse intersection of the sheets $\mathcal{V}_1$ and $\mathcal{V}_2$, so $v_1'(z_0) \neq v_2'(z_0)$ and hence there is at most one t for which $(0,t)$ is a stationary point. In fact, there is exactly one t if and only if r/s is between $z_0 w_0 v_1'(z_0)$ and $z_0 w_0 v_2'(z_0)$. Writing $v_j'(z_0) = (1/u_j)'(z_0) = -(u_j'/u_j^2)(z_0)$, we have

$$z_0 w_0 v_j'(z_0) = -u_j'(z_0)\frac{z_0}{w_0} = \mathbf{dir}_j(z_0, w_0).$$

Our assumption that r/s is interior to $\mathbf{dir}(z_0, w_0)$ is precisely what we need to guarantee there is exactly one stationary point, $(0, t_0)$ with $t_0 \in (-1, 1)$.

In order to apply Theorem **??** we must also check that at $(0, \pm 1)$, the gradient of f is not orthogonal to the boundary of $\mathcal{N} \times [-1, 1]$. In fact it is the θ derivative we have just shown to be non-vanishing there, and since the tangent space to the boundary is in the $\partial/\partial\theta$ direction at $(0, \pm 1)$, the requirement is met. Using first Lemma **??** and (**??**), then Theorem **??** with the determination of $\psi(0, t_0)$ in (**??**),

$$\begin{aligned} a_{r,s} &\sim \frac{s}{4\pi} z_0^{-r} w_0^{-s} \int_{\mathcal{N}} \int_{-1}^{1} \exp(-sf(\theta,t))\psi(\theta,t)\, dt\, d\theta \qquad (7.26) \\ &\sim \frac{s}{4\pi} z_0^{-r} w_0^{-s} \frac{2\pi}{s} \frac{2}{w_0^2} \frac{G(z_0,w_0)}{H_{ww}(z_0,w_0)} \mathcal{H}^{-1/2} \\ &= z_0^{-r} w_0^{-s} \frac{G(z_0,w_0)}{w_0^2 H_{ww} \mathcal{H}^{1/2}} \end{aligned}$$

where $\mathcal{H}$ is the determinant of the Hessian of f at $(0, t_0)$.

We now compute the Hessian of f at $(0, t_0)$. The function $f(0, \cdot)$ is constant at $\theta = 0$ so the pure t partials vanish at $\theta = 0$. In particular, $f_{tt}(0, t_0) = 0$. This means that the Hessian is nondegenerate if and only if the mixed partial $f_{\theta,t}(0, t_0)$ is non-vanishing, and the determinant of the Hessian is the negative of the square of the mixed partial. To obtain the leading term asymptotic, we therefore do not need to calculate $f_{\theta,\theta}(0, t_0)$. The mixed partial is computed by setting $\theta = 0$ in (**??**) and differentiating.

$$\begin{aligned} f_{\theta,t}(0, t_0) &= \left.\frac{d}{d_t}\right|_{t=t_0} \left[-iz_0 w_0 \left(\frac{1+t}{2}\, v_1'(z_0) + \frac{1-t}{2}\, v_2'(z_0) \right) \right] \\ &= \frac{-iz_0 w_0}{2} \left(v_1'(z_0) - v_2'(z_0) \right) \\ &= \frac{iz_0}{2w_0} \left(u_1'(z_0) - u_2'(z_0) \right) . \end{aligned}$$

Consequently,

$$\mathcal{H} = \frac{z_0^2}{4w_0^2} \left(u_1'(z_0) - u_2'(z_0) \right)^2 . \tag{7.27}$$

Since $(u_1' - u_2')^2$ is a symmetric function of u_1' and u_2', it may be expressed in terms of the partial derivatives of H.

Lemma 7.13 *Let $h = u \prod_{j=1}^{k} (w - u_j(z))$ for some analytic functions u_j with $u_j(z_0) = w_0$ and u not vanishing at (z_0, w_0). Then for $1 \le j \le k$, the j^{th} elementary symmetric function of $u_1', \ldots, u_k'$ is given by*

$$e_j(u_1'(z_0), \ldots, u_k'(z_0)) = (-1)^j \binom{k}{j} \frac{h_{j,k-j}}{h_{0,k}} ,$$

where $h_{i,j}$ denotes the partial derivative i times in the first coordinate and j times in the second coordinate at (z_0, w_0). Equivalently,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} h_{j,k-j} \, (u_i'(z_0))^j = 0.$$

PROOF: Partially differentiate h a total of k times, $k-j$ times with respect to w and j times with respect to z. Each factor $w - u_i(z)$ vanishes at (z_0, w_0), so the only nonzero contributions to $h_{j,k-j}(z_0, w_0)$ come from differentiating each term once with respect to either z or w. Thus $h_{j,k-j}(z_0, w_0)$ is $(k-j)!j!u(z_0, w_0)$ times the sum over all subsets S of cardinality j of $\{1, \ldots, k\}$ of $\prod_{i \in S} u_i'(z_0)$. Dividing by this by $h_{0,k}(z_0, w_0) = k!u(z_0, w_0)$ proves the lemma. $\square$

Applying this in the case $k = 2$ with $h = H$ and $u = \dfrac{G(z,w)}{\chi(z,w)\prod_{i=1}^{k}(-u_i(z))}$ gives

$$u_1'(z_0) + u_2'(z_0) = -\frac{H_{zw}(z_0, w_0)}{2H_{ww}(z_0, w_0)}$$

$$u_1'(z_0)u_2'(z_0) \quad = \quad \frac{H_{zz}(z_0, w_0)}{H_{ww}(z_0, w_0)} .$$

Hence (with all derivatives evaluated at (z_0, w_0)),

$$(u_1' - u_2')^2 = (u_1' + u_2')^2 - 4u_1'u_2' = 4\frac{H_{wz}^2 - H_{zz}H_{ww}}{H_{ww}^2} .$$

Plugging this into (**??**) gives

$$\mathcal{H} = \frac{z_0^2(H_{wz}^2 - H_{zz}H_{ww})}{w_0^2 H_{ww}^2} .$$

Substituting this into equation (**??**) in turn gives

$$a_{r,s} \sim z_0^{-r} w_0^{-s} \frac{G}{zw\sqrt{H_{wz}^2 - H_{zz}H_{ww}}}$$

evaluated at (z_0, w_0). Finally, the plateau result (Corollary **??**) implies that the error term here is of smaller exponential order, leading to the statement of the theorem. $\square$

7.4 Arbitrary d and k

In order to apply this is several variables, we require the following computational device which represents the residue sum as a difference quotient. A proof may be found in Devore and Lorentz (1993), p. 121 (7.7) and (7.12), though for completeness we give a sketch here. Fix any $d \geq 2$ and let $\triangle$ denote the $d-1$ simplex, that is the subset of $\mathbf{R}^d$ defined by

$$\{(x_1, \ldots, x_d) : \sum_{j=1}^{d} x_j = 1 \text{ and } 0 \leq x_j \leq 1 \text{ for all } 1 \leq j \leq d\}.$$

Let $\mathbf{d}\lambda$ denote Lebesgue measure on $\triangle$.

Lemma 7.14 *Let h be a function of one complex variable, analytic in a neighborhood of 0, with derivatives denoted $h^{(j)}$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \triangle$, let v_α denote the convex combination $\alpha_1 v_1 + \cdots + \alpha_d v_d$. Then*

$$\sum_{j=1}^{d} \frac{h(v_j)}{\prod_{r \neq j} v_j - v_r} = \int_\triangle h^{(d-1)}(v_\alpha)\ \mathbf{d}\lambda(\alpha)$$

both as formal power series in d variables $v_1, \ldots, v_d$ and in a neighborhood of the origin in C^d.

Proof: Both sides are linear in h so it suffices to consider the case where $h(z) = z^j$, $j = 1, 2, \ldots$. In this case the LHS may be written as a rational function of $v_1, \ldots, v_d$, whose denominator is the Vandermonde determinant $\prod_{r \neq s}(v_r - v_s)$. The numerator is $\sum_{t=1}^{d} v_t^j \prod_{r \neq s \neq j}(v_s - v_r)$, which is symmetric in $v_1, \ldots, v_d$ and vanishes whenever the quantities $v_1, \ldots, v_d$ are not distinct. Hence the numerator is divisible by the Vandermonde determinant and the LHS is a polynomial, evidently homogeneous of degree $j - d + 1$ and vanishing if $j < d - 1$. The RHS is also a homogeneous polynomial of degree $j - d + 1$. Hence the equality for all j follows from the equality summed over all j, so we now consider the case $h(z) = 1/(1-z)$.

In this case the LHS is given by

$$\frac{1}{\prod_{r=1}^{d}(1 - v_r)} \sum_{r=1}^{d} \prod_{s \neq r} \frac{1 - v_s}{v_r - v_s}.$$

Clearing the denominator of the sum (which is the Vandermonde determinant again), one sees that the numerator is a polynomial of degree $d(d-1)$ which is again symmetric and vanishing when some $v_r = v_s$, hence is divisible by the Vandermonde determinant. Thus the sum is a polynomial of degree 0. Evaluating at $v_1 = 1, v_r = 0$ for $r > 1$, shows that the value is 1. Since the summation is identically 1, it follows that the LHS is equal to $\prod_{r=1}^d (1 - v_r)^{-1}$. As a formal power series, the coefficient of every term $v_1^{j_1} \cdots v_d^{j_d}$ is equal to 1. Now evaluate the coefficient of $v_1^{j_1} \cdots v_d^{j_d}$ on the RHS. By homogeneity, the coefficient when $h(z) = 1/(1-z) = \sum_{n=0}^\infty z^n$ is the same as the coefficient when $h(z) = z^{j+d-1}$ for $j = j_1 + \cdots + j_d$. Here $h^{(d-1)}(z) = (j+d-1)\cdots(j+1)z^j$ and the RHS reduces to

$$\binom{j}{j_1, \ldots, j_d} \int_\triangle \alpha_1^{j_1} \cdots \alpha_d^{j_d} \, \mathbf{d}\lambda(\alpha_1, \ldots, \alpha_d) \, .$$

The integral is the normalizing coefficient for a Gamma distribution, and is given by $j_1! \, j_2! \cdots j_d!/(j+d-1)!$, showing that the coefficient is 1 and completing the proof of the lemma. □

Applying this yields

Corollary 7.15

$$R = \sum_{j=1}^k \text{Res}(w_d^{-r_d - 1} F; w_d = u_j(\widehat{\mathbf{w}})) = \int_\triangle h^{(d-1)}(v_\alpha(\widehat{\mathbf{w}})) \, \mathbf{d}\lambda(\alpha)$$

where $h(x) = x^{r_d + k - 1}/\chi(\widehat{\mathbf{w}}, 1/x)$ *and* $v_\alpha(\mathbf{y}) := \sum_{j=1}^k \alpha_j v_j(\mathbf{y})$.

Proof: Since all the poles are simple,

$$\begin{aligned} R &= \sum_{j=1}^k \text{Res}\left(\frac{w_d^{-r_d-1}\chi(\mathbf{w})}{\prod_{r \neq j}(1 - v_r(\widehat{\mathbf{w}})w_d)}; w_d = u_j(\widehat{\mathbf{w}})\right) \\ &= \sum_{j=1}^k \frac{(-1/v_j(\widehat{\mathbf{w}}))v_j(\widehat{\mathbf{w}})^{r_d+1}\chi(\widehat{\mathbf{w}}, u_j(\widehat{\mathbf{w}}))}{\prod_{r \neq j}(1 - v_r(\widehat{\mathbf{w}})u_j(\widehat{\mathbf{w}}))} \\ &= -\sum_{j=1}^k \frac{v_j(\widehat{\mathbf{w}})^{r_d+k-1}\chi(\widehat{\mathbf{w}}, 1/v_j(\widehat{\mathbf{w}}))}{\prod_{j=1}^k v_j(\widehat{\mathbf{w}}) - v_r(\widehat{\mathbf{w}})} \, . \end{aligned}$$

Setting $h(x) = x^{r_d+k-1}/\chi(\widehat{\mathbf{w}}, 1/y)$ and applying the theorem yields the corollary. $\square$

Finally, we plug this into Lemma **??** to see that

$$\Xi = (2\pi)^{1-d}\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}} \int_{\tilde{\mathcal{N}}} \int_{\triangle} h^{(k-1)} v_\alpha(\widehat{\mathbf{w}}(\widehat{\theta}))\, \mathbf{d}\lambda(\alpha)\, d\widehat{\theta}\,.$$

This positions us to prove

Theorem 7.16 *Let $F = G/H$ be meromorphic in a neighborhood of $D(\mathbf{z})$ and suppose that $\mathbf{z}$ is an isolated multiple point of $\mathcal{V}$ of multiplicity k. Assume further that the intersection of the tangent planes to the sheets $\mathcal{V}_j$ near $\mathbf{z}$ is a single point, that $\mathbf{r}$ is in the interior of $\mathbf{dir}(\mathbf{z})$, and that the multiplicities of the sheets are all 1. Let $\mathcal{N}$ be a neighborhood of the origin in $\mathbf{R}^{d-1}$ and let $\triangle$ be the $(k-1)$-simplex in $\mathbf{R}^k$. Define a function $f : \mathcal{N} \times \triangle \to \mathbf{C}$ by*

$$f(\widehat{\theta}, \alpha) = -\log[v_\alpha(\widehat{\mathbf{w}}(\widehat{\theta}))z_d] + i\sum_{j=1}^{d-1} \frac{r_j}{r_d}\theta_j\,.$$

Under a few more conditions,

$$a_{\mathbf{r}} \sim (2\pi)^{1-d}\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}}...$$

8 Cone points

9 Classification for two-variable meromorphic functions with nonnegative coefficients

We began with the question of general computation of asymptotics for multivariate generating functions. We then restricted our attention to meromorphic functions, for which the zero variety of the denominator was the key to analysis via iterated Cauchy integration. For purposes of classification some natural questions are:

1. what are all possible local geometries of minimal points of $\mathcal{V}$?

2. which of these can be handled by variants of the methods we have seen so far?

3. are these sufficient to yield a good approximation to $a_{\mathbf{r}}$ no matter what the direction, $\mathbf{dir}(\mathbf{r})$, for any meromorphic generating function?

To make the last question more concrete, consider the simplest possible example, namely binomial coefficients, where $F = 1/(1-z-w)$ and $\mathcal{V}$ is a complex line. There are no singular points here, but how do we know that as (z, w) varies over minimal points of $\mathcal{V}$, the direction $\mathbf{dir}(z, w)$ will cover all of $\mathbf{RP}^1$?

In the two variable case, assuming nonnegativity of coefficients, this question will be answered affirmatively by Theorem **??**. After that we will discuss the degree to which we have a complete classification in higher dimensions, or in case the coefficients have mixed signs.

9.1 Puiseux series

We begin by observing, without restriction on the dimension or signs of the coefficients, that cusps may never be locally minimal. To properly define our terms, consider the power series expansion about a point $\mathbf{z} \in \mathcal{V}$ where all the first partials of H vanish. The expansion of $H(\mathbf{x})$ near $\mathbf{z}$ is then a sum of terms of degrees 2 and higher. We call $\mathbf{z}$ a *homogeneous point* of degree k if this expansion contains terms $(x_j - z_j)^k$ for each $j = 1, \ldots, d$, and contains no terms of total degree less than k. A point where the first partials vanish that is not homogeneous of any order is commonly known as a cusp. `Discussion of Puiseux series here`

Lemma 9.1 *If* $\mathbf{z}$ *is a locally minimal point of* $\mathcal{V}$ *with nonzero coordinates, and* F *is meromorphic in a neighborhood of* $\mathbf{z}$ *then* $\mathbf{z}$ *is homogeneous.*

Proof: Passing to $F(z_1 x_1, \ldots, z_d x_d)$ if necessary, we may assume $\mathbf{z} = \mathbf{1}$. Setting $x_j = 1$ for all but one index j, we cannot obtain the zero function (by minimality), and so some term in the expansion around **1** is a pure power of $(x_j - 1)$, and we denote the minimal degree such term by $c_j(x_j - 1)^{k_j}$. If **z** is not a homogeneous point, then there is some j for which some monomial has total degree lower than k_j. Assume without loss of generality that $j = d$. The function $F(x, x, \ldots, x, y)$ then has a minimal degree pure $y - 1$ term $c_0(y-1)^k$, $k := k_d$, and some term $c'(x-1)^a(y-1)^b$ with $a + b < k$. In other words, the Newton Polygon of $F(x, \ldots, x, y)$ around $(1, 1)$ has a support line passing through $(0, k)$ with slope $-p/q$ in lowest terms, and $p > q$. It is well known that we may describe the solutions $y(x)$ of the equation

$$F(1 + x, \ldots, 1 + x, 1 + y) = 0$$

as follows. Write

$$H := (y-1)^k(c_0 + c_1(y-1)^{-p}(x-1)^q + c_2(y-1)^{-2p}(x-1)^{2q} + \cdots + c_s(y-1)^{-sp}(x-1)^{sq})$$

for the polynomial collecting all the terms on this support line. Then for each q^{th} root of unity, ω, and each root λ of $\sum c_{s-j}\lambda^j = 0$, there is a solution $y = \lambda^{1/p}x^{q/p}(\omega + o(1))$ as $x \to 0$. The standard proof involves showing that perturbing H by terms of higher homogeneous degree affects the solutions only by factors of $(1 + o(1))$ as $x \to 0$.

Varying x over the set $|\pi - \arg(x)| \leq \pi/4$, we see that the solutions $y(x)$ must sometimes be in this set as well. For those x, the points $(1+x, \ldots, 1+x, 1+y)$ will be in $\mathcal{V} \cap D(\mathbf{1}) \setminus T(\mathbf{1})$, violating minimality of **1**. By contradiction, we have shown that no monomial in the expansion around **1** has lower total degree than any pure power term, hence **1** is minimal.
□

9.2 Classification theorem

The variety determined by the terms of the power series about **z** that have leading homogeneous degree is called the *tangent cone.* A test for whether a degree k homogeneous point is a multiple point is whether the degree k part of the expansion factors completely into linear factors (equivalently, whether the tangent is a union of hyperplanes). If not, we call **z** a *cone point.* The bad news is that we do not yet know how to deal with cone points. The good news is that cone points do not exist in two dimensions (every homogeneous polynomial in two complex variables factors into linear terms).

To summarize, if F is meromorphic in a neighborhood of its domain of convergence in $\mathbf{C}^2$, then every $(z, w) \in \mathcal{V}$ is either geometrically a smooth point, or is a multiple point. For smooth points, Theorem **??** and the generalizations given in section **??** assure us that asymptotics may be computed in directions $\mathbf{dir}(z, w)$. For multiple points, Theorem **??** in the simplest case and the extensions (higher multiplicity in Pemantle and Wilson 2000b and non-transverse intersections in Pemantle and Wilson 2000c) show that asymptotics may be computed for all directions $\mathbf{dir}(z, w)$. The following theorem then completes the classification.

Theorem 9.2 *Let* $F = G/H = \sum a_{r,s} z^r w^s$ *be the quotient of analytic functions* $G, H : \mathbf{C}^2 \to \mathbf{C}$. *Suppose that the coefficients* $a_{r,s}$ *are all nonnegative, and that* $F(z, 0)$ *and* $F(0, w)$ *are not entire. Then for every direction* $\alpha \in \mathbf{RP}^1$ *there is a minimal* $\mathbf{z} \in \mathcal{V}$ *with* $\alpha \in \mathbf{dir}(\mathbf{z})$.

Proof: Let (x, y) be any point on the boundary of $\log \mathcal{D}$. For $u < e^x$ and $v < e^y$ the power series for F is convergent at (u, v). As $u \uparrow e^x$ and $v \uparrow e^y$ therefore, $F(u, v)$ is finite and increasing. On the other hand, the power series for F is not absolutely convergent on $T(e^x, e^y)$, since we know F to have some singularity on this torus. Hence $F(u, v) \uparrow \infty$ as $(u, v) \uparrow (e^x, e^y)$. Since F is meromorphic, it must have a pole at (e^x, e^y), hence $(e^x, e^y) \in \mathcal{V}$ and is a minimal point of $\mathcal{V}$. As (x, y) varies over the boundary of $\log \mathcal{D}$, we let $\gamma \subseteq \mathcal{V}$ denote the curve traced out by this minimal point.

Pick any $\alpha \in \mathbf{RP}^1$. The convex set $\log \mathcal{D}$ has horizontal and vertical support hyperplanes (by non-entirety of $F(z, 0)$ and $F(0, w)$), and therefore has a support hyperplane normal to α; let (x, y) be a point of intersection of this support plane with $\log \mathcal{D}$. We have just seen that $(z(\alpha), w(\alpha)) := (e^x, e^y)$ is a minimal point of $\mathcal{V}$. If (z, w) is a smooth point of $\mathcal{V}$ then $\mathbf{dir}(z, w) = \{\alpha\}$.

Assume now that (z, w) is not a smooth point. By Lemma **??**, (z, w) is a homogeneous point, and since $d = 2$, (z, w) is a multiple point. Then $\mathbf{dir}(z, w)$ is the set of normals to $\log\mathcal{D}$ at (x, y), so again $\alpha \in \mathbf{dir}(z, w)$. This finishes the proof. $\square$

9.3 Progress on singularities in three or more dimensions: classification and applications

1. Node points are still OK

2. Possibility of other homogeneous singularities

3. Random tiling examples

10 Effective computation

11 Obtaining multivariate generating functions

11.1 Standard boundary conditions

1. Rehash motivating example from Larsen and Lyons

2. Outline the general case of standard BC's

3. Give a formulation of the solution in terms of symmetric functions, if possible

11.2 Symmetry boundary conditions

Summarize work of Flatto et al.

11.3 Queuing models and left-continuous random walks

1. Discuss queuing models leading to left-continuous random walks in more than one dimension

2. Show how to reduce these to Riemann-Hilbert problems, following Taylor et al

3. Solve some of these

References

[1] Aldous, D. (1999). Unpublished manuscript.

[2] Arnold, V., Guseuin-Zade, S. and Varchenko, A. (1988). Singularities of differentiable maps, Vol. II: Monodromy and asymptotics of interals. *Monographs in Mathematics*, **83**. Birkhäuser: Boston.

[3] Bender, E. and Richmond, L. B. (1983). Central and local limit theorems applied to asymptotic enumeration II: multivariate generating functions. *JCT A* **34** 255 - 265.

[4] Bleistein, N. and Handelsman, R. (1986). Asymptotic expansions of integrals, second edition. Dover: Mineola, NY.

[5] Burton, R. and Pemantle, R. (1993). Local statistics, entropy and limit theorems for uniform spanning trees and domino tilings, via transfer-impedances. *Ann. Probab.* **21** 1329 - 1371.

[ChySal96] Chyzak, F. and Salvy, B. (1998). Non-commutative elimination in Ore Algebras proves multivariate identities. *J. Symbol. Comput.* **26** 187 - 227.

[6] Cohn, H., Elkies, N. and Propp, J. (1996). Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.* **1985** 117 - 166.

[7] Comtet, L. (1974). Advanced combinatorics. Reidel: Boston.

[8] Devore, R. and Lorentz, G. (1993). Constructive approximation. Springer-Verlag: New York.

[9] Doubilet, P., Rota, G.-C. and Stanley, R. (1972). On the foundations of combinatorial theory (VI): the idea of generating function. In: Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 2: Probability Theory, 267 - 318. University of California Press: Berkeley.

[10] Flajolet, P. and Odlyzko, A. (1990). Singularity analysis of generating functions. *SIAM J. Disc. Math.* **3** 216 - 240.

[11] Flatto, L. and Hahn, S. (1984). Two parallel queues created by arrivals with two demands, I. *SIAM J. Appl. Math.* **44** 1041 - 1053.

[12] Flatto, L. and McKean, H. (1977). Two queues in parallel. *Comm. Pure Appl. Math.* **30** 255 - 263.

[13] Furstenburg, H. (1967). Algebraic functions over finite fields. *J. Algebra* **7** 271 - 277.

[14] Gao, Z. and Richmond, L. B. (1992). Central and local limit theorems applied to asymptotic enumeration IV: ultivariate generating functions. *J. Comput. Appl. Math.* **41** 177 - 186.

[15] Gessel, I., Ionescu, A. and Propp, J. (1995). Counting constrained domino tilings of Aztec diamonds. *Preprint.*

[16] rauert, H. and Remmert, R. (1979). Theory of Stein spaces. Translated from the German by Alan Huckleberry. Grundlehren der Mathematischen Wissenschaften 236. Springer-Verlag: Berlin.

[17] riffiths, P. and Harris, J. (1994). Principles of algebraic geometry. Wiley: New York.

[18] Henrici, P. (1977). Applied and computational complex analysis, Vol. 2. John Wiley and Sons: New York.

[19] Hörmander, L. (1983). The analysis of partial differential operators, I. Grundlehre der Math. Wiss. # 256. Springer: Berlin.

[20] Hörmander, L. (1990). An introduction to complex analysis in several variables, third edition. North-Holland: New York.

[21] Jockusch, W., Propp, J. and Shor, P. (1995). Random domino tilings and the arctic circle theorem. *Preprint.*

[22] Kenyon, R. and Pemantle, R. (2000). Universal decay laws for correlations of tile placements in exactly solvable models. *In preparation.*

[23] Larsen, M. and Lyons, R. (1999). Coalescing particles on an interval. *J. Theor. Prob.* **12** 201 - 205.

[Lip88] Lipshitz, L. (1988). The diagonal of a D-finite power series is D-finite. *J. Algebra* **113** 373 - 378.

[24] Milnor, J. (1969). Morse Theory, 3^{rd} Edition. Annals of Math. Studies no. 51. Princeton University Press: Princeton.

[25] Odlyzko, A. (1995). Asymptotic enumeration methods. In: Handbook of combinatorics, Vol. II, Graham, Grötschel and Lovász, editors. Elsevier: New York.

[26] Pemantle, R. and Wilson, M. (2000a). Asymptotics for multivariate sequences, part I: smooth points of the singular variety. *In preparation.*

[27] Pemantle, R. and Wilson, M. (2000b). Asymptotics for multivariate sequences, part II: multiple points. *In preparation.*

[28] Pemantle, R. and Wilson, M. (2000c). Asymptotics for multivariate sequences, part III. *In preparation.*

[29] Stanley, R. (1978). Generating functions. In: Studies in combinatorics, M.A.A. Studies in Mathematics Vol. 17, Gian-Carlo Rota, Ed. MAA: Providence.

[30] Stanley, R. (1986). Enumerative combinatorics, vol. 1. Wadsworth & Brooks/Cole: Monterey.

[31] Stanley, R. (1999). Enumerative combinatorics, vol. 2. Cambridge Studies in Advanced Mathematics no. 62. Cambridge University Press: Cambridge.

[32] Stein, E. (1993). Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, Vol. 43. Princeton University Press: Princeton.