Crossed products of restricted enveloping algebras

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Abstract

Let K be a field of characteristic $p > 0$, let L be a restricted Lie algebra and let R be an associative K-algebra. It is shown that the various constructions in the literature of crossed product of R with $u(L)$ are equivalent. We calculate explicit formulae relating the parameters involved and obtain a formula which hints at a noncommutative version of the Bell polynomials.

Crossed products of group algebras have played an important part in the study of group algebras and Galois theory of rings, while considerable use has also been made of the analogous constructions for universal enveloping algebras of Lie algebras. The more general notion of crossed product of a Hopf algebra, which subsumes the above cases, was introduced in full generality in [3] and independently in [2]. The object of this paper is to clarify the special case in which the Hopf algebra in question is the restricted enveloping algebra of a restricted Lie algebra, as was done for ordinary enveloping algebras in [5].

We show that three constructions so far given in the literature define the same object up to isomorphism. In order to establish this it is necessary to compute explicit formulae relating the parameters which influence the twisting of the p-mapping. Specializing to the case where the coefficient ring is commutative we obtain a formula which deserves further clarification.

1 Definitions

Throughout this paper K is a field of characteristic $p > 0$ and L a restricted Lie algebra. At least three definitions of crossed product by a restricted enveloping algebra have been given in the literature. First, there is the construction found in, for example, the paper [1]. Let R be an associative K-algebra and L a restricted Lie K-algebra. Fix a vector space embedding $^{-} : L \to u(L)$ and a linear map $\partial \colon L \to \mathrm{Der}_K R$. Then $S = R *_{t,t'} u(L)$ is an associative K-algebra with the K-linear structure of $R \otimes_K u(L)$ such that multiplication extends that of R and

(i) $r\bar{x} = \bar{x}r + \partial(x)(r)$

(ii) $\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} + t(x, y)$

(iii) $\bar{x}^p = \overline{x^{[p]}} + t'(x)$

for all $r \in R$, all $x, y \in L$, and some functions $t \colon L \times L \to R$ and $t' \colon L \to R$.

Associativity of S is equivalent to certain rather complicated relations between the functions t, t' and ∂; however as remarked in [1], associativity is the more natural condition. The difficulty stems from the fact that an elementary description of restricted Lie algebra cocycles, in terms of functional equations on L, is considerably more complex than in the ordinary Lie algebra case. For now we note only that if either t or t' is zero then rather simple conditions on the other may be deduced. For example, consider the case $\partial = 0$, in which case S is called a twisted restricted enveloping algebra. If $t = 0$ then a straightforward computation shows that for associativity it is necessary and sufficient that t' be a p-semilinear map, whereas if $t' = 0$ then one checks easily that it is necessary and sufficient that t be a Lie cocycle.

Given the functions t, t', uniqueness is clear from this definition, but existence has not been shown. The second definition, completely analogous to that in [4], has the advantage that existence, as well as associativity, is clear. Denote the multiplicative identity of R by ι, and let R^{-} denote R with its induced restricted Lie algebra structure. Given an extension $0 \to R^{-} \to M \to L \to 0$ of restricted Lie K-algebras, form the usual restricted enveloping algebra $u(M)$. Denoting the multiplication in $u(M)$ by $\times$, we define the crossed product to be $S = u(M)/I$, where I is the ideal generated by $\iota - 1$ and all $r_1 r_2 - r_1 \times r_2$ for $r_1, r_2 \in R$. Denote this construction by $S = R \times_e u(L)$, where e is the extension above.

The third definition is obtained by specializing the general construction of Doi and Takeuchi, Blattner, Cohen and Montgomery. We give the general definition here. Let H be a Hopf algebra and R an algebra which is weakly acted on by H. That is, there is a linear map $H \otimes R \to R$ given by $h \otimes r \mapsto h \cdot r$, such that $h \cdot 1 = \epsilon(h)1, h \cdot (rs) = \sum (h_1 \cdot r)(h_2 \cdot s)$ and $1 \cdot h = h$, for all $h \in H$ and $r, s \in R$. Given an invertible map $\sigma \colon H \otimes H \to R$, define $R \#_\sigma H$ to be the algebra with the additive structure of $R \otimes H$ and multiplication

$$(r \otimes h)(s \otimes k) = \sum r(h_1 \cdot s)\sigma(h_2, k_1) \otimes h_3 k_2 \tag{1}$$

for all $h, k \in H$ and all $r, s \in R$.

It is known (see [6]) that the crossed product is associative with identity $1 \otimes 1$ if and only if σ is a normalized Hopf cocycle and R satisfies a twisted H-module condition.

We interpret the generalities above in the specific case $H = u(L)$. The comultiplication in $u(L)$ is of course given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$. Then (1) yields

$$(1 \otimes x)(1 \otimes y) = 1 \otimes xy + \sigma(x, y) \otimes 1 \qquad \text{for all } x, y \in L \tag{2}$$

and

$$(r \otimes 1)(1 \otimes x) = (1 \otimes x)(r \otimes 1) + (x \cdot r) \otimes 1 \qquad \text{for all } r \in R,\ x \in L. \tag{3}$$

The condition that σ be a 2-cocycle is equivalent to the identity:

$$\sigma(xy, z) = x \cdot \sigma(y, z) + \sigma(x, yz) \qquad \text{for all } x, y, z \in L. \tag{4}$$

The normalization condition yields $\sigma(1,1) = 1$ and $\sigma(x, 1) = 0 = \sigma(1, x)$ for all x in the augmentation ideal of $u(L)$.

Theorem 1.1. *Let R be an algebra and L a restricted Lie algebra. Then the following are equivalent for an algebra S:*

(1) *S is associative and $S \cong R *_{t,t'} u(L)$ for some t, t'*

(2) *$S \cong R \times_e u(L)$ for some extension e of R^- by L*

(3) *$S \cong R \#_\sigma u(L)$ with σ a normalized cocycle and R a twisted $u(L)$-module.*

Proof. The proof is a modification of [5, Theorem 2.8] and is only sketched. The main difference is in the implication (3) $\Rightarrow$ (1), which we prove in Section 2.1.

Assume (1). Then clearly the subspace $M = \bar{L} \dotplus R \cdot 1$ is a restricted Lie algebra extension of R^- by L. As in [5] we have an epimorphism $u(M) \to R * u(L)$ and the kernel of this map is easily seen to be the ideal I in the definition of $R \times_e u(L)$ by using Jacobson's theorem (analogue of the PBW theorem for restricted Lie algebras). Hence (1) $\Rightarrow$ (2).

Assume (2). Then the restricted Lie extension $0 \to R^- \to M \to L \to 0$ yields a cleft extension $u(R^-) \subseteq u(M)$ of Hopf algebras and hence a crossed product $u(M) \cong u(R^-)\#_{\sigma'} u(L)$. Since I is generated by its intersection with R the cocycle σ' yields a cocycle $\sigma : u(L) \times u(L) \to R$ and it follows that $R \times_e u(L) \cong R\#_\sigma u(L)$. Thus (2) $\Rightarrow$ (3).

Assume (3). Since S is associative we need only verify the properties (i) - (iii) of the definition of $R * u(L)$. Let $\bar{x} = 1 \otimes x$ for $x \in L$. Property (i) has already been seen to hold in equation (3), and we have $\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} + \sigma(x, y) - \sigma(y, x)$ for all $x, y \in L$, which gives (ii). For (iii) we need to compute $\bar{x}^p - \overline{x^{[p]}}$ and show this belongs to $R \otimes 1$. This is done in the next section (see Theorem 2.1 there). Thus (3) $\Rightarrow$ (1). $\square$

2 Explicit calculations

The main point of this section is to complete the proof of Theorem 1.1 by showing that $(1 \otimes x)^p - (1 \otimes x^p) \in R \otimes 1$. We do this by computing an explicit formula for the twisting function t' in terms of the cocycle σ.

In the case of $U(L)$ as treated in [5] it was noted that given a 2-cocycle σ for $U(L)$, then the Lie cocycle τ associated with the extension of R^- by L is simply given by $\tau(x, y) = \sigma(x, y) - \sigma(y, x)$ and that nothing further need be said in this direction. In the other direction it was shown with more effort that for $U(L)$ the values of σ can be computed inductively from the action of L on R and the values of τ on L. In our situation the relation between σ and t' is not obvious, though it is clear from above that $t(x, y) = \sigma(x, y) - \sigma(y, x)$. We calculate a formula for t' in terms of σ below. As for recapturing σ from δ, t and t', a close examination of Montgomery's paper cited above shows that the argument there will carry over to the restricted case with only minor changes, and we do not include it here.

I would like to thank Shaun Cooper for helpful conversations regarding

the material of this section, in particular subsection 2.2.

2.1 General case

We first set up some notation. Let m be a positive integer. The *composition* (ordered partition) corresponding to the sum $m = p_1 + \cdots + p_r$ of positive integers will be written as an ordered multiset $P = \{p_1, \ldots, p_r\}$, and we write $|P| = m$.

We introduce the abbreviation $\sigma(n)$ for $\sigma(x^{n-1}, x)$. We also define, for a composition $P = \{p_1, \ldots, p_r\}$,

$$\sigma(P) = \sigma(p_1) \cdots \sigma(p_r).$$

Here we use the usual convention that if $r = 0$ then $\sigma(P) = 1$.

Theorem 2.1. *Let $R\#_\sigma u(L)$ with σ a 2-cocycle for $u(L)$. Then for all $x \in L$,*

$$(1 \otimes x)^p = (1 \otimes x^p) + \left[\sum_P \prod_{k=1}^{r} \binom{\sum_{i=1}^{k} p_i - 1}{p_k - 1} \sigma(P) \right] \otimes 1,$$

where the sum is over all compositions P of p.

Proof. The normalization condition implies that $\sigma(1) = 0$. Also equation (1) yields

(5)
$$\overline{x^j}\bar{x} = \overline{x^{j+1}} + \sum_{k=0}^{j} \binom{j}{k} \sigma(j-k+1)\overline{x^k} \qquad \text{for all } x \in L \text{ and all } j \geq 0.$$

Clearly $\bar{x}^n = \sum_{j=0}^{n} c_{nj}\overline{x^j}$ for some $c_{nj} \in K$. We first obtain a recurrence relation for the coefficients c_{nj}, by using (5) compare $\bar{x}^{n-1}$ and $\bar{x}^n$:

$$\begin{aligned}
\bar{x}^{n-1}\bar{x} &= \sum_{j=0}^{n-1} c_{n-1,j}\overline{x^j}\bar{x} \\
&= \sum_{j=0}^{n-1} c_{n-1,j} \left(\overline{x^{j+1}} + \sum_{k=0}^{j} \binom{j}{k} \sigma(j-k+1)\overline{x^k} \right) \\
&= \sum_{j=0}^{n-1} c_{n-1,j}\, \overline{x^{j+1}} + \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} \binom{j}{k} \sigma(j-k+1)\overline{x^k}.
\end{aligned}$$

Comparing coefficients gives the desired relation, using the convention that $c_{nj} < 0$ if $j < 0$:

(6)
$$c_{nj} = c_{n-1,j-1} + \sum_{i=j}^{n-1} \binom{i}{j} c_{n-1,i}\sigma(i+1-j) \qquad \text{if } n \geq 0 \text{ and } 0 \leq j \leq n.$$

Note that this relation determines the c_{nj} completely once the value $c_{00} = 1$ is specified.

On examining small values of n one is led to conjecture the formula

$$c_{nj} = \sum_{|P|=n-j} a_{jP}\, \sigma(P) \tag{7}$$

for some $a_{jP} \in K$. Here the sum is over all compositions P of $n-j$.

Equation (7) clearly holds for $n = 0$. We substitute into the recurrence relation (6). The right hand side is equal to

$$\sum_{|P|=(n-1)-(j-1)} a_{j-1,P}\, \sigma(P) + \sum_{i=j}^{n-1} \binom{i}{j} \sum_{|Q|=n-1-i} a_{iQ}\, \sigma(Q)\sigma(i+1-j).$$

The first sum is over compositions of $n-j$ and so is the second since $\sigma(Q)\sigma(i+1-j)$ has the form $\sigma(P)$ with $|P| = n-j$. Thus the expression (7) for the c_{nj} is correct. We now solve for the a_{jP}. Formally comparing coefficients of $\sigma(P)$ we obtain the relation

$$a_{jP} = a_{j-1,P} + a_{p_r-1+j,P'} \binom{p_r - 1 + j}{j} \qquad \text{if } 0 \leq j \leq n. \tag{8}$$

Note that this relation determines the a_{jP} uniquely once the value $a_{0\{\}} = 1$ is specified. Here of course we have $a_{jP} = 0$ if $j < 0$. Note also that since $\sigma(1) = 0$, the sum can be taken only over all compositions of $n-j$ each of whose parts is at least 2. Finally, observe that if a_{jP} satisfy this formula then the corresponding c_{nj} certainly satisfy (6).

On close examination one sees the pattern $a_{jP} = \binom{j+|P|}{j} a_{0P}$ and this combined with successive applications of equation (8), using the fact that $\sum_k p_k = |P|$, leads us to the formula

$$a_{jP} = \binom{j+|P|}{|P|} \prod_{k=1}^{r} \binom{\sum_{i=1}^{k} p_i - 1}{p_k - 1}.$$

This holds for the initial condition and so it suffices to check that equation (8) is satisfied. For a composition $P = \{p_1, \ldots, p_r\}$ we write

$$\Pi_s = \prod_{k=1}^{s} \binom{\sum_{i=1}^{k} p_i - 1}{p_k - 1}.$$

Substituting our proposed values into the right side of (8) we obtain

$$\begin{aligned}
& \binom{j-1+|P|}{|P|}\Pi_r + \binom{p_r-1+j}{j}\binom{j-1+|P|}{|P|-p_r}\Pi_{r-1} \\
= & \left[\binom{j-1+|P|}{|P|} + \frac{\binom{p_r-1+j}{j}\binom{j-1+|P|}{|P|-p_r}}{\binom{|P|-1}{p_r-1}}\right]\Pi_r \\
= & \left[\binom{j-1+|P|}{|P|} + \frac{(p_r-1+j)!(j-1+|P|)!(p_r-1)!(|P|-p_r)!}{j!(p_r-1)!(|P|-p_r)!(j-1+p_r)!(|P|-1)!}\right]\Pi_r \\
= & \left[\binom{j-1+|P|}{|P|} + \binom{j-1+|P|}{|P|-1}\right]\Pi_r \\
= & \binom{j+|P|}{|P|}\Pi_r
\end{aligned}$$

which equals the left side of (8) with our proposed a_{jP}.

It follows that if $n = p$, the characteristic of K, then $a_{jP} = 0$ in R for $0 < j < p$, and $j = p$ implies $a_{jP} = 1$. The result follows. □

An explicit example: with $p = 7$ we have

$$\begin{aligned}
\bar{x}^7 &= \overline{x^{[7]}} + \sigma(x^6, x) + 6\sigma(x^4, x)\sigma(x, x) + 15\sigma(x, x)\sigma(x^4, x) \\
&+ 15\sigma(x^3, x)\sigma(x^2, x) + 20\sigma(x^2, x)\sigma(x^3, x) \\
&+ 24\sigma(x^2, x)\sigma(x, x)^2 + 36\sigma(x, x)\sigma(x^2, x)\sigma(x, x) + 45\sigma(x, x)^2\sigma(x^2, x).
\end{aligned}$$

2.2 R commutative

In this subsection we show that a much simpler formula for t' can be derived in the special case when R is commutative.

Note that in the above example we have $6 + 15 = 21, 15 + 20 = 35$ and $24+36+45 = 105$ and these are all divisible by 7. This is no accident as we show below.

Theorem 2.2. *Let $S = R\#_\sigma u(L)$ with R commutative. Then for all $x \in L$,*

$$(1 \otimes x)^p = 1 \otimes x^p + \sigma(x^{p-1}, x).$$

Proof. We proceed as in the proof of Theorem 2.1 with a few notational modifications.

Recall that a *partition* is a composition $\{p_1, \dots, p_r\}$ in which $p_1 \leq p_2 \leq \cdots \leq p_r$. There is a natural action of the symmetric group S_r on the set of compositions of length r, given by $\tau(\{p_1, \dots, p_r\}) = \{p_{\tau(1)}, \dots, p_{\tau(r)}\}$. Each orbit has a unique representative which is a partition.

Since R is commutative, all the values of σ commute. Thus in equation (7) the sum $\sum_P a_{jP}\, \sigma(P)$ can be rewritten as $\sum_Q b_{jQ}\, \sigma(Q)$, where the second sum is over all partitions of $n - j$. For each partition Q, the coefficient b_{jQ} is equal to the sum $\sum_P a_{jP}$ where P runs over all compositions in the orbit of Q.

The recurrence relation we derive is more complicated:

$$b_{jQ} = b_{j-1,Q} + \sum_{k=1}^{r} \binom{q_k - 1 + j}{j} b_{q_k-1+j,Q_k} \tag{9}$$

where Q_k denotes the partition obtained by deleting one occurrence of q_k from Q.

For a partition $Q = \{q_1, q_1, \cdots, q_1, q_2, \cdots, q_2, \cdots, q_r\}$ we shall write e_k for the multiplicity of q_k. By trial and error one conjectures that

$$b_{jQ} = \binom{j + |Q|}{|Q|} w(Q)$$

where $w(Q) = \frac{|Q|!}{(q_1!)^{e_1} \cdots (q_r!)^{e_r} e_1! \cdots e_r!}$. Since this is clear for the initial condition, we again proceed by showing that the putative b_{jQ} satisfy the recurrence relation (9). We use the easily established formula $w(Q_k) = w(Q)\frac{(|Q|-q_k)!}{|Q|!}(q_k)!e_k$. The right side of (9) then equals

$$
\begin{aligned}
& w(Q)\binom{j-1+|Q|}{|Q|} + \sum_{k=1}^{r}\binom{q_k-1+j}{j}\binom{|Q|-1+j}{|Q|-q_k} w(Q_k) \\
= \; & w(Q)\binom{j-1+|Q|}{|Q|} + \frac{(|Q|-1+j)!}{j!(|Q|-q_k)!}\sum_{k=1}^{r}\frac{w(Q_k)}{(q_k-1)!} \\
= \; & w(Q)\left[\binom{j-1+|Q|}{|Q|} + \frac{(|Q|-1+j)!}{j!|Q|!}\sum_{k=1}^{r} q_k e_k\right] \\
= \; & w(Q)\left[\binom{j-1+|Q|}{|Q|} + |Q|\frac{(|Q|-1+j)!}{j!|Q|!}\right] \\
= \; & w(Q)\left[\binom{j-1+|Q|}{|Q|} + \binom{j-1+|Q|}{|Q|-1}\right] \\
= \; & w(Q)\binom{j+|Q|}{|Q|}
\end{aligned}
$$

and this equals the left side of (9) with the proposed b_{jQ}.

If n is equal to the characteristic p of K, then for $1 \le j \le p-1$ the factor $\binom{|j|+|Q|}{|Q|} = \binom{p}{j} = 0$ in K. If $j = 0$ then unless $Q = \{p\}$ the factor $w(Q)$ is zero in K, and if $j = p$ then $w(Q) = 1$. This yields the desired result. $\square$

2.3 Comments

Comparison of the two formulae for t' derived in the previous subsections yields the following interesting formula. Let $P = \{p_1, p_2, \ldots, p_r\}$ be a composition of n. Then

$$
(10) \qquad \sum_{\tau \in S_r} \prod_{k=1}^{r} \binom{\sum_{i=1}^{k} p_{\tau(i)} - 1}{p_{\tau(k)} - 1} = \binom{n}{p_1; p_2; \cdots; p_r}.
$$

One combinatorial proof of this formula was shown to me by Marston Conder and I thank him for permission to reproduce it here. The right side of (10) counts the number of words of length n in the noncommuting variables $x_1, \ldots, x_r$ in which each x_i occurs p_i times. For each $\tau \in S_r$ the product on the left of (10) counts the number of such words in which, as we traverse from right to left, the ith new variable encountered is $x_{\tau(i)}$, whence the result follows.

Obviously knowing *a priori* how to replace the left side of the formula with the right would obviate the work of subsection 2.2. However we have included that subsection for possible other applications, and also because it gives a more natural proof. The expressions $w(Q)\sigma(Q)$ involved there are essentially values of the so-called Bell polynomials (see [7]) for details). It is possible that examination of our derivation of (10) will lead to a noncommutative analogue of these polynomials.

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