BELL'S PRIMENESS CRITERION FOR $W(2n+1)$

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Abstract. On the basis of experimental work involving matrix computations, we conjecture that a criterion due to Bell for primeness of the universal enveloping algebra of a Lie superalgebra applies to the Cartan type Lie superalgebras $W(n)$ for $n = 3$ but does not apply for odd $n \geq 5$. The experiments lead to a rigorous proof, which we present.

1. Introduction

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $L = L_0 \dot{+} L_1$ with a graded bilinear product mapping $[\ ,\ ]\colon L \times L \to L$ which satisfies certain identities. A good general reference is [Sch79]. In particular the restriction to L_1 of the product map yields a symmetric bilinear map. A result due to Bell ([Bel90]) shows that if the *product matrix* which represents this map is nonsingular then the universal enveloping algebra $U(L)$ is a prime ring.

The finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero have been classified by V. Kac in [Kac77]. There is an important structural division of such algebras into those of classical type and those of Cartan type. It is known ([Bel90], [KK96]) that Bell's criterion holds for all but one family of the classical simple algebras. In recent papers ([Wil96], [Wil]), the first author has attempted to determine whether Bell's criterion applies to the Cartan type simple Lie superalgebras, and has shown that the algebras in the families of $W(2n)$ and $H(n)$ also satisfy the criterion, and that $S(2n+1)$ does not. The proofs in these cases, though not trivial, were of a more straightforward character than in the present paper.

Here we dispose of one of the remaining cases by showing that $W(n)$ does not satisfy Bell's criterion if n is odd and $n \geq 5$. While this has no obvious ring-theoretic ramifications, the greater complexity of this case leads to an interesting interplay between experimental and rigorous mathematics, and suggests further work. In fact the algebras $W(2n+1)$ provide the first "naturally occurring" case where Bell's criterion fails for a nontrivial reason.

In section 2 we introduce the basic notation and background. Subsection 2.1 can be safely omitted at a first reading, but the others are essential

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for the rest of the paper. Section 3 presents our experimental results and section 4 our theorems and proofs.

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2. Definitions

2.1. The algebra $W(n)$. A good general reference for this subsection is [Sch79].

Let K be a field of characteristic zero and let $\Lambda = \Lambda(V)$ be the exterior (Grassmann) algebra of the vector space $V = K^n$. Then Λ is an associative superalgebra of dimension 2^n where the $\mathbb{Z}_2$-grading is induced by the usual $\mathbb{Z}$-grading given by degree.

Let $W = W(n) = D(\Lambda)$, the Lie superalgebra of superderivations of Λ. Then $W = \oplus_r W_r$ is naturally $\mathbb{Z}$-graded and this is consistent with the $\mathbb{Z}_2$-grading. Here the graded component W_r consists of all superderivations which map V into Λ_{r+1}, so the highest degree actually occurring is $n-1$ and the lowest is -1.

For homogeneous $\partial \in W$ and $x, y \in \Lambda$, we have $\partial(xy) = \partial(x)y \pm x\partial(y)$ where the $-$ occurs if and only if both x and ∂ are odd. Every element of W restricts to a linear map $V \to \Lambda$. Conversely every element of W arises in this way and we have the isomorphism of vector spaces $W \cong \Lambda \otimes_K V^*$, where V^* denotes the linear dual of V. We shall use this identification in the rest of the paper. Under this isomorphism the element $a \otimes f$ is identified with the superderivation taking $v \in V$ to $af(v) \in \Lambda$. One obtains the multiplication formula for *odd* elements

$$[a \otimes f, b \otimes g] = af(b) \otimes g + bg(a) \otimes f. \tag{1}$$

2.2. Computations in $\Lambda \otimes V^*$. In this subsection we interpret everything in subsection 2.1 in terms of a specific basis for $\Lambda \otimes V^*$. We shall use the formulas obtained throughout the remainder of the paper.

The exterior algebra $\Lambda(V)$ is the free anticommutative algebra on V. In other words it is generated by V and all relations are consequences of the basic identity $vw = -wv$ for all $v, w \in V$. Of course this implies that $v^2 = 0$ for all $v \in V$.

In this paper ordered sets will always be written as lists $\langle i_1, \ldots, i_r \rangle$. A subset of a set will not automatically inherit any ordering which its superset may happen to have.

Fix an ordered basis $\langle v_1, \ldots, v_n \rangle$ for V. For each subset I of $N = \langle 1, \ldots, n \rangle$, choose an order $i_1 < i_2 < \cdots < i_r$ of I and define $v_I = v_{i_1} v_{i_2} \cdots v_{i_r}$. The set of all such v_I (where we define $v_\emptyset = 1$) forms a basis for Λ. Here the choice of ordering of I is completely arbitrary; changing the order of I only changes the corresponding v_I by a factor of ± 1. For definiteness, unless otherwise stated we shall assume I to be ordered in natural (increasing) order as a subset of N.

We shall need the easily established formula which is valid for any ordering of I.

$$(-1)^{|I|-p(I,i)} v_{d(I,i)} v_i = v_I = (-1)^{1+p(I,i)} v_i v_{d(I,i)} \qquad \text{if } i \in I. \tag{2}$$

Here by $d(I,i)$ we mean the ordered set I with the element i (if it appears) deleted. This set is considered to inherit its order from I.

Let $\langle \partial_1, \dots, \partial_n \rangle$ be the dual basis to $\langle v_1, \dots, v_n \rangle$, i.e. $\partial_i(v_j) = \delta_{ij}$. For any choice of orderings of the I, the set of all $v_I \otimes \partial_i$ is a basis of $\Lambda \otimes V^*$. For our later computations we shall always use use the following choice. If $i \notin I$ then we order I naturally as a subset of N. However if $i \in I$ we order I naturally, except that we insist that i be the last element of I. Thus if I' is the complement $I \setminus \{i\}$ we have $v_I \otimes \partial_i = v_{I'} v_i \otimes \partial_i$, where I' is in natural (increasing) order. Note that the ordering of I depends on i here, so that in basis elements $v_I \otimes \partial_i$ and $v_I \otimes \partial_j$ the set I may be ordered differently.

Given an ordered set I and an integer i, let $p(I,i)$ denote the position of i in I if it occurs and zero otherwise. Explicitly,

$$p(I,i) = \begin{cases} s, & \text{if } I = \langle i_1, \dots, i_r \rangle \text{ and } i = i_s \\ 0, & \text{if } i \notin I. \end{cases}$$

The degree of a basis element $v_I \otimes \partial_i$ is $|I| - 1$, and such an element is called odd or even according as its degree is either odd or even. Note that the maximum degree occurring is $n-1$ and the minimum is -1. It follows from all our definitions and identifications that the multiplication formula for *odd* elements becomes

(3)

$$[v_I \otimes \partial_i, v_J \otimes \partial_j] = (-1)^{1+p(J,i)} \chi_J(i) v_I v_{d(J,i)} \otimes \partial_j + (-1)^{1+p(I,j)} \chi_I(j) v_J v_{d(I,j)} \otimes \partial_i.$$

Here χ_J denotes the characteristic function of the set J. Note that it is immediate from (3) and anticommutativity that the product is zero if $|I \cap J| \geq 2$.

From now on we shall not distinguish between $W(n)$ and $\Lambda \otimes V^*$ and we shall use the description above of the latter for all computations.

2.3. Product matrix. Let $L = L_0 \dot{+} L_1$ be a finite-dimensional Lie superalgebra and let $\{x_1, x_2, \dots, x_N\}$ and $Y = \{y_1, \dots, y_M\}$ be ordered bases for, respectively, L_1 and L_0. The subspaces L_0 and L_1 are called respectively the even and odd parts of L. The product matrix represents the bilinear pairing $[\ ,\ ]$, so that with respect to these bases the i,j entry of the product matrix is the product $[x_i, x_j]$. The matrix is considered to be defined over the commutative polynomial algebra $K[Y]$ (in fact its entries are linear combinations of the variables $y_1, \dots, y_M$).

For $L = W(n)$ we use the basis defined above. Thus the rows and columns are indexed by the pairs (I,i) corresponding to the basis elements $v_I \otimes \partial_i$.

As an example, let $L = W(3)$. Here the basis elements for the even part are $y_{ij} = v_{\langle i \rangle} \otimes \partial_j$, $z_1 = v_{\langle 2,3,1 \rangle} \otimes \partial_1$, $z_2 = v_{\langle 1,3,2 \rangle} \otimes \partial_2$ and $z_3 = v_{\langle 1,2,3 \rangle} \otimes \partial_3$.

The basis elements for the odd part are $x_i = \partial_i$ and $x_{ijk} = v_{\langle i,j \rangle} \otimes \partial_k$ and these are ordered as follows:

$$x_1 < x_2 < x_3 < x_{211} < x_{122} < x_{123} < x_{311} < x_{132} < x_{133} < x_{231} < x_{322} < x_{233}.$$

Thus the product matrix, which we denote by $\mathbf{W(3)}$, is

$$\begin{bmatrix}
0 & 0 & 0 & -y_{21} & y_{22} & y_{23} & -y_{31} & y_{32} & y_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & y_{11} & -y_{12} & -y_{13} & 0 & 0 & 0 & y_{31} & -y_{32} & y_{33} \\
0 & 0 & 0 & 0 & 0 & 0 & y_{11} & -y_{12} & -y_{13} & -y_{21} & y_{22} & -y_{23} \\
-y_{21} & y_{11} & 0 & 0 & 0 & 0 & 0 & z_2 & -z_3 & 0 & -z_1 & 0 \\
y_{22} & -y_{12} & 0 & 0 & 0 & 0 & -z_2 & 0 & 0 & z_1 & 0 & z_3 \\
y_{23} & -y_{13} & 0 & 0 & 0 & 0 & z_3 & 0 & 0 & 0 & z_3 & 0 \\
-y_{31} & 0 & y_{11} & 0 & -z_2 & z_3 & 0 & 0 & 0 & 0 & 0 & z_1 \\
y_{32} & 0 & -y_{12} & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_2 \\
y_{33} & 0 & -y_{13} & -z_3 & 0 & 0 & 0 & 0 & 0 & z_1 & z_2 & 0 \\
0 & y_{31} & -y_{21} & 0 & z_1 & 0 & 0 & 0 & z_1 & 0 & 0 & 0 \\
0 & -y_{32} & y_{22} & -z_1 & 0 & z_3 & 0 & 0 & z_2 & 0 & 0 & 0 \\
0 & y_{33} & -y_{23} & 0 & z_3 & 0 & z_1 & z_2 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

One can compute (using a computer algebra system such as Maple) that this matrix is in fact nonsingular, i.e. $W(3)$ satisfies Bell's criterion. The first author showed in [Wil96] that the even Witt algebras $W(2n)$ satisfy Bell's criterion. Proving such a result relies on finding a generally applicable specialization. However, though one can find many specializations which work for $W(3)$, it is unclear how to generalize any of them even from $n = 3$ to $n = 5$.

3. Experimental data

3.1. Probabilistic methods. The first author has written Maple code, used for all computations in this subsection, which generates the product matrices for all Cartan type simple Lie superalgebras. The code is available via WWW from the URL `http://www.math.auckland.ac.nz/~wilson/bellcrit.html`.

The rather straightforward methods used in previous papers yield nothing, so we resort to experiment. Maple shows easily that the (12×12) product matrix of $W(3)$ is nonsingular. We turn our attention to the product matrix $\mathbf{W(5)}$ of $W(5)$. Experimentally, we must first decide if we think $\mathbf{W(5)}$ is likely to be singular or not; then hunt for a possible proof. A computer algebra program such as Macsyma or Maple might attempt to determine the singularity of $\mathbf{W(5)}$ by direct elementary methods. However $\mathbf{W(5)}$ is too large (an 80 by 80 matrix, whose entries include 80 different variables) for this to be successful. One way to simplify the computation is by specialization; give each variable an (integer) value, and study the resulting numerical matrix. It is fairly clear that the rank of the specialized matrix cannot exceed that of $\mathbf{W(5)}$ itself. (The rank of a matrix is the size of its largest non–singular square submatrix; a submatrix of $\mathbf{W(5)}$ is singular iff its determinant (as a polynomial in our 80 variables) is the 0 polynomial.) In particular, if we find a non–singular specialized matrix, we may conclude that $\mathbf{W(5)}$ is non–singular. It is unclear how to choose values for the variables so that the rank of the specialized matrix will be large (ideally, equal to the rank of $\mathbf{W(5)}$). Most "regular–looking" choices have too much symmetry to give a large rank.

In the absence of any cleverer ideas, a reasonable thing to do is to choose values at random in some way. This gives not just one specialization, but many – a different one each time we try it.

Early on, then, the authors attempted to calculate the ranks of randomly specialized versions of $\mathbf{W(5)}$. The variables were given independent random values sampled from a probability distribution μ; distributions μ we used included:

(i) The values 0 and 1, each taken with probability 1/2. This has the advantage of simplifying computation.
(ii) The values -1,0 and 1, each taken with probability 1/3.
(iii) The values $-80, \ldots, 80$, taken with equal probability.

100 specializations were performed using each method and the rank of the resulting matrices computed. The results are shown in the table below.

Method	Rank < 75	Rank $=75$
(i)	14	86
(ii)	2	98
(iii)	0	100

In no case did the rank of a specialization exceed 75. We are thus provided with no firm conclusion; if we are to take anything from this exercise, it is a belief that $\mathbf{W(5)}$ may well be singular. However, it is not clear *a priori* how much faith one should place in these results. For a sufficiently generic matrix they would appear compelling, but the structure of the matrix in question may have a large effect on the data. It is conceivable that specializations exist which give the matrix full rank, but that they are generated only with low (or zero) probability by our random methods. In (i), for example, each specialization will give the value 0 to about half the variables and the value 1 to the rest. Might not achieving full rank require the "1s" to be in a strong majority ?

Fortunately there is an argument which can lay most of our fears to rest. We are really attempting to determine whether the determinant of $\mathbf{W(5)}$, a polynomial in our 80 variables, is the 0 polynomial. We can make use of the following known result:

Proposition. *Let Q be a polynomial in n variables; suppose that Q is not* *identically* 0. *Let I be a finite subset of the coefficient field of Q, with $|I| \geq c\, deg(Q)$. Then the number of elements of I^n which are zeros of Q is at most $c^{-1}|I|^n$.*

Proof. See [Sch80]. □

For us $\deg(Q) \leq 80$, and if we take $I = \{-80, \ldots, 80\}$ (c.f. (iii)), then the inequality in this result is satisfied with $c = 2$. So if $\mathbf{W}(\mathbf{5})$ is non–singular, each random specialization of the sort in (iii) has probability at least 1/2 of detecting this fact; that we failed to detect it in 100 tries means that we have witnessed a very rare event (one with probability smaller than 2^{-100} ($\approx 10^{-30}$)). It thus appears that $\mathbf{W}(\mathbf{5})$ is very probably singular.

Similar support can be given for the assertion that the rank of $\mathbf{W}(\mathbf{5})$ is exactly 75; we omit the details here. While this kind of probabilistic argument does not constitute proof in the traditional sense, it is quite sound enough for further experimental investigations to be based on its conclusion. For more on arguments of this type, see [CS78].

3.2. The nullspace. Additional exact rank computations were made to supplement the lower bounds found in the previous section. We used MACSYMA for all the computations discussed in this section. Proving the singularity of a 80×80 matrix with 80 variables is a daunting task. Even the fact that half the matrix entries are zero may not help very much. Examples of expanded determinants like ours can have 2^{79} terms.

There is one special situation that could be efficiently exploited, however. In all other nontrivial cases where Bell's criterion does not hold, this is caused purely by the zero-pattern of the product matrix — its expanded determinant has no nonzero terms. Now this fact can be demonstrated by a $O(n^{5/2})$ algorithm [HK73] applied to a 0-1 matrix with the same zero pattern as the matrix of interest. Hoping to exploit this fact, we formed a general 80×80 matrix having the same zero pattern as our candidate. When the variables in this matrix were randomly specialized, the calculated determinants were not zero. Thus, there was no hope that the zero pattern alone could make our candidate singular. Hence if indeed $\det \mathbf{W}(\mathbf{5}) = 0$, this is caused by some interesting cancellation in the expanded determinant.

One must avoid having too many variables in a symbolic computation. Intermediate computations involving many variables may very well exhaust computer memory even if the final answer would be quite compact. To avoid this situation, we randomly specialized the variables and performed all arithmetic over the ring $\mathbb{Z}_{9973}$. The prime 9973 was chosen for the convenience of having displayed integers having at most four digits.

When we asked for not merely the rank of our specialized matrices, but for their nullvectors, we were fortunate to find the 80-tuples representing the nullvectors all began with at least 55 zeros. We therefore undertook to prove, if we could, that the last 25 columns of the unspecialized matrix has rank of only 20, implying a rank deficiency of at least 5 for the entire matrix.

In the partitioning of $\mathbf{W(5)}$ introduced in the next section, the last 25 columns consist of the block $\mathbf{W_{-1,3}}$ with 5 rows involving 50 variables, the block $\mathbf{W_{1,3}}$ with 50 rows involving only 5 variables, and additional rows of zeros, which we disregarded.

Naturally, the first block, $\mathbf{W_{-1,3}}$, was avoided as long as possible because it involves 50 variables. We wanted to show the remaining nonzero rows, which form $\mathbf{W_{1,3}}$, were of rank 15, because it would then follow that the rank of all of the rows in the last 25 columns could not exceed 20.

Concentrating, then, on $W_{1,3}$ which has only 5 variables, we further reduced the task to finding a (right) nullvector using only some of its rows because the random specialization indicated these sufficed to obtain rank 15. The resulting nullvector was then demonstrated to nullify all of $\mathbf{W_{1,3}}$. Since the nullvector was found to depend on 10 free parameters, we had proven the rank of $\mathbf{W_{1,3}}$ to be 15, as we had expected.

Summarizing, we showed that the rank of all of the rows in the last 25 columns could not exceed 20. Hence, neither could the column rank exceed 20. As a result the entire matrix can not have rank exceeding 75. But in the previous section, we saw that the rank was at least 75.

With hindsight, we see that we erred on the side of caution. In less than 4 seconds of computing time on our workstation, MACSYMA finds the rank of $\mathbf{W_{1,3}}$ to be 15. In addition, one can find an explicit row dependence, but its form, with 55 original variables and 5 free parameters, makes it difficult to interpret and generalize.

At this stage we have proved that $\mathbf{W(5)}$ does not satisfy Bell's criterion. It remains to see whether the argument above will generalize to $\mathbf{W(n)}$, for odd $n > 5$. To do this we have to exhibit the row dependencies explicitly. This is carried out in the next section.

4. Proofs

In the light of the above it is easy to conjecture that the product matrices for odd $n \geq 5$ are singular. This is proved below, by finding an upper bound for the rank of the submatrix $\mathbf{W_{.,n-2}}$ as suggested by our experimental work.

A rather detailed analysis of the structure of the product matrix is required, and the particular basis we use plays a crucial role. Of course, this basis was not the one first used, but was discovered in the course of the analysis. The fact that we use the same basis elements for the rows and columns means that the product matrix is symmetric.

4.1. Detailed structure of the product matrix. From now on assume that $n \geq 3$ is **odd**. Then the highest odd degree occurring in W is $n-2$ and the highest even one $n-1$. Grouping the basis elements by increasing degree we obtain a block structure to the product matrix. We let $\mathbf{W_{r,s}}$ denote the product submatrix formed by all products of $\mathbf{W_r}$ with $\mathbf{W_s}$, let $\mathbf{W_{r,.}}$ denote the horizontal concatenation of all $\mathbf{W_{r,s}}$, and let $\mathbf{W_{.,s}}$ denote the vertical concatenation of all $\mathbf{W_{r,s}}$. Then the product matrix has the structure

$$\mathbf{W(n)} = \begin{pmatrix} 0 & \mathbf{W_{-1,1}} & \mathbf{W_{-1,3}} & \dots & \mathbf{W_{-1,n-4}} & \mathbf{W_{-1,n-2}} \\ \mathbf{W_{1,-1}} & \mathbf{W_{1,1}} & \dots & \dots & \mathbf{W_{1,n-4}} & \mathbf{W_{1,n-2}} \\ \mathbf{W_{3,-1}} & \dots & \dots & \dots & \mathbf{W_{3,n-4}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{W_{n-2,-1}} & \mathbf{W_{n-2,1}} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We shall bound above the rank of each block $\mathbf{W_{r,n-1-r}}$.

Fix an odd r with $1 \leq r \leq n-2$. The component W_{n-1} has basis consisting of all $z_k = v_N \otimes \partial_k$ with $k \in N$, so every nonzero entry in $\mathbf{W_{r,n-1-r}}$ is a linear combination of the z_k.

Let $I, J \subseteq N$ with $|I| = r+1, |J| = n-r$. We now obtain conditions on (I,i) and (J,j) in order that the entry of $\mathbf{W_{r,n-1-r}}$ in the row indexed by (I,i) and the column indexed by (J,j) be nonzero. This entry is of course equal to $[v_I \otimes \partial_i, v_J \otimes \partial_j]$. We say that (I,i) and (J,j) are *linked* in this situation. We shall not pursue the obvious graph-theoretical interpretation of this term.

It follows from the multiplication formula (3) that a necessary condition for linking is that $I \cap J = \{i\}$ or $I \cap J = \{j\}$. These two possibilities are in fact mutually exclusive, since

(4)
$$[v_I \otimes \partial_i, v_J \otimes \partial_i] = 0 \qquad \text{if } |I| \text{ and } |J| \text{ are even and } I \cap J = \{i\}.$$

To see this, we compute:

$$\begin{aligned} [v_I \otimes \partial_i, v_J \otimes \partial_i] &= -v_I v_{d(J,i)} \otimes \partial_i - v_J v_{d(I,i)} \otimes \partial_i \\ &= \Big(-v_{d(I,i)} v_i v_{d(J,i)} - v_{d(J,i)} v_i v_{d(I,i)}\Big) \otimes \partial_i \\ &= \Big(v_{d(I,i)} v_{d(J,i)} v_i - v_{d(I,i)} v_{d(J,i)} v_i\Big) \otimes \partial_i \\ &= 0. \end{aligned}$$

In summary, (I,i) and (J,j) are linked if and only if $i \neq j$ and $I \cap J = \{i\}$ or $I \cap J = \{j\}$. The corresponding entry in $\mathbf{W(n)}$ equals $\pm z_k$ for some $k \in N$, and is given exactly by

$$[v_I \otimes \partial_i, v_J \otimes \partial_j] = \begin{cases} v_{I \setminus \{i\}} v_J \otimes \partial_j & \text{if } I \cap J = \{i\}, \\ v_{J \setminus \{j\}} v_I \otimes \partial_i & \text{if } I \cap J = \{j\}. \end{cases} \tag{5}$$

The cases where $i \in I$ and $i \notin I$ behave rather differently and we examine each separately in more detail.

1)$\mathbf{i} \notin \mathbf{I}$. Here we must have $I \cap J = \{j\}$. For each $j \in I$ there is exactly one such J and in fact we have $v_J \otimes \partial_j = v_{N \setminus I} v_j \otimes \partial_j$ by our basis convention. Thus the corresponding entry in the product matrix is

$$v_{N \setminus I} v_I \otimes \partial_i.$$

Note that this is independent of J and j and so a row indexed by such a pair (I,i) has precisely $|I|$ nonzero entries all of which are the same. Furthermore for a fixed I the nonzero entries occur in the same columns for all i.

2)$\mathbf{i} \in \mathbf{I}$. There are 3 subcases.

(i)$\mathbf{I} \cap \mathbf{J} = \{\mathbf{j}\}$. We have $v_J \otimes \partial_j = v_{N\setminus I} v_j \otimes \partial_j$ and $v_I \otimes \partial_i = v_{d(I,i)} v_i \otimes \partial_i$ and so the entry in the product matrix is

$$v_{N\setminus I} v_{d(I,i)} v_i \otimes \partial_i.$$

(ii)$\mathbf{I} \cap \mathbf{J} = \{\mathbf{i}\}, \mathbf{j} \in \mathbf{J}$. Here $v_I \otimes \partial_i = v_{N\setminus J} v_i \otimes \partial_i$ and the corresponding entry is

$$v_{N\setminus J} v_{d(J,j)} v_j \otimes \partial_j.$$

(iii)$\mathbf{I} \cap \mathbf{J} = \{\mathbf{i}\}, \mathbf{j} \notin \mathbf{J}$. Here the corresponding entry is

$$v_{N\setminus J} v_J \otimes \partial_j.$$

4.2. Estimating ranks. After the preliminaries of the previous subsection we can now prove a key lemma.

Lemma 4.1. *The rank of* $\mathbf{W_{r,n-1-r}}$ *is at most* $\binom{n+1}{r+1}$.

Proof. Fix $A \subseteq N$ with $|A| = r$. For each $k \in B = N \setminus A$, consider the submatrix S_k of $\mathbf{W_{r,n-1-r}}$ formed by all rows indexed by pairs $(A \cup \{k\}, i)$ as i ranges over B. By the analysis above, the columns which correspond to the nonzero entries in S_k are indexed by pairs of the 4 types $(B,j), j \in A$; (B,k); $(B,j), j \in B \setminus \{k\}$; $(B \setminus \{k\} \cup \{j\}, j), j \in A$.

Let F be the function field $= K(z_1, \ldots, z_n)$. The rows where $i \neq k$ span a 1-dimensional F-subspace since we are in case 1 above. Thus using suitable row operations over F we may assume that such rows contain only ones and zeroes. Furthermore the ones occur precisely in the columns of the 2nd and 4th types above.

We now compute the remaining entries of S_k, namely those in the row with $i = k$. For the columns of the first type we are in case 2(iii) above and the entry is $v_A v_B \otimes \partial_j$. This is equal to $\epsilon(A) z_j$ where $\epsilon(A) = \pm 1$. For the column of the second type the entry is of course zero by equation (4).

For the columns of the third type we are in case 2(ii) and the entry is $v_A v_{d(B,j)} v_j \otimes \partial_j$. This can be rewritten using equation (2) as $(-1)^{|B|-p(B,j)} v_A v_B \otimes \partial_j$ and this is equal to $(-1)^{p(B,j)} v_A v_B \otimes \partial_j$ since $|B| = n-1-r$ is even. We can write this as $\varepsilon(A,j) z_j$ where $\varepsilon(A,j) = \pm 1$.

Finally for columns of the fourth type we are in case 2(i). The corresponding entry is $v_{d(B,k)} v_A v_k \otimes \partial_k$. This simplifies to $v_A v_k v_{d(B,k)} \otimes \partial_k$ by anticommutativity and then to $(-1)^{1+p(B,k)} v_A v_B \otimes \partial_k$ by equation (2). In terms of the notation of the previous case this is equal to $-\epsilon(A,k) z_k$.

Thus S_k may be represented by the following table.

	$(B,j), j \in A$	(B,k)	$(B,j), j \in B \setminus \{k\}$	$((B \setminus \{k\}) \cup \{j\}, j), j \in A$
$i = k$:	$\epsilon(A) z_j$	0	$\epsilon(A,j) z_j$	$-\epsilon(A,k) z_k$
$i \neq k$:	0	1	0	1

The first row of the table represents one row of S_k whereas the second row represents $n-r-1$ rows. Similarly, each column of the table may represent many columns of S_k.

By adding $\epsilon(A,k)z_k$ times any of the rows with $i \neq k$ to the row with $i = k$ we convert S_k to a matrix which may be represented by the following table.

	$(B,j), j \in A$	$(B,j), j \in B$	$((B \setminus \{k\}) \cup \{j\}, j), j \in A$
$i = k$:	$\epsilon(A) z_j$	$\epsilon(A,j) z_j$	0
$i \neq k$:	0	δ_{kj}	1

In particular, note that if we keep A fixed and perform the above procedure for each $k \in B$ in turn, all the rows with $i = k$ are now identical, so form a rank 1 submatrix.

Now allow A to vary. Each row of $\mathbf{W_{r,n-1-r}}$ which is indexed by some (I,i) with $i \in I$ appears precisely once in the above construction. Thus the total contribution to the rank of $\mathbf{W_{r,n-1-r}}$ by such rows is at most equal to the number of A, namely $\binom{n}{r}$. As noted above, for a given I then the rows indexed by (I,i) with $i \notin I$ are the same for all i. Thus the total contribution to the rank by rows with $i \notin I$ is at most equal to the number of I, namely $\binom{n}{r+1}$. Hence $\mathbf{W_{r,n-1-r}}$ has rank at most $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$. □

We illustrate the above proof in our example $n = 3$. Take $A = \{1\}$. Then the submatrix S_2 when represented as above yields

	x_{231}	x_{322}	x_{233}	x_{311}
x_{122}	z_1	0	z_3	$-z_2$
x_{123}	0	z_3	0	z_3

while S_3 is represented as

	x_{231}	x_{233}	x_{322}	x_{211}
x_{132}	0	z_2	0	z_2
x_{133}	z_1	0	z_2	$-z_3$

.

The main result follows directly:

Theorem 4.2. *If n is odd, $W(n)$ satisfies Bell's criterion only for $n = 3$.*

Proof. The case $n = 1$ is trivial and the associated 1×1 product matrix is zero. Now assume that $n \geq 3$. The submatrix $\mathbf{W_{\cdot,n-2}}$ (the rightmost "column" of the product matrix) consists of 2 nonzero blocks and has dimensions $(n2^{n-1}) \times n^2$. Since the rank of $\mathbf{W_{-1,n-2}}$ is at most n, it follows from Lemma 4.1 that the rank of $\mathbf{W_{\cdot,n-2}}$ is at most $n + \binom{n+1}{2} = n(n+3)/2$. Thus the rank of $\mathbf{W(n)}$ is at most $n2^{n-1} - n^2 + n(n+3)/2 = n2^{n-1} - n(n-3)/2$.

For $n \geq 5$ this is strictly less than $n2^{n-1}$. We know the criterion holds for $n = 3$. □

Note that for $n = 5$ the bound in the proof yields the correct answer 75. For $n = 3$ the bound also gives the right answer 12. One can show using Lemma 4.1 that the bound is not sharp for $n \geq 7$. We do not have a conjecture for the exact value of the rank when $n \geq 7$.

5. Comments and future work

The converse of Bell's criterion is not yet known to be either true or false, though false seems (intuitively) most likely. In light of this, it would be of interest to know whether $U(W(n))$ is prime for odd $n \geq 5$. We have made no progress on this question.

The latest details on the verification of Bell's criterion can be accessed via WWW at `http://www.math.auckland.ac.nz/~wilson/Research/bellcrit.html`.

Added in proof. The first two authors have shown that if n is even, $S(n)$ and $\widetilde{S}(n)$ satisfy Bell's criterion — the details will appear elsewhere. Thus all the Cartan type Lie superalgebras have now been accounted for.

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