ASSOCIATIVE ALGEBRAS SATISFYING A SEMIGROUP IDENTITY

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Abstract. Denote by $(R, \cdot)$ the multiplicative semigroup of an associative algebra R over an infinite field, and let $(R, \circ)$ represent R when viewed as a semigroup via the circle operation $x \circ y = x + y + xy$. In this paper we characterize the existence of an identity in these semigroups in terms of the Lie structure of R. Namely, we prove that the following conditions on R are equivalent: the semigroup $(R, \circ)$ satisfies an identity; the semigroup $(R, \cdot)$ satisfies a reduced identity; and, the associated Lie algebra of R satisfies the Engel condition. When R is finitely generated these conditions are each equivalent to R being upper Lie nilpotent.

1. Introduction and statement of results

A well-known result due to Levitzki ([Lev46]) states that every finitely generated bounded nil ring is nilpotent. Not long ago, Zel'manov proved the Lie-theoretic analogue: every finitely generated Lie ring satisfying the Engel condition is nilpotent ([Zel90]). The corresponding problem in the category of groups is the famous Burnside problem. The construction by Adian and Novikov of infinite finitely generated groups of finite exponent provided a negative solution to this problem (see [Adi79]).

The Burnside problem has some natural generalizations. For example, the problem of whether or not every Engel group is locally nilpotent remains open ([Sha94]). Because every nilpotent group is known to satisfy a semigroup identity ([Mal53, NT63]), a weaker version of this problem has also been posed: does every Engel group satisfy a semigroup identity ([MK95, Problem 2.82])? Even the following question ([Rhe]) remains open: can an Engel group contain a free (noncommutative) subsemigroup?

Recently, the present authors settled the ring-theoretic analogues of these problems.

Recall that R satisfies the *Engel identity* of degree n if and only if

$$e_n := [x, \underbrace{y, y, \dots, y}_{n}]$$

is identically zero in R; whereas, R is said to be *upper Lie nilpotent* if the descending central series of associative ideals $\{\gamma^i(R)\}$ in R defined by $\gamma^1(R) = R, \gamma^{i+1}(R) = \langle [\gamma^i(R), R] \rangle$ reaches zero in finitely many steps. In addition to the usual multiplicative semigroup, $(R, \cdot)$, R forms a semigroup, denoted by $(R, \circ)$, under the circle operation $x \circ y = x + y + xy$. We proved in [RW] that every finitely generated associative ring R satisfying the Engel condition is upper Lie nilpotent. From this result we were able to infer that whenever R satisfies an Engel identity then both the associated circle and multiplicative semigroups of R must satisfy a so-called Morse identity.

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Define sequences f_n and g_n by $f_1(x,y) = xy, g_1(x,y) = yx$, and

$$f_{n+1}(x,y,x_3,\dots,x_{n+2}) = f_n(x,y,x_3,\dots,x_{n+1})x_{n+2}g_n(x,y,x_3,\dots,x_{n+1}),$$

$$g_{n+1}(x,y,x_3,\dots,x_{n+2}) = g_n(x,y,x_3,\dots,x_{n+1})x_{n+2}f_n(x,y,x_3,\dots,x_{n+1}),$$

for all $n \geq 1$. The nth *Malcev identity* ([Mal53]) is the semigroup identity

$$f_n(x,y,x_3,\dots,x_{n+1}) = g_n(x,y,x_3,\dots,x_{n+1}),$$

while the nth *Morse identity* $u_n(x,y) = v_n(x,y)$ ([Mor21]) is the nth Malcev identity with $x_3 = \cdots = x_{n+1} = 1$.

Consequently, neither $(R,\cdot)$ nor $(R,\circ)$ can contain a free subsemigroup if R satisfies an Engel identity.

The problem of characterizing finitely generated groups satisfying an arbitrary semigroup identity has been studied by several authors (see, for example, [LL69], [SS93] and [Sha93]). Because this class of groups contains the Burnside groups, this problem is highly nontrivial — especially in light of the recent construction by Olshanskii and Storozhev of a 2-generated group satisfying a semigroup identity that is not a periodic extension of a locally soluble group ([OS96]).

In this article we study associative algebras that satisfy an arbitrary semigroup identity. In fact, we obtain a partial converse to our former result.

Throughout the remainder of this paper, K will denote an infinite commutative domain and R an associative K-algebra on which the action of K is torsion-free (this occurs, for example, when K is an infinite field). All identical relations in algebraic objects will be assumed to be nontrivial unless otherwise stated. A semigroup S satisfies an identity if and only if there are distinct words u, v in the free semigroup on

$$X = \{x = x_1, y = x_2, x_3, x_4, \dots\}$$

so that $u = v$ in S. The semigroup identity is *left reduced* if the first letters of u and v are different, *right reduced* if the last letters of u and v are different and simply *reduced* if it is both left and right reduced. In other words, $u = v$ is reduced if and only if uv^{-1} and $v^{-1}u$ are reduced words in the free group on X. If $(R,\cdot)$ (respectively $(R,\circ)$) satisfies an identity we often say that R satisfies a *semigroup identity* (respectively, a *circle semigroup identity*). Clearly each of these corresponds to a polynomial identity in R. A generalization of a multiplicative semigroup identity in R is a *binomial identity*, a polynomial identity of the form $\alpha_1 u_1 + \alpha_2 u_2 = 0$, where u_1, u_2 are monomials and $\alpha_1, \alpha_2 \in K$. The various types of reduced binomial identities are defined in the obvious way.

V. Tasić and the first author proved in [RT] that R is Lie nilpotent of class at most n if and only if $(R,\circ)$ satisfies the nth Malcev identity. The main result in the present article further demonstrates the close relationship between the Lie structure of R and semigroup properties of R:

Theorem 1.1. *Let R be a K-algebra. Then the following statements are equivalent.*

(i) *R satisfies a circle semigroup identity;*
(ii) *R satisfies a reduced semigroup identity;*
(iii) *R satisfies a reduced binomial identity;*
(iv) *R satisfies an identity of the form $\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0$, $\alpha_i \in K, \alpha_0 \neq 0, \alpha_n \neq 0$;*
(v) *R satisfies an Engel identity; and,*
(vi) *$(R,\circ)$ satisfies a Morse identity.*

Furthermore, for any two conditions A, B from (i)–(vi), our proof gives (sometimes theoretical) bounds for the degree of the identity in B in terms of the degree of the identity in A. In particular, these bounds do not depend on R, K or the characteristic of K. Notice, too, that since every finite semigroup (in particular $(R, \circ)$ where R is a finite ring) satisfies an identity, some hypothesis on the coefficient ring K is required. The following example demonstrates that the distinction between reduced and arbitrary multiplicative semigroup identities is also necessary.

Example 1.2. Let R be the subalgebra of the matrix algebra $M_2(K)$ spanned by the matrix units e_{11} and e_{12}. Then $[R,R] \subseteq Ke_{12}$, so R satisfies the semigroup identity $[x,y]z = xyz - yxz = 0$. R does not satisfy any Engel identity, since $[e_{11}, e_{12}] = e_{12}$. Thus, by Theorem 1.1, R does not satisfy any reduced semigroup identity, nor any circle semigroup identity.

Theorem 1.3. *Let R be a K-algebra where* $\operatorname{char} K = p > 0$*. Then the following statements are equivalent.*

(i) *R satisfies a semigroup identity;*
(ii) *R satisfies a binomial identity;*
(iii) *R satisfies an identity of the form $\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0$, $\alpha_i \in K$; and,*
(iv) *R satisfies an identity of the form $y^m e_m y^m = 0$.*

We remark that the characteristic zero analogue of Theorem 1.3 is stated in [GM82]; however, their result corresponding to our implication (iv) ⇒ (i) is not proved and does not seem obvious to the present authors.

The fact that R is non-unital is essential to Example 1.2, as indicated by the following proposition.

Proposition 1.4. *Let R be a unital K-algebra. If R satisfies a semigroup identity then R satisfies the corresponding reduced semigroup identity.*

Theorem 1.5. *There exists a function f, depending only on natural numbers d and n, such that if a K-algebra R satisfies a circle semigroup identity of degree n and R is generated over K by d elements then R is upper Lie nilpotent of index at most $f(d,n)$.*

2. Semigroup identities

Our hypotheses on K were chosen to imply, by the usual Vandermonde determinant argument, that every homogeneous component of a polynomial identity for R is also a polynomial identity for R (see [Row88, 6.4.14]). We shall use this key fact freely, without explicit mention.

By a *partial linear identity* we shall mean an identity of the form

$$\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0,$$

with $\alpha_i \in K$. Such an identity will be called *left reduced* if $\alpha_0 \neq 0$, *right reduced* if $\alpha_n \neq 0$ and *reduced* if it is both left and right reduced.

Proposition 2.1. *Let R be a K-algebra.*

(i) *If a semigroup S satisfies an identity in $x, y, x_3, \ldots$ which is left reduced, right reduced or reduced, then S satisfies an identity, of the same type, in x and y only.*

(ii) *If R satisfies a binomial identity then R is bounded nil or R satisfies a semigroup identity.*
(iii) *If R satisfies a binomial identity which is left reduced, right reduced or reduced then R satisfies a partial linear identity of the same type.*
(iv) *If R satisfies the identity $y^n = 0$ then R satisfies $e_{2n-1} = 0$.*

Proof. Suppose without loss of generality that our left reduced identity has the form

$$xx_{i_1} \cdots x_{i_m} = yx_{j_1} \cdots x_{j_n}.$$

Recall that we identify $x = x_1$ and $y = x_2$. Substituting $x_i = xy^i, i \geq 3$, we obtain a left reduced identity in x and y only. If the original identity were right reduced as well then $x_{i_m} \neq x_{j_n}$. Thus, by an appropriate permutation of the variables, we obtain an equivalent identity of the form

$$x_{k_1} \cdots x_{k_m} x = x_{l_1} \cdots x_{l_n} y.$$

Substituting $x_i = xy^i, i \geq 3$ into this identity and then concatenating on the right with the 2-variable left reduced identity yields the 2-variable reduced identity:

$$xx_{i_1} \cdots x_{i_m} x_{k_1} \cdots x_{k_m} x = yx_{j_1} \cdots x_{j_n} x_{l_1} \cdots x_{l_n} y.$$

This, and symmetry, proves (i).

Next, given a binomial identity $\alpha_1 u_1 + \alpha_2 u_2 = 0$ holding in R, set all variables equal, to y say. If the identity is not homogeneous then separating components shows that R is bounded nil. On the other hand, if it is homogeneous then $(\alpha_1 + \alpha_2)y^n = 0$ for some n, so that either R is bounded nil or $\alpha_1 = -\alpha_2$, in which case $u_1 - u_2 = 0$ holds in R. This proves (ii).

In order to prove (iii), suppose that R satisfies a given binomial identity and observe from (i) and (ii) that either R is bounded nil, in which case R satisfies a partial linear identity by (iv) below, or R satisfies a semigroup identity of the form $u(x, y) - v(x, y) = 0$. Thus we may assume that R is not bounded nil, and hence that the semigroup identity is homogeneous. We assert that the homogeneous component of degree 1 in x of the identity $u(x + y, y) - v(x + y, y) = 0$ is a partial linear identity. To see why it is nontrivial, write $u = au', v = av'$, where a has length m and $u' = v'$ is a left reduced equation. If (as we may assume without loss of generality) u' starts with x and v' with y then in the expansion of $u(x + y, y)$ there is precisely one monomial starting with $y^m x$, whereas no monomial in the expansion of $v(x + y, y)$ begins with $y^m x$. This, and symmetry, yields (iii).

To prove the well-known fact (iv), let l, r denote respectively the K-linear operators of left and right multiplication by y. Then, since l and r commute,

$$e_m = (r - l)^m(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} l^i r^{m-i}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} y^i x y^{m-i}. \tag{1}$$

Thus if $m = 2n - 1$ and R satisfies $y^n = 0$ then every term in the sum on the right is zero.

□

Proposition 1.4 is a consequence of the following result.

Proposition 2.2. *Let R be a K-algebra.*

(i) *If $(R, \circ)$ satisfies a semigroup identity then $(R, \cdot)$ satisfies the same identity.*

(ii) *If $(R, \circ)$ satisfies a semigroup identity then $(R, \circ)$ satisfies the corresponding reduced identity.*

(iii) *If R is unital then $(R, \circ) \cong (R, \cdot)$.*

Proof. Let S be the unital hull of R, that is, $S = R$ if R is unital and $S = K1 \oplus R$ if R is nonunital. The map $\iota \colon r \mapsto 1 + r$ is an injective semigroup map from $(R, \circ)$ into $(S, \cdot)$ which is onto if (and only if) $R = S$. This proves (iii). The image under ι of an identity in $(R, \circ)$ is an identity in $(1+R, \cdot) \subseteq (S, \cdot)$. Only the bottom degree homogeneous component of this identity involves 1 and the other homogeneous components yield identities in $(R, \cdot)$. The highest degree component is precisely the original identity, yielding (i).

Assume that $u(x, y) = v(x, y)$ is an identity for $(R, \circ)$ of degree n. Write $u = au'b, v = av'b$ where $u' = v'$ is a reduced equation. We show that $u' = v'$ also holds in $(R, \circ)$. It suffices, by symmetry and by induction on the maximum length of a and b, to prove this in the case when $a = x$ and b is empty. The identity $xu'(x, y) = xv'(x, y)$ in $(R, \circ)$ is equivalent to the polynomial identity

$$(1 + x)u'(1 + x, 1 + y) - (1 + x)v'(1 + x, 1 + y) = 0$$

in R. Let m be an odd integer with $m \geq n + 1$. Then multiplying the last identity on the left by $1 - x + \cdots + (-1)^m x^m$ yields the polynomial identity

$$(1 + x^{m+1})u'(1 + x, 1 + y) - (1 + x^{m+1})v'(1 + x, 1 + y) = 0.$$

Separating homogeneous components and using the fact that x^{m+1} has higher x-degree than u' and v', we obtain the polynomial identity

$$u'(1 + x, 1 + y) - v'(1 + x, 1 + y) = 0$$

in R, which is equivalent to $u' = v'$ holding in $(R, \circ)$. This proves (ii). □

The following lemma is crucial to our main theorems and is best possible in view of Example 1.2s. A simpler argument, as in [GM82], is available in characteristic zero. That argument fails in positive characteristic, where the situation is more delicate.

Lemma 2.3. *Suppose that R satisfies $y^m \alpha y^k = 0$ where $\alpha = \sum_i \alpha_i y^i x y^{n-i}$.*

(i) *If α is right reduced then R satisfies $y^{m+n} e_n y^k = 0$.*

(ii) *If α is left reduced then R satisfies $y^m e_n y^{n+k} = 0$.*

(iii) *If α is reduced then R satisfies $y^m e_{3n-1} y^k = 0$.*

Proof. By symmetry, the proof of (ii) is entirely analogous to that of (i). If the conclusions of (i) and (ii) hold then the conclusion of (iii) follows from equation (1):

$$y^m e_{3n-1} y^k = y^m \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} y^i e_n y^{2n-1-i} y^k = 0.$$

Thus it suffices to prove the conclusion of (i).

First assume that $m = k = 0$. Make the substitution $y \mapsto y(y + 1)$. Expanding $\sum_{i=0}^{n} \alpha_i y^i (y+1)^i x y^{n-i} (y+1)^{n-i} = 0$ by the binomial theorem and separating homogeneous components yields identities $v_0 = 0, \ldots, v_n = 0$ for R, where v_r is homogeneous of degree $n + r$ in y. We claim that

$$\sum_{r=0}^{n} (-1)^r v_r y^{n-r} = \alpha_n y^n e_n. \tag{2}$$

To establish equation (2), it suffices to show that the coefficients of y^axy^{2n-a} on each side are equal, whenever $0 \le a \le 2n$.

First note that by equation (1), the coefficient of y^axy^{2n-a} in y^ne_n is $(-1)^{a-n}\binom{n}{a-n}$ if $a \ge n$ and 0 otherwise. With the usual convention on binomial coefficients, the expression $(-1)^{a-n}\binom{n}{a-n}$ is valid for all a. Using the same convention we may sum over all values of any index occurring.

Now we calculate the coefficient of y^axy^{2n-a} in v_ry^{n-r}, or, what is the same, the coefficient of y^axy^{n+r-a} in v_r. The binomial theorem expansion above shows that the coefficient of y^sxy^t is precisely $\sum_{i+j=n}\alpha_i\binom{i}{s-i}\binom{j}{t-j}$. Putting $s=a$ and $t=r+n-a$, we obtain the desired coefficient as $\sum_i \alpha_i\binom{i}{a-i}\binom{n-i}{r-(a-i)}$.

It follows that

$$\begin{aligned}
&\sum_r(-1)^r\sum_i\alpha_i\binom{i}{a-i}\binom{n-i}{r-(a-i)}\\
=\ &\sum_i\alpha_i\binom{i}{a-i}\sum_r(-1)^r\binom{n-i}{r-(a-i)}\\
=\ &\sum_i\alpha_i\binom{i}{a-i}(-1)^{a-i}\sum_s(-1)^s\binom{n-i}{s}\\
=\ &(-1)^{a-n}\binom{n}{a-n}\alpha_n
\end{aligned}$$

since the inner sum has the value zero unless $n-i=0$, and 1 otherwise. This proves (i) in the case $m=k=0$.

In the general case, where m and k are not necessarily zero, the substitution $y \mapsto y(y+1)$ into the original identity yields an identity

$$\sum_{r+s+t\le m+n+k} c_{rst}y^s(y^mv_ry^k)y^t = 0, \tag{3}$$

for some coefficients $c_{rst} \in K$. For $0 \le a \le n$, consider the homogeneous component of (3) of degree $m+n+k+a$ in y. The only v_r occurring have $r \le a$ and the only term involving v_a is precisely $y^mv_ay^k$. By induction on a, $y^mv_ry^k = 0$ is an identity in R for all $r < a$ and hence so is $y^mv_ay^k = 0$. We may now proceed exactly as in the special case above and the conclusion follows.

□

2.1. **Unital algebras.**

In case R is unital, more information can be obtained. Note that $e_n(x,y) = x(\operatorname{ad} y)^n = x(\operatorname{ad}(y+1))^n = e_n(x,y+1)$. Thus by substituting $y \mapsto y+1$ into the result of (i) or (ii) in Lemma 2.3 and separating out the component of degree n in y we obtain $e_n = 0$ in R.

In the rest of this subsection (which is not essential to the main results of the paper) we give a characterization (for unital K-algebras) of the Engel identities.

For each $m \ge 0$, let W_m be the K-submodule of $K\langle x,y\rangle$ with basis all monomials y^ixy^j such that $i+j=m$, and let $V_n = \sum_{m=0}^n W_m$ and $V = \sum_{n\ge0} V_n$. Note that W_0 is spanned by the monomial x, and that for $n \ge 1$, e_n is a reduced element of W_n.

Define the difference operator Δ on V by $\Delta\alpha(x,y) = \alpha(x,y+1) - \alpha(x,y)$. Note that $\Delta\colon V_n \to V_{n-1}$, and that the homogeneous component of degree $n-1$ in y of $\Delta\alpha$ is simply the formal partial derivative $\partial\alpha/\partial y$ with respect to y (that is, the unique K-derivation of $K\langle x,y\rangle$ sending y to 1 and x to 0).

Proposition 2.4. *Let R be a unital K-algebra, and $\alpha \in W_n$.*

(i) $\Delta\alpha = 0$ *if and only if α is a scalar multiple of e_n.*
(ii) *If* $\operatorname{char} K = 0$ *then $\partial\alpha/\partial y = 0$ if and only if α is a scalar multiple of e_n.*

Proof. Given $\alpha(x,y) = \sum_{i=0}^{n} \alpha_i y^i x y^{n-i}$, expand $\alpha(x,y+1)$ by the binomial theorem. The coefficient of $y^s x y^t$ in $\alpha(x,y+1)$ is given by

$$[y^s x y^t] = \begin{cases} \sum_i \alpha_i \binom{i}{s}\binom{n-i}{t}, & s+t < n \\ 0, & s+t = n. \end{cases} \tag{4}$$

Now $\Delta\alpha = 0$ if and only if the coefficients of all monomials $y^i x y^j$, for $i+j \le n-1$, are zero. This gives a system of linear equations in the $n+1$ unknowns $\alpha_0, \dots, \alpha_n$. We claim that the coefficient matrix M has rank exactly n. Indeed, by equation (4), the submatrix of rows corresponding to the components of $xy^m, 0 \le m \le n-1$ has the form

$$\begin{bmatrix} * & 1 & 0 & 0 & \cdots & 0 \\ * & * & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ * & * & \dots & * & 1 & 0 \\ * & * & * & \dots & * & 1 \end{bmatrix}$$

which shows that the rank is at least n. However the rank is not $n+1$, since, as observed above, the coefficient vector $\alpha_i = (-1)^i \binom{n}{i}$ of e_n is in the kernel of M. This proves (i).

To prove (ii), it suffices to show that in characteristic zero, the submatrix of M consisting of all rows corresponding to monomials $y^s x y^t$ with $s+t = n-1$ has rank n. By equation (4), this submatrix has the form

$$\begin{bmatrix} n & 1 & 0 & 0 & \cdots & 0 \\ 0 & n-1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & n-2 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 & n \end{bmatrix}.$$

Since $\operatorname{char} K = 0$ the submatrix consisting of the first n columns is nonsingular and (ii) follows. □

3. Proofs of Theorems

We first prove Theorem 1.1. The implication (ii) ⇒ (iii) is obvious and (i) ⇒ (ii) and (iii) ⇒ (iv) follow from Proposition 2.2 and Proposition 2.1 respectively. By Lemma 2.3, (iv) and (v) are equivalent. Suppose then that R satisfies $e_n = 0$. Let $x, y \in R$. By [RW], the subalgebra T of R generated by x and y is Lie nilpotent of class m depending on n only. An easy induction on m shows that T, and hence R, satisfies the Morse identity, in the circle sense, of degree m. Indeed,

$$u_m - v_m = [u_1, v_1, v_2, \dots, v_{m-1}].$$

This proves (v) $\Rightarrow$ (vi). The last implication (vi) $\Rightarrow$(i) is obvious.

We now prove Theorem 1.3. The implication (i) $\Rightarrow$ (ii) is obvious, (ii) $\Rightarrow$ (iii) follows from Proposition 2.1, and (iii) $\Rightarrow$ (iv) can be deduced from Lemma 2.3. If $\operatorname{char} K = p > 0$ then (iv) $\Rightarrow$ (i) since by increasing m if necessary we may assume that $m = p^t$, so that

$$y^{p^t} x y^{2p^t} - y^{2p^t} x y^{p^t} = y^{p^t} e_{p^t} y^{p^t} = 0.$$

Finally, Theorem 1.5 follows from the quantitative form of Theorem 1.1 and [RW, Theorem].

4. COMMENTS

The following questions arise naturally from the work in this paper. The converses were shown to hold in [RW]. Here R is an arbitrary ring.

- If a ring R satisfies a reduced semigroup identity, does R necessarily satisfy an Engel identity?
- If a ring R satisfies a reduced circle semigroup identity, does R necessarily satisfy an Engel identity?

Even the case when R is unital appears difficult.

In [GM82] it was shown (using arguments special to characteristic zero) that the K-algebra R satisfies a partial linear identity if and only if the algebra of 2×2 upper triangular matrices over K is not in the variety generated by R. Perhaps this is true in all characteristics.

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