# Coordination via Polling in Plurality Voting Games under Inertia

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**Abstract.** We discuss a general model for strategic voting in plurality elections under uncertainty, using the concept of *inertia* to deal with information and risk attitude of agents. The model is flexible enough to be used for human societies and artificially designed multiagent systems. Under some further assumptions we show that a sequence of pre-election polls can help agents to coordinate on an equilibrium outcome. This mechanism is closely related to the STV voting rule and gives insight into the political science principle known as Duverger's law.

**Keywords:** Strategic Voting, Plurality System, Inertia, Pre-election Polls, Equilibrium, Duverger's law

## 1 Introduction

Voting as a preference aggregation method is widely used in human society and in artificially designed systems of software agents. A large amount of recent research has considered the situation where a single individual or a coalition of small size attempts to manipulate an election result in its favour, assuming that the remaining agents act sincerely. Such an assumption is often justified if the goal is to prove computational hardness results. However, if we wish to compare voting rules for their suitability from the viewpoint of mechanism design, it is necessary to understand how the rules behave under fully strategic behaviour.

The plurality rule is the most widely used voting rule despite substantial criticism from social choice theorists. One point in its favour is its simplicity and space-efficiency: an agent need only report a single alternative instead of submitting a full preference order, a list of utilities, or a binary approval vector, as is the case with most other rules. However, even such a simple rule becomes complicated when strategic voting behaviour is considered. We wish to study plurality voting under the assumption that all agents act strategically, as a starting point for the study of more general classes of rules.

Voting games have notoriously many equilibria and are hard to understand for this reason. As a mechanism designer, we may wish to help agents coordinate on a particular equilibrium outcome. For large electorates, it is not realistic to assume complete common knowledge on the part of all agents; again, a mechanism designer may wish to increase the publicly known information in order to achieve a better overall result.

A commonly used device that addresses these issues, especially for plurality elections, is to use publicly announced pre-election polls. Such polls, which amount to an approximate simulation of an election with the same agents and alternatives, increase the commonly known information among agents and may influence their strategic behaviour, leading (if all goes well) to a better overall outcome. Several authors from political science and economics disciplines have discussed the influence of pre-election polls in plurality elections, both empirically and theoretically (we discuss some of them in section 5). The key topic of interest is what is called "Duverger's law", a general political science principle stating that plurality voting tends to favor two-party competition [12]. Most of the models that have been used, with few exceptions (e.g. [6]), concern static equilibria, classifying them as "duvergerian" or "non-duvergerian", and do not discuss the dynamic process of converging to equilibria via the use of polls.

### 1.1 Our Contribution

We give a model for plurality elections under uncertainty that allows for heterogeneous agents. We introduce the concept of an agent's *inertia*, which combines both uncertainty and risk-aversion. Although it is not an exact generalization of known models in the literature, it is rather general and seems realistic enough to be used for both human society and for designed systems of autonomous agents. For the present article, we specialize to the case of 3 alternatives, place some restrictions on agent strategies, and consider some particular distributions of inertia. We present some numerical and analytic results on convergence to equilibria, both duvergerian and non-duvergerian. The results show that in addition to the actual preferences of the agents, the distribution of inertia plays a large role in the equilibrium outcome.

## 2 Our model

We assume that each agent has a fixed preference order, a total order on all alternatives (indifference is not allowed). These agents act rationally. There is a (possibly infinite) sequence of polls. Each agent who participates in a poll truthfully reports the same result that he would have reported if that poll were, in fact, the election.

*Remark 1.* The truthfulness condition is perhaps a strong assumption. However, it is standard in previous work in political science contexts. Furthermore, a designer of a multiagent system could enforce this condition, either by explicitly requiring it (e.g. programming the agents, in a cooperative context) or by not announcing the exact time of the election (in other words, each poll could be the real election, and so each agent has an incentive to report his real move in the simultaneous election game).

**Definition 1.** *We define* $s_{i,k}$ *as the score of alternative* $i$ *in poll* $k$*. This is the fraction of agents in poll* $k$ *who report* $i$ *as their preferred alternative.*

*Remark 2 (Partial information).* As the voting system is plurality, polls only report the top alternative. For example, for 3 alternatives $a, b$ and $c$, we have 6 different possible preference orders, namely *abc*, *acb*, *bac*, *bca*, *cab* and *cba*. We denote the number of agents with these preference orders by $(v_1, \ldots, v_6)$ respectively. However, polls just report the total support of the alternative. Therefore, the announced result in poll $k$ is $s_{a,k} = v_{1,k} + v_{2,k}$, $s_{b,k} = v_{3,k} + v_{4,k}$ and $s_{c,k} = v_{5,k} + v_{6,k}$. In the rest of this paper, for simplicity, we denote $v_{5,1}$ and $v_{6,1}$ by $v_5$ and $v_6$ respectively.

If the polls report more information, for example, the whole voting situation then agents have complete information and there would be low inertia.

**Definition 2.** *For each poll, each agent has a real number from the interval* $[0, 1]$*, called his* inertia*.*

**Definition 3.** *An agent with inertia* $\varepsilon$*, is* certain *the score of alternative* $j$ *is more than the score of alternative* $i$ *in poll* $k$*, if*

$$(1 + \varepsilon)s_{i,k} < (1 - \varepsilon)s_{j,k} \tag{1}$$

*otherwise, he is* doubtful*.*

*Remark 3.* The idea of inertia combines both credible information (to which it is inversely related) and risk-aversion (to which it is directly related). The lower it is, the more likely an agent is to act strategically in a given situation. Note that the inertia of an agent depends not just on the information given to the agent (that is, the sequence of poll scores), but also on his willingness to act on such information. The latter may be influenced by the perceived reliability of the information, as well as the attitude to risk of the agent.

If inertia of an agent is zero, then he will always be certain that $j$ is ahead of $i$ provided that such a result is reported. Of course, results can be reported only to finite precision, so it is possible that $j$ in fact, beats $i$ but such agent is not certain of that fact.

On the other hand, an agent with inertia equal to 1 will always be doubtful of every possible claimed comparison of scores.

## 3 Game Analysis

The setup above is still very general and allows for a wide range of outcomes. In this paper, we shall consider only the case where agents have a strong bias toward voting sincerely. They will only vote insincerely if they are certain that their preferred alternative has no chance to win, based on the announced poll result. Furthermore, they will then vote for the next highest alternative that has a chance of winning, and remain with that one until convinced that it is a certain loser, etc. If the only alternative that the agent feels is not a certain loser is the alternative lowest in that agent's preference order, then the agent votes for the second highest scoring alternative.

*Remark 4.* This kind of assumption has been made in some papers, for example [5], This behaviour may be required by the system designer in the case of cooperative agents.

An approximation to this situation occurs when the utilities of the alternatives are highly incomparable, and an agent with the preference order of $c_1 c_2 \dots c_m$ has utility relation $u_{v,3} \ll u_{v,2} \ll \cdots \ll u_{v,1}$.

*Example 1 (3 alternatives).* A rational *cba* agent who is certain (according to Definition 3) that the score of alternative $c$ is less than that of alternative $b$ (we can say this agent has *abandoned* $c$) will change his vote to $b$ in the next poll. He knows that no $a$ and $b$ supporters have incentive to change their votes in favour of $c$. Therefore, he thinks $c$ is a loser, and he votes for $b$ in other polls.

Even if *bac* agents feel that $b$ will lose, there is no incentive for them to vote in favour of $a$. In both cases, they get the same utility. Therefore, the total score of $a$ and $b$ is increasing.

In the case of more than 3 alternatives, the supporters of the last alternative in the first poll abandon that alternative first. The point is, this happens at different times (that is, after different numbers of polls) for different agents, depending on their inertia. In each poll, the current supporters of the two top alternatives do not abandon their alternative at the next step.

Suppose that the first poll result is $s_{c_m,1} < s_{c_{m-1},1} < \cdots < s_{c_2,1} < s_{c_1,1}$. Consider an agent with inertia $\varepsilon$. In the second poll, if he is a $c_m-$supporter he should validate all the conditions below to become certain that $c_m$ is not a loser; if a $c_{m-1}-$supporter, all but the first condition; and so on (a $c_2$-supporter need check only the last condition and a $c_1$-supporter need not check anything).

Otherwise, he should vote for his next preferred alternative who is not a loser.

$$
\begin{aligned}
(1+\varepsilon)s_{c_m,1} &> (1-\varepsilon)s_{c_{m-1},1} \\
(1+\varepsilon)(s_{c_{m-1},1} + s_{c_m,1}) &> (1-\varepsilon)s_{c_{m-2},1} \\
(1+\varepsilon)(s_{c_{m-2},1} + s_{c_{m-1},1} + s_{c_m,1}) &> (1-\varepsilon)s_{c_{m-3},1} \\
&\cdots \\
(1+\varepsilon)\sum_{l\geq 2} s_{c_l,1} &> (1-\varepsilon)s_{c_1,1}
\end{aligned}
$$

### 3.1 Convergence to Equilibrium

In the special case where inertia is identically zero for all polls and all agents, different scenarios can happen. If there is a tie between 2 alternatives, neither of which is the winner, depending on the voting situation, the results can be different, otherwise, agents vote for one of those tied alternatives according to their preference order. If there is no tie,

- When $m = 3$, just one more poll is needed to converge to a duvergerian equilibrium where only the top two alternatives receive any votes.
- For $m$ alternatives, we need at most $m - 2$ polls to converge to a duvergerian equilibrium. In each poll, the least favoured alternative

is omitted. Convergence can occur more quickly, if the top two alternatives are far ahead of all others. The extreme case is where one alternative receives an absolute majority. In this case, the convergence occurs in the next poll.

*Remark 5.* There is a connection with the voting method STV (single transferable vote, or instant runoff). When $m = 3$, if inertia is identically zero then, our assumptions mean that the plurality election is actually just STV (using Hare's rule). For general inertia and general $m$, we could fix some $\beta > 0$ and require that the election system automatically deletes the alternative whose support becomes less than $\beta$ for the next poll. If we assume that 2 alternatives do not reach at this bound simultaneously, we again simulate STV. However, our procedure is more general, as several alternatives may be eliminated at one step. We thank Edith Elkind for bringing some of these points to our attention.

To get a better idea about the different cases that can happen, we consider several cases numerically and one case analytically in detail.

### 3.2 Uniform uncertainty, 3 alternatives

We consider a 3-alternative election with a large number of agents, with a uniform inertia distribution on [0,1] for all agent types and for all polls. We assume the inertia of each agent remains constant during the sequence of polls. We describe the initial setup via a quadruple formed from the first poll result $s_{a,1}, s_{b,1}, s_{c,1}$ and the true percentage $v_6$ of type *cba* agents. Without substantial loss of generality, we assume that $s_{c,1} < s_{b,1} < s_{a,1}$ and we approximate the discrete uniform by a continuous one for purposes of computation.

All $c$ supporters who believe that $c$ is a loser change their votes in favour of their second alternative. The percentage of type $t$ agents (*cab* and *cba*) who vote in favour of alternative $i$ ($a$ and $b$ respectively) in poll $k+1$ is denoted by $\alpha_{t,i,k}$.

Note that the assumption of a common inertia distribution for all agents and polls implies that for all $k$, $\alpha_{cab,a,k} = \alpha_{cba,b,k} = \alpha_k$.

**Proposition 1.** *For a uniform distribution of inertia for all agents during the sequence of polls and initial result* $V = (s_{a,1}, s_{b,1}, s_{c,1}, v_6)$

$$\alpha_k = \frac{1}{1 + \frac{2^k \left(\frac{s_{c,1}-v_6}{s_{b,1}+v_6}\right)^k (s_{b,1}+v_6-2s_{c,1})}{(s_{b,1}-s_{c,1})\left(-2^k\left(\frac{s_{c,1}-v_6}{s_{b,1}+v_6}\right)^k + \left(1-\frac{v_6}{s_{c,1}}\right)^k\right)}} \tag{2}$$

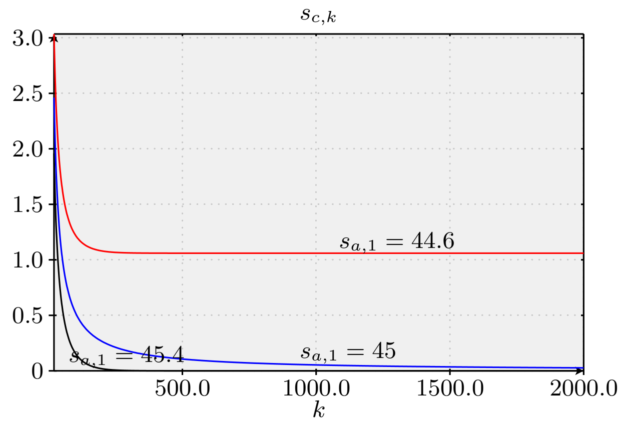


**Fig. 1.** Uniform inertia distribution and $V = (s_{a,1}, 35, 100 - s_{a,1} - 35, 5)$

*Proof.* According to the order of alternatives in the first poll and Definition 3, a $c$ supporter concludes that $c$ is a loser and changes if $(1+\varepsilon)s_{c,k} < (1-\varepsilon)s_{b,k}$.

Therefore, $\alpha_k = p\{\varepsilon < \frac{s_{b,k}-s_{c,k}}{s_{b,k}+s_{c,k}}\}$. The score of alternatives $a$, $b$ and $c$ in poll $k$ is given by:

$$s_{a,k} = s_{a,1} + \alpha_{k-1}v_5 \qquad s_{b,k} = s_{b,1} + \alpha_{k-1}v_6 \tag{3}$$

$$s_{c,k} = s_{c,1} - \alpha_{k-1}v_6 - \alpha_{k-1}v_5 \tag{4}$$

Therefore,

$$\alpha_k = p\{\varepsilon < \frac{s_{b,1} - s_{c,1} + \alpha_{k-1}(s_{c,1} + v_6)}{s_{b,1} + s_{c,1} - \alpha_{k-1}(s_{c,1} - v_6)}\} \text{ and } \alpha_1 = p\{\frac{s_{b,1} - s_{c,1}}{s_{b,1} + s_{c,1}}\} \tag{5}$$

The stated solution formula for this recurrence is readily established by induction.

**Proposition 2.** *The score of the last alternative in the first poll (which we denote by c) satisfies*

$$\lim_{k\to\infty} s_{c,k} = \begin{cases} 0 & \mathit{if}\, s_{b,1} + v_6 \geq 2s_{c,1} \\ (\frac{2s_{c,1}-v_6-s_{b,1}}{s_{c,1}-v_6})s_{c,1} & \mathit{if}\ \ s_{b,1} + v_6 < 2s_{c,1} \end{cases} \tag{6}$$

*Proof.* The score of alternative $c$ after $k+1$ polls is

$$s_{c,k+1} = (1 - \alpha_k)s_{c,1} \tag{7}$$

According to the Proposition 1, if we converge $\alpha_k$ to infinity, we have,

$$\lim_{k\to\infty} \alpha_k = \begin{cases} 1 & s_{b,1} + v_6 \geq 2s_{c,1}; \\ \frac{s_{b,1}-s_{c,1}}{s_{c,1}-v_6} & s_{b,1} + v_6 < 2s_{c,1}. \end{cases}$$

The result follows immediately.

*Remark 6.* The convergence to zero is exponentially fast with the exponential rate decreasing as we approach the boundary between the two cases, and at the boundary it is subexponential.

Figure 1 shows three special cases (the boundary case and 2 different cases in its neighbour).

**Theorem 1 (duvergerian equilibrium).** *In a plurality voting game with 3 alternatives and initial result* $V = (s_{a,1}, s_{b,1}, s_{c,1}, v_6)$ *and uniform distribution of inertia, the polling sequence yields a duvergerian equilibrium if and only if* $s_{b,1} + v_6 \geq 2s_{c,1}$.

*Proof.* Follows immediately from Proposition 2.

The result above is for a very special inertia distribution and explicit analysis of this type is not possible for general distributions. In the next section, we investigate some different distributions via numerical simulations. Intuitively, we expect that distributions skewed to the left (with more agents of low inertia) will converge to the $\varepsilon \equiv 0$ case more quickly. It seems to us that inertia should decrease for each agent over the sequence of polls, as information should increase but risk attitude should remain constant. In this case, we again expect quicker convergence to the zero inertia situation.

*Example 2.* Consider the initial result $V = (s_{a,1}, s_{b,1}, s_{c,1}, v_6) = V = (45, 35, 20, 5)$. According to Theorem 1, we have a limiting duvergerian equilibrium for uniform inertia distribution. Numerical results in Figure 1 also confirm this result. When we change the inertia distribution to be triangular with apex 0.5, we have the result in Figure 2. As we see in Figure 1, the speed of converging is very low but changing the inertia distribution accelerates the speed.

## 4 Numerical Results

We consider the continuous triangular distribution $T(p)$ whose density function's graph has vertices at $(0, 0), (p, 2)$ and $(1, 0)$.

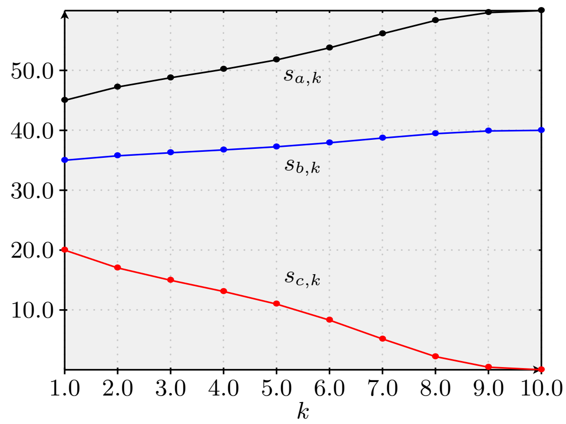


**Fig. 2.** $V = (45, 35, 20, 5)$ and $T(0.5)$ inertia distribution

In Figure 2, we have 5% *cba* agents, we increase this to 10% in Figure 3 and we see the expected faster convergence. Changing the score of $c$ to 25, we see the result in Figure 4. This seems to yield a non-duvergerian equilibrium, and it appears that the score of $c$ converges to 22. Until now, we saw the effect of the initial results of polls on the equilibrium outcome. We can also see the effect of the inertia distribution on the fixed initial result $V = (40, 35, 25, 10)$ in Figures 4 and 5, which deal with $T(0.5)$ and $T(0.3)$ distributions.

## 5 Related Work

Our work here was derived from first principles and is not really closely related to anything we have found in the literature, but we give a brief discussion of some possibly relevant references. We divide them into two categories. First, papers that discuss the strategic voting effect of polls and Duverger's law. These come from the fields of political science and economics. Second, papers that study equilibria in plurality voting games, from a more algorithmic viewpoint.

In the first category, there is a stream of research studying Duverger's law by means of theoretical models, such as [2], [11], [10] and [6]. Our model belongs in this class, but is more general (although not a strict generalization). There is also some experimental and agent-based simulation work [8, 7, 1].

There are several important differences between our work and these papers. One of the differences is related to the different amount of infor-

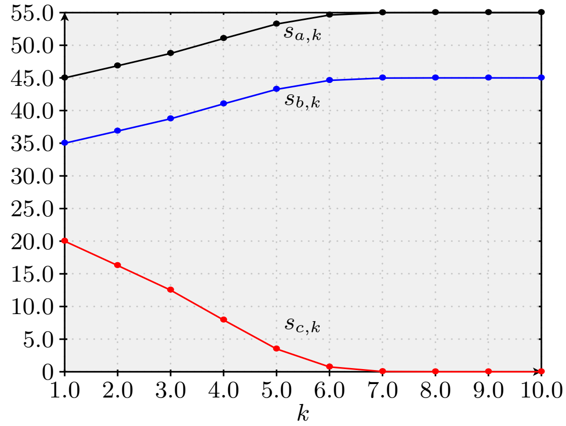


**Fig. 3.** $V = (45, 35, 20, 10)$ and $T(0.5)$ inertia distribution

mation. Besides the partial information about the voting situation, the other extra feature considered in the present paper is inertia about the announced result of polls. To our knowledge, all previous papers which have considered uncertainty (and many have not) have only considered a common probability distribution for all agents over the possible voting situations. The other difference is that we assume agents have a strict preference over alternatives and all preference orders are allowed. However, some of these papers such as [10] and [6] consider very special cases where 2 of 3 alternatives belong to the same party. The agents supporting the other party are indifferent between these 2 alternatives.

In the second category, we find [9] which studies the conditions under which the plurality game is guaranteed to converge to a Nash equilibrium (it considered a special model in which agents are allowed to change their votes if they cannot coordinate their actions). The other paper in this category is [4] that studies the dominance solvability of plurality voting games, stating results in terms of the proportion of agents who agree on the least popular alternative in a sequence of subsets of the original set of alternatives. As each subset is derived from the previous one by deleting the least popular alternative, it is to some extent similar to the omission of trailing alternatives during our sequence of polls. In [3], the equilibria of simultaneous and sequential plurality games with abstention are discussed. The paper characterizes the profiles that can yield pure Nash equilibria in a simultaneous game.

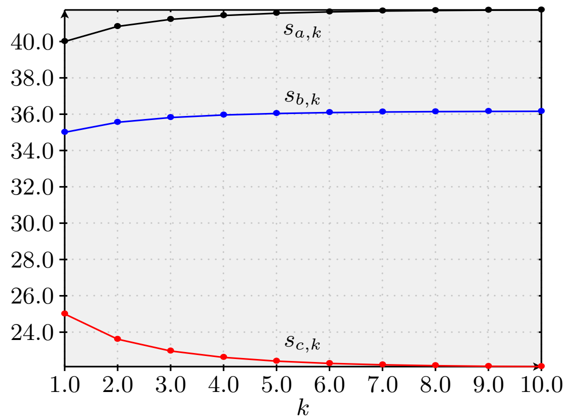


**Fig. 4.** $V = (40, 35, 25, 10)$ and $T(0.5)$ inertia distribution

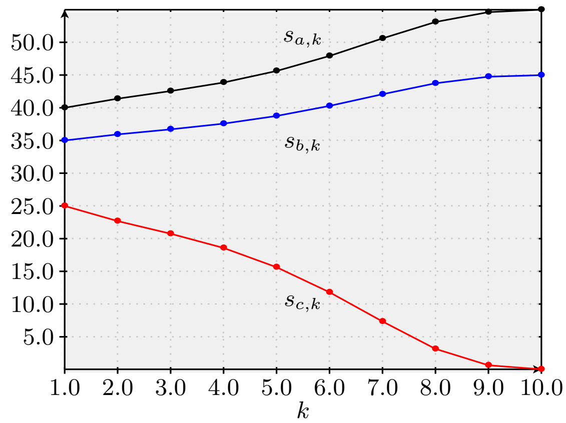


**Fig. 5.** $V = (40, 35, 25, 10)$ and $T(0.3)$ inertia distribution

## 6 Conclusion and Future Work

Voting games, even for fairly simple systems such as plurality with 3 alternatives, have a large number of equilibria. In our setup, pre-election polls act as a coordination device to reduce the number of possible outcomes (often, yielding a unique outcome). The quantity that we call inertia plays a large role and must be understood or controlled in order to apply these results to human elections or artificially designed systems in which preference or judgement aggregation is to be achieved through plurality voting.

Obvious possible ideas for future work involve relaxing some of the many assumptions made in the model, such as the bias toward sincere voting. Other ideas which we intend to pursue include:

- Try to make realistic yet general assumptions on the distribution of inertia of agents and its evolution through the poll sequence, and prove qualitative results about limiting equilibria.
- Show that the restrictions on strategies imposed in our model, in fact, lead to a good overall outcome in the sense of utilitarian social welfare, for example.
- Formalize the model in terms of game theory (involving standard concepts such as repeated games, games of incomplete information, and stability of equilibria).
- Investigate the connection between the polling model and STV in more detail. For example, can STV with Coombs' rule be interpreted in terms of the antiplurality (veto) rule?

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