Iterated Regret Minimization in Voting Games

Miranda Emery and Mark C. Wilson

Abstract

The game-theoretic solution concept Iterated Regret Minimization (IRM) was introduced recently by Halpern and Pass. We give the first application of IRM to simultaneous voting games. We study positional scoring rules in detail and give theoretical results demonstrating the bias of IRM toward sincere voting. We present comprehensive simulation results of the effect on social welfare of IRM compared to both sincere and optimal voting. The results fit into a broader research theme of the welfare consequences of strategic voting .

1 Introduction

The concept of regret has a long history in decision theory [12]. The idea that decision-makers act to minimize regret is the main competitor to the idea that they aim to maximize utility, particularly in situations where probabilities are not easily available. The concept has also proved useful in game theory, where the *minimax regret solution* is used, and is widely used in machine learning. In social choice theory, regret has been used to measure the performance of voting rules under sincere voting [10]. All these applications can be formulated to involve a single decision-maker "playing against nature", and do not involve strategic considerations of other players' strategies.

Many models of strategic voting have been studied in recent decades. Much of this work concerns sequential voting, whereas we deal here with simultaneous voting. Research on simultaneous voting games has mostly focused on the plurality rule. The most commonly used solution concept from game theory, namely Nash equilibrium, can lead to very strange situations when applied to voting games. Not only are there hugely many equilibria, some are of very low desirability from the point of view of social choice. For example, for most voting rules used in practice or studied by researchers, the situation where voters are unanimous in preferences yet all vote for their least preferred candidate is a Nash equilibrium. In this context, game-theoretic solution concepts such as dominance solvability, strong equilibrium, coalition-proof equilibrium, and other refinements of Nash equilibrium have been studied [5], [11], [13], [14].

Iterated Regret Minimization (IRM) as a solution concept for strategic games was introduced by Halpern and Pass [7], who considered several well-known games where the outcome produced by IRM accords better with intuition than the Nash equilibrium. IRM is analogous to dominance solvability in that it is defined by iterated deletion of strategies, fewer assumptions on rationality are required than for Nash equilibrium, and it leads in most cases to a definite prediction for the outcome of a game. However it is more widely applicable than dominance solvability for voting games. A few other papers have explored IRM in various contexts. In [4] a unified approach to iterated deletion solution concepts from the viewpoint of regret is given. IRM has been applied to some restricted classes of games, including game graphs [6].

1.1 Our contribution

We give what to our knowledge are the first results on IRM as applied to voting games, with special emphasis on positional scoring rules. We give examples of some poor quality outcomes that can occur with this solution concept. On the other hand, we show that in addition to the good features listed above, IRM leads to a strong bias toward sincere voting, and seems very worthy of study on this point alone. We explore computationally the quality of the solution using various measures of overall societal welfare.

Specifically, we show that the IRM outcome under plurality or antiplurality coincides with the sincere outcome, regardless of the tiebreaking scheme or voter utilities over candidates. We carry out detailed simulations and measure the effect on overall welfare of IRM compared to both sincere and optimal voting, and discuss in detail the interpretation of these results.

Our results support a growing belief that widespread strategic voting may in fact be socially desirable if and only if the information about other voters possessed by each voter is sufficiently high, and that IRM is close to the boundary between high and low information in this interpretation.

2 Preliminaries

We define IRM for voting games and refer the reader to [7] for more details. We use a set of m candidates $C = \{c_1, \ldots, c_m\}$, and a set of n voters $V = \{v_1, \ldots, v_n\}$. Each voter v_i has a utility function u_i which induces a sincere preference order (a strict linear ordering of C) and the collection of all of these is the **preference profile**. The actions available to v_i are of the form α where α is a preference order (not necessarily sincere). This gives the **strategy profile** which we denote by π. In this article we do not consider mixed strategies, as is common when studying voting games, whose payoffs are typically ordinal.

Note that we require that the utility function distinguish between candidates, so that indifference is not allowed. Much of the basic theory below still works when indifference is allowed, but not all.

Definition 2.1. *The* **scoring rule** *defined by a weight vector $w = (s_1, \ldots, s_m)$, with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$, elects all candidates with maximal score. The score of candidate a is defined to be $\sum_v s_{i(v;a)}$, where $i(v;a)$ is the rank of a in the vote expressed by voter v.*

We consider four specific scoring rules: plurality, anti-plurality, Borda and k-approval, defined by $(1, 0, 0, \ldots, 0)$, $(1, 1, \ldots, 1, 0)$, $(m-1, m-2, \ldots, 1, 0)$ and $(1, 1, \ldots, 1, 0, \ldots, 0)$ (there are k 1's) respectively.

In order to have a definite result (and not to have to deal with the utility to a player of a set of winners, a rather controversial subject), we always use a fixed tiebreaking order on the candidates (alphabetical unless otherwise specified).

Once we have an idea of payoffs, a given rule and set of players and candidates specifies a **game form**. We shall use different models of payoffs. Purely ordinal payoffs do not allow us to compute regret, because we cannot compare the difference between our first and second with the difference between our second and third candidate, for example. A commonly used assumption is that of **Borda utilities** whereby the payoff to a voter when her ith highest candidate is elected is precisely $m - i$. We could also generate utilities from the interval $[0, 1]$, which we do for some results.

Definition 2.2. *Let α be an action and π a strategy profile. The* **partial profile** *π_{-i} is formed from π by deleting the action of voter v_i. We first define the* **regret of α with respect to π_{-i}**, *denoted $\mathbf{R}(i, \pi_{-i}, \alpha)$. When v_i votes α, this leads to some candidate c winning the election. There are other possible winners if v_i votes differently (keeping π_{-i} fixed), and we let b be the one of highest utility to v_i. Then*

$$\mathbf{R}(i, \pi_{-i}, \alpha) = u_i(b) - u_i(c) \geq 0.$$

We next define the **maximum regret of** α *by*

$$\mathbf{MR}(i, \alpha) := \max_{\pi_{-i}} \mathbf{R}(i, \pi_{-i}, \alpha).$$

Finally, the **minimax regret** *for player i is*

$$\mathbf{MMR}(i) := \min_{\alpha} \mathbf{MR}(i, \alpha).$$

Example 2.3. *(regret with respect to a partial profile) Consider a scenario under the Borda rule with three candidates A, B, C, a voter v with the preference BAC, and a partial profile where A has 0 points, B has 1 point and C has 2 points. We use Borda utilities and an alphabetical tiebreaking rule. Consider v's six possible actions of voting ABC, ACB, BAC, BCA, CAB, and CBA. The candidates elected by these votes in this partial profile are respectively A, C, B, B, C, and C. Thus the best possible outcome for v is B and the regrets of BAC and BCA in this partial profile are 0. The vote that elects A has a regret of 1, while the three that elect C all have regret 2.*

Definition 2.4. Iterated regret minimization *(IRM) is the solution concept defined by iterative removal of all strategies that do not minimize maximum regret. At iteration j, each player v_i updates her action set S_i^j by removing all actions that do not minimize maximum regret among all actions in S_i^{j-1}, assuming that each other player v_k is playing actions in S_k^{j-1}. Note that S_i^0 consists of all possible actions, for each i. This iterative process must terminate. In that case, if S_i is the set of actions remaining to player v_i, then IRM returns the set of all winners of elections specified by profiles $(s_1, \ldots, s_n)$ with $s_i \in S_i$.*

Remark 2.5. *IRM is a rather pessimistic solution concept in that it presupposes very little knowledge about others' preferences. However it is not as pessimistic as the usual* minimax regret solution*, which is what is computed by the first round of the IRM process.*

Example 2.6. *(IRM winner)*

Consider a scenario under plurality with $m = 3$, $n = 3$. Tiebreaking is alphabetical and we use Borda utilities. The voters v_1, v_2, and v_3 respectively have preferences ABC, CAB, and CBA.

In the first iteration, each voter v_i considers the two other voters, both of whom can place either A, B, or C first. This gives nine different partial profiles, and there are three actions that v_i can take in each of these profiles. The table below gives the winning candidate for each combination of partial profile and vote.

v_i's	*Partial profile*								
vote	*AA*	*AB*	*AC*	*BA*	*BB*	*BC*	*CA*	*CB*	*CC*
A	*A*	*A*	*A*	*A*	*B*	*A*	*A*	*A*	*C*
B	*A*	*B*	*A*	*B*	*B*	*B*	*A*	*B*	*C*
C	*A*	*A*	*C*	*A*	*B*	*C*	*C*	*C*	*C*

For each of the three voters the maximum regret is found for each vote, based on that voter's preference. Note that the maximum possible regret for any action with respect to any partial profile is 2.

When v_1 faces the partial profile AC, A will win the election unless v_1 places C first, in which case C wins. This implies that for v_1, placing C first has maximum regret 2. Similarly, placing B first has maximum regret 1 (attained for example in the partial profile BA), and placing A first has maximum regret 0. Thus v_1 is left only with votes that place A first. Similarly, v_2 and v_3 place C first. The IRM winner is found in a single iteration, because all actions remaining to all players give the same result, namely that C wins.

IRM is analogous to dominance solvability, where all dominated strategies are iteratively removed, all players acting (i.e. thinking) simultaneously at each iteration.

In [7] a formal epistemic characterization of IRM is given. Intuitively, each player makes no assumptions about other players, with high probability, but with probability ε they are assumed to be rational (in the sense of applying IRM themselves), with probability ε^2 they are assumed to be rational and to believe that they are playing rational players, etc. Equivalently, players use a lexicographic minimization strategy: first minimize regret with respect to all strategies by other players, then minimize regret with respect to all strategies by rational players, etc.

In each iteration, voters know only how many other players there are, and what actions are available to those players; from this information each voter can find all possible partial profiles. In subsequent iterations each voter can infer which votes the other voters have eliminated and which have been retained.

The issue of how much information players have about others is crucial for IRM: under full common knowledge it reduces to maximizing expected utility.

3 Theoretical results

Although in Example 2.6 the IRM winner was the same as the sincere winner, this certainly does not always happen (as with most strategic voting solution concepts).

Example 3.1. *(the sincere loser can be the IRM winner)*

Consider the following scenario under the Borda rule with Borda utilities and alphabetical tie breaking.

Voters	*Preference order*
v_1	$ACDB$
v_2	$BCAD$
v_3	$BDCA$
v_4	$CDAB$
v_5	$DABC$

The sincere ranking is C, D, A, B. Here C and D each have score 8, and C wins the tie, while A and B each have score 7, with A winning the tie. Following the first two iterations of IRM, the first four voters have already eliminated all but their sincere votes, while v_5 is left with $DABC$ and $DBAC$. If v_5 votes sincerely then his least preferred candidate C will win, while the insincere vote causes candidate B to win, and so B is the IRM winner. This is a case where all four candidates were close to being tied and so all possible outcomes have an almost equal payoff for each player. Further, despite being the lowest ranked candidate under sincere voting, B is preferred to the sincere winner by a majority of voters (in other words B is witness to the fact that C is not a Condorcet winner).

IRM appears to yield a single winner far more often than dominance solvability. However, it does not always do so.

Example 3.2. *(IRM doesn't always give a unique winner, even for scoring rules) Consider a scenario under the Borda rule with candidates A, B, C, D, and voters v_1, v_2, v_3, v_4, v_5. The sincere preferences of the voters are respectively $ABDC, ACDB, BADC, BDAC, CBDA$. Borda utilities are used and the tiebreaking rule is alphabetical.*

Following the first iteration, each voter is left with two votes – those that place their most preferred candidate first and least preferred last, with the remaining two candidates in either order. In the second iteration there are 32 profiles and two of these elect D as the winner, while the rest elect A or B. For v_2, there are no partial profiles in which switching C and D changes the winner and so both votes have a regret of 0. For v_4, voting $BADC$ can cause

A to win where otherwise D wins, and so the vote $BADC$ has a regret of 1. On the other hand, changing from $BADC$ to $BDAC$ always leads to the same or a better result for this voter and so $BDAC$ has a regret of 0.

For the remaining voters, both votes have a regret of 1. For v_1, we see that voting $ADBC$ causes D to win in two profiles where otherwise B would have won. On the other hand, if v_5 votes insincerely while the remaining voters vote sincerely, then v_1 can make A win by voting $ADBC$, and B win by voting $ABDC$. Thus both votes have the same regret. For v_3 and v_5 we have a similar result where one vote almost always leads to a better outcome, except in the two profiles where D wins.

Thus in the second iteration the vote $BADC$ is eliminated for v_4 while all other votes are retained. In the third iteration no further votes are eliminated and so we have three possible winners: A, B, D.

One useful fact about IRM is that under some assumptions on utilities, all voters with the same preference order adopt the same pure strategy. Note that when using Borda utilities, all voters with the same preference order will have the same payoffs for each outcome. This enables much simpler computation of the IRM outcome.

Proposition 3.3. *If two voters have the same payoffs over all outcomes, then those voters eliminate the same votes in every iteration, and hence adopt the same pure strategy.*

Proof. Each voter in each iteration bases his decision on the possible votes the other voters could cast, and the payoffs he receives. In the first iteration, every other voter is choosing from the same set of votes (i.e. all possible votes) and so the voter's decision is based only on the number of other voters, and his payoffs. Hence two voters with the same payoffs will eliminate the same votes in the first iteration. The result follows by induction on the number of iterations. □

For most commonly used voting rules, ranking one's most preferred candidate above one's least preferred makes sense for both sincere and strategic voting. This is also the case for IRM. Note, however, that actions violating this constraint are not weakly dominated in general, because non-monotonic rules exist. One of these is Instant Runoff Voting (IRV), the single-winner version of STV. This rule successively eliminates the plurality loser, deleting this candidate from all voters' preference orders, until a single winner remains.

Definition 3.4. *Let v_i be a voter and π_{-i} a partial profile. Votes α and β are called* **π_{-i}-equivalent** *for i if the winner is the same whether v_i votes α or β when faced with π_{-i}. The votes are* **equivalent** *if they are equivalent for each choice of v_i and π_{-i}.*

Definition 3.5. *A voting rule is* **weakly positively responsive** *(WPR) if the set S below is nonempty, and whenever there are precisely 2 winners x, y in a partial profile π_{-i}, adding any single extra vote $\alpha \in S$ makes x the sole winner.*

S is the set of all votes α satisfying

- *in α, x is ranked above y;*
- *α is not equivalent to the vote obtained from α by swapping x and y.*

Example 3.6. *Each scoring rule is WPR (S consists of all votes giving x a strictly greater score than y).*

Definition 3.7. *Let v be a voter with sincere preference order $c_1, \ldots, c_m$. An expressed preference order by v is* **extremely insincere** *if it places c_m above c_1. A vote by v* **attains the maximum possible regret** *if it elects c_m when faced with some partial profile, where another vote would have elected c_1.*

Recall that a voting rule is **neutral** if it is symmetric with respect to all candidates. It is **Pareto optimal** if whenever x is ranked above y by all voters, y is not a winner.

Proposition 3.8. *Suppose that a voting rule is neutral, Pareto optimal, and weakly positively responsive. Then in the first iteration, every extremely insincere vote attains the maximum possible regret.*

Proof. Suppose that $n-1$ is even, and let A and C be voter v's most (respectively) least preferred candidates. Consider a partial profile in which half the voters rank A first and C second, and half rank C first and A second. By Pareto optimality and neutrality, A and C are tied winners and no other candidate is a winner (in the partial profile). By WPR, there is a vote in which v ranks A above C and A wins. Thus ranking C over A has maximum possible regret for v.

If $n-1$ is odd we use a different construction that depends on the tiebreaking rule. First construct a partial profile as in the previous case, with $n-2$ voters. If A wins in a tie then the extra vote in the partial profile will have C above A, whereas if C wins in a tie then the extra vote in the partial profile has A above C. In each case v can make A win by ranking A above C, but C will win if v ranks C above A. $\square$

Example 3.9. *(sometimes, all votes attain maximum possible regret)*

Consider a voter v with preference ABC under IRM with Borda utilities and alphabetical tiebreaking, where $m = 3$, $n = 9$ and the rule is IRV. Consider the partial profile where there are 3 ABC voters, 3 CAB voters and 2 BCA voters. If v votes sincerely then B will be eliminated, and C will win overall. However if v votes BAC then C will be eliminated and the winner will be A. Thus for v, the sincere vote has maximum possible regret. In fact in this situation all six of v's possible votes have this regret and so v eliminates nothing in the first iteration. Thus if in fact we have 9 voters each with ABC as their sincere preference order, each will perform the same computation and retain all possible votes. Thus IRM will return all candidates as winners.

For scoring rules, we can say more, because some other votes have lesser regret.

Definition 3.10. *Let v be a voter with preference order $c_1, \dots, c_m$. A vote by v is* **extreme-fixing** *if it gives c_1 the highest possible score and c_m the lowest possible score.*

Theorem 3.11. *When carrying out IRM using a scoring rule, the votes not eliminated at the first iteration are precisely the extreme-fixing ones.*

Proof. Let $u_1, \dots, u_m$ denote the utilities of voter v with preference order $c_1, c_2, \dots, c_m$. We have four cases, as the proof differs depending whether the number of voters n is odd or even, and depending which candidate is favoured by the tie-breaking rule. For each case the result can be shown by constructing either one or two profiles, each of which eliminate some set of votes, such that the complete set of votes eliminated consists precisely of the ones that are not extreme-fixing. This is done by showing that the votes in this set can all cause the voter's lowest ranked candidate to win, while a different vote makes the highest ranked candidate the winner, thus giving them the maximum possible regret of $u_1 - u_m$. Finally we show that the maximum regret of the remaining votes is $u_2 - u_m$. This means that all votes with regret $u_1 - u_m$ are eliminated.

Let the score vector be $s = \{s_1, s_2, \dots, s_m\}$. We construct a new vector $t = \{t_1, t_2, \dots, t_x\}$ where $x \leq m$ that contains only the unique scores. For example, under plurality, antiplurality, and k-approval, $t = \{1, 0\}$, while under the Borda rule, $t = s$ because all scores are unique. Let p_i be the score of candidate i in the partial profile under consideration.

Case 1: n odd; c_1 beats c_m in a tie As c_1 will win when c_1 and c_m are tied, we need to construct a profile where $p_m = p_1 - t_x + t_1$. This ensures that the only possible way for v to make c_1 the winner is to give c_1 the highest possible score, and c_m the lowest possible score. Any other positioning of these two candidates will leave c_m with a strictly higher score than c_1. Firstly, we have two groups of voters, each of size $(n-3)/2$. The first group places c_1 first and c_m second, and the second group has c_m first and c_1 second. All other candidates are permuted among the $n-3$ voters. This ensures that c_1 and c_m are the only possible winners following the addition of v's votes, as the scores of the other candidates will not be high enough. At this point c_1 and c_m have an equal number of points. One of the two remaining votes places c_1 last and c_m at some position i corresponding to the score t_j, and the other has c_m first and c_1 at position i. This gives us $p_m - t_1 - t_j = p_1 - t_x - t_j \rightarrow p_m = p_1 - t_x + t_1$ as required.

Case 2: n odd; c_m beats c_1 in a tie In this case we need to construct a profile where $p_m < p_1 - t_1 + t_x$. However if p_m is too low, it may be possible that some vote that gives c_1 a slightly lower score, or c_m a slightly higher score, will still elect c_1 as the winner. To ensure this is not the case, we consider two different partial profiles. In both, as above, we have two groups in which c_1 and c_m are alternately first and second, with all other candidates permuted. The two remaining votes in the first partial profile are as follows: one voter places c_1 last and c_m at position i, while the other places c_m in a position corresponding to the score t_2, with c_1 at position i. In this partial profile we have $p_m - t_2 - t_j = p_1 - t_x - t_j \rightarrow p_m = p_2 - t_x + t_1$, ensuring that v must give c_1 the highest possible score t_1 in order for c_1's final score to be strictly higher than c_m's final score. However, depending on the values of the score vector, c_1 may still win when v gives c_1 the highest score but gives c_m a score higher than t_x.

Hence we consider a second similar partial profile that differs only in the final two votes. In this partial profile we have one of these two voters place c_m first and c_1 at position i, while the other voter places c_1 in a position corresponding to the score t_{x-1}, with c_m at position i. With similar reasoning to above, this shows that v must give c_m the lowest possible score in order for c_1 to win overall.

Case 3: n even; c_1 beats c_m in a tie As in Case 1 above, we need to construct a profile where $p_m = p_1 - t_x + t_1$. We have again the two groups of voters placing c_1 and c_m in either first or second place, this time each of size $(n-2)/2$. This leaves one remaining voter, who places c_1 last and c_m first, giving $p_m - t_1 = p_1 - t_x \rightarrow p_m = p_1 - t_x + t_1$ as required.

Case 4: n even; c_m beats c_1 in a tie Analogously to Case 2, we need to consider two different partial profiles. With both partial profiles we start as always with the two groups of voters, each of size $(n-2)/2$. The remaining vote in the first partial profile places c_1 last and c_m at a position corresponding to score t_2. Thus we have $p_m - t_2 = p_1 - t_x \rightarrow p_m = p_2 - t_x + t_1$, and so v must give c_1 the highest possible score as otherwise c_1 and c_m could tie, in which case c_m would win, or c_m could have a strictly higher score than c_1, also leading to c_m winning.

In the second partial profile the final voter places c_m first and c_1 in a position corresponding to the score t_{x-1}. From this it follows that votes that do not give c_m the lowest possible score will have the maximum possible regret.

In order to show that the maximum regret of the votes that give c_1 the highest possible score and c_m the lowest possible score is $u_2 - u_m$, we can consider a set of partial profiles similar to those above, but with c_1 and c_2 switched. That is, these profiles describe situations in which votes that give c_2 the highest possible score and c_m the lowest possible score will elect c_2, with this being the best possible outcome, while all other votes will elect c_m and thus have a regret of $u_2 - u_m$. Because scoring rules are positively responsive, it is not possible for votes giving c_1 the highest possible score and c_m the lowest possible score to get a higher regret than this, because adding support to c_1 can not change it from the winner to a loser.

□

Corollary 3.12. *IRM always elects the sincere winner under each scoring rule when $m \leq 3$, and under plurality and antiplurality for every m.*

Proof. For plurality the highest possible score is 1, given only to the the candidate placed first. The lowest possible score is 0, given to all other candidates. Thus under the plurality rule, all votes that do not place the voter's highest ranked candidate first are eliminated in the first iteration. Because permuting the remaining candidates has no effect on their scores and thus no effect on the outcome of the election, every vote receives a regret of 0 in the second iteration, and the process halts. Thus the IRM outcome is always the same as the sincere outcome under plurality. A similar line of reasoning for anti-plurality leads to a situation where all voters are left with only those votes that place their least preferred candidate last, producing an outcome identical to the sincere outcome. □

In order for the results on iterated regret minimization to be meaningful, the process must produce results that differ from those produced by other solution concepts. This is now easy to see, because Nash equilibrium, strong equilibrium, coalition-proof equilibrium, dominance solvability, and maximin utility solutions can all involve insincere voting under the plurality rule. For completeness and to aid the reader's intuition we give some explicit examples.

Example 3.13. *(IRM differs from Nash) Consider the plurality rule with $n \geq 3$. The unanimous profile where all voters have the same preference order ρ is a Nash equilibrium where the top-ranked candidate wins, which is also the outcome under IRM. However, there are many other Nash equilibria, including the profile where all voters submit the reverse of ρ and the bottom-ranked candidate in ρ wins. On the other hand, IRM strategies are not always Nash equilibrium strategies. Consider the profile under plurality (where A beats B beats C in a tie) having 2 ABC voters, 2 BAC voters and 1 CBA voter. Under IRM, A wins. However this is not a Nash equilibrium as the CBA voter can switch her vote to B to make B the winner.*

Example 3.14. *(IRM differs from dominance solvability)*

A game is **dominance solvable** *if iteratively eliminating strongly dominated actions leads to a unique outcome. A vote A strongly dominates a vote B for some voter v if in every possible partial profile of the other voters, casting A leads to a winner that is more highly ranked by v than the candidate who wins when B is cast. It is known that plurality voting games are not often dominance solvable (necessary and sufficient conditions are given in [5]), so that this solution concept must differ from IRM in general.*

For example, consider a profile with 2 ABC voters, 2 BAC voters and 1 CBA voter. Borda utilities are used and the tiebreaking rule is alphabetical. Under plurality, this is not dominance solvable, since in the first iteration the option of voting for A is eliminated for the voter with preference CBA, but no further votes can be removed for any voter. However, IRM produces a unique outcome for this game.

There also exist elections where both dominance solvability and IRM yield unique predictions, but the corresponding winners are different. For example, consider the profile with 2 ABC voters, 3 BAC voters and 2 CAB voters. Again we use Borda utilities and the *alphabetical tiebreaking rule. Borda utilities are used and the tiebreaking rule is alphabetical. Applying iterated removal of dominated strategies under plurality, we have in the first iteration that each voter eliminates the vote for their least preferred candidate. In the second iteration, if the CAB voters vote for their most preferred candidate C then B will win, while if they vote for A, A will win. Thus they eliminate the vote for C and we have A as the final winner.*

Under IRM, only the sincere votes are left after the first iteration, and the winner is B.

Example 3.15. *(IRM differs from strong Nash and coalition-proof equilibrium) A game is in a* strong equilibrium *if no subset of players exists that can profitably deviate. A game is in a* coalition-proof equilibrium *if there is no possible deviation by a set of players that is self-enforcing, meaning that once this deviation has taken place, no further deviations are possible. Under plurality, the winner of an election that is in a strong equilibrium is always the Condorcet winner for that profile [11]. If no Condorcet winner exists, then there is also no strong equilibrium. On the other hand, a candidate is the IRM winner if and only if it is the sincere winner. The plurality rule is not Condorcet consistent, and so there exist profiles where the sincere winner differs from the Condorcet winner, thus for these profiles, the strong equilibrium result and the IRM result differ.*

For example, the following profile under the plurality rule is both a strong equilibrium and a coalition-proof equilibrium. There are 2 *ABC voters,* 2 *BCA voters and* 3 *CBA voters. The winner of this election is the Condorcet winner B.*

The BCA voters have no incentive to change as their most preferred candidate is already winning. The ABC voters and the CBA voters are too few separately to change the outcome of the election, but will not form a coalition together. This is because the CBA voters will not be part of a coalition that makes A the winner, as A is their least preferred candidate, and similarly the ABC voters will not switch their vote to C, as C is their least preferred candidate. Thus there are no possible successful deviating coalitions, and this is both a strong and a coalition-proof equilibrium. This profile could not be produced by iterated regret minimization, as the IRM winner will be the sincere winner C.

4 Experimental results

We carried out detailed numerical computations. The parameters were:

- voting rules: Borda, 2-approval, IRV;
- utilities: randomly chosen in interval $[0, 1]$;
- social welfare measures: utilitarian, egalitarian, net satisfaction;
- values of m (number of candidates) and n (number of voters).

We generated 10000 elections for each given rule and regret utility measure, except for $m = 4$ and $n = 30$, which we restricted to 1000 scenarios for reasons of running time. Profiles were generated from the IC (uniform) distribution. This was done by using the Mersenne Twister random generator to generate utilities for each candidate, for each voter. These utilities were then ordered to give a preference order over the candidates. In cases where a voter had the same utility for two candidates (a very rare event) the profile was discarded.

For each rule, we used the same utility measure to compute regret in order to compute the welfare. The welfare measures used on each profile were as follows. The utilitarian social welfare U is the sum of utilities over all voters, while the egalitarian social welfare E is the minimum over all voters. In cases where more than one winner was produced by IRM, the winner with the lowest welfare was used.

Table 1: Summary statistics with $m = 4$: Borda (B) and 2-approval (2A) rule and utilities. % S gives percentage of simulated elections where IRM winner is sincere winner.

Util/rule	n	Mean N	Mean $\overline{U}$	Mean $\overline{E}$	% S
B/B	5	0.010	-0.004	-0.016	90.4
B/B	30	0.0003	-0.0002	0.0	99
2A/B	5	-0.014	-0.027	-0.020	88.9
2A/B	30	-0.00001	-0.001	-0.00001	100
2A/2A	5	0.013	-0.013	0.0	100
2A/2A	30	0.0	0.0	0.0	100

Net satisfaction: the difference N between the number of voters who preferred the IRM outcome to the sincere outcome, and the number who preferred the sincere outcome to the IRM outcome.

Utilitarian difference: the difference $\overline{U}$ between U under the IRM outcome and the maximum possible value of U (the "socially optimal outcome").

Egalitarian difference: the difference $\overline{E}$ between E under the IRM outcome and the maximum possible value of E.

In order to make meaningful comparisons, we normalized the welfare measures by the maximum possible value. For example, the maximum Borda welfare score of a candidate is $n(m-1)$, and the maximum when random utilities are generated is n. Our normalized welfare measures below all lie between -1 and 1.

We computed, in addition to the sincere and IRM winner, the optimal winner with respect to each measure of social welfare.

To compute regret of actions, we used a straightforward algorithm based on the definition. In order to speed up the algorithm, we used compilation functions as in [15]. For example, for scoring rules, only the scores of each candidate in a partial profile are relevant when regret computations are being made. We therefore generated possible score vectors directly. Since the scoring rules we dealt with have differences of 0 or 1 between successive elements of the weight vector, possible score vectors correspond to *compositions* (ordered sums of nonnegative integers) of $(n-1)\sum_i s_i$ with no part larger than $(n-1)\max s_i$. These can be generated lexicographically in a standard way.

We implemented the algorithm above in Java 1.7.0_25 via Eclipse. Source code and raw data files for all experimental output are available from the first author on request. Data analysis was done via a standard spreadsheet program.

The results for Borda rules with Borda utilities are summarised in Table 1, where m is the number of candidates and n is the number of voters.

We computed more detailed results when $m = 4$. Figure 1 displays the mean net satisfaction for $m = 4$ and $1 < n \leq 20$.

Table 2 gives the IRM winners by sincere ranks for the Borda rule and 2-approval rule using all three measures of utility.

A subset of these computations was carried out for IRV. In these cases, interesting results can be obtained even when $m = 3$. Figure 2 shows the fraction of IRM winners at each position using both utilitarian and egalitarian Borda ranking. Note that sincere ranking is not computed, because the standard version of IRV does not rank all candidates.

Figure 1: Mean net satisfaction ($m = 4$). Horizontal axis: n. Vertical axis: N.

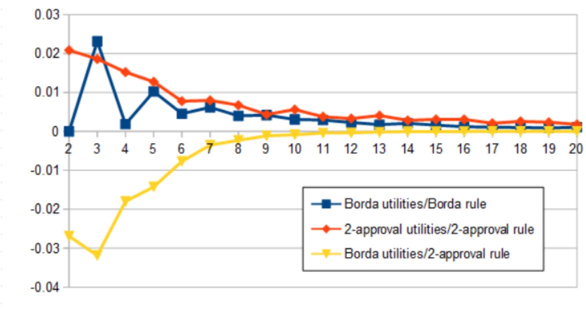
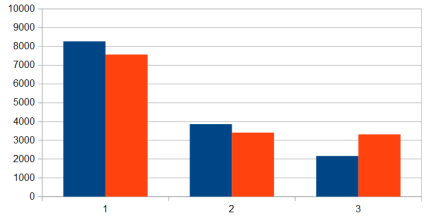


Table 2: Sincere rank of IRM winner: $m = 4, n = 5$.

Utility/rule	Sincere rank			
	1	2	3	4
Borda/Borda	9037	941	69	7
Borda/2-approval	8885	1002	151	60
2-approval/Borda	8285	2408	295	27
2-approval/2-approval	8992	981	150	36
Random/Borda	9037	941	69	7
Random/2-approval	8885	1002	151	60

Figure 2: Rank of IRM winner: Borda utilities, IRV, $m = 3, n = 5$. Left bars, U; right, E.



4.1 Interpetation

4.1.1 Sincerity

The first point to note is that sincere voting occurs very often, and as n increases for m fixed, it becomes more frequent, while as m increases with n fixed, it becomes less common. Thus the overwhelming number of data points lead to values of 0 for net satisfaction, for example. This bias toward sincerity is common to all solution concepts where players do not form binding coalitions: voting sincerely makes sense if one cannot change the outcome.

However, IRM leads more often than other solution concepts to sincere outcomes even for small values of m and n, and not all voters need vote sincerely for this to happen. This is because of the pessimistic nature of regret minimization. The following example is one in which the IRM winner is equal to the sincere winner but not all voters vote sincerely.

Example 4.1. *(why sincere outcomes are so common)*

The rule is Borda and Borda utilities are used. Tie-breaking is alphabetical.

Voters	*Preference order*
v_1	*ACDB*
v_2	*ADBC*
v_3	*BCAD*
v_4	*CBAD*
v_5	*DBCA*

In the first iteration every voter eliminates all votes that do not have their least preferred candidate last and most preferred candidate first, leaving each with two votes. Of the $2^5 = 32$ *possible combinations of votes, all but 4 of them elect the sincere winner A. The first voter* v_1 *eliminates the sincere vote and votes insincerely, while the remaining four voters eliminate the insincere vote and vote sincerely.*

Sincere voting was less common with IRV than with scoring rules, which accords with intuition. Being not positively responsive, this rule can easily lead to scenarios in which sincere voting hurts an otherwise winning candidate.

Table 2 shows that the level of sincerity in the outcomes is robust to the choice of utilities. For both Borda and 2-approval the number of winners of each sincere rank is the same regardless of whether Borda utilities or random utilities are used. Because there were only four candidates, two of which are always fixed after the first iteration, in most scenarios the majority of possible profiles from the second iteration onwards elect the same winner, with a small proportion possibly electing another candidate. When this is the case, the source of positive regret for all voters is the difference in utility between these two candidates and so the results will be the same regardless of whether Borda utilities or random utilities are used.

4.1.2 Welfare

We first discuss the Borda/Borda case. For many of the scenario sets tested, the mean net satisfaction was positive despite the mean utilitarian difference being negative, implying that there are scenarios where most voters prefer the IRM winner, but those who prefer the sincere winner prefer it by a wider margin. As an example of this, consider the following profile with four candidates and five voters.

Example 4.2. *(net satisfaction vs utilitarian difference)*

The rule is Borda and Borda utilities are used. Tie-breaking is alphabetical.

Voters	*Preference order*
v_1	$ADCB$
v_2	$BADC$
v_3	$BCAD$
v_4	$DBAC$
v_5	$DCAB$

The sincere winner is D with 9 points, while the IRM winner is A, with 8 points. Thus net satisfaction is 0.2*, since 3 voters prefer A to D. However the two voters who prefer D to A contribute* -0.8 *to utilitarian difference while the three other voters contribute* 0.6*, so* $U = -0.2$.

The utilitarian difference and egalitarian difference can by definition never be positive in this case, but we do find in Table 1 that the average utilitarian difference is very close to 0 for all scenario sets. The magnitude of the maximum value for the net satisfaction was never bigger than the magnitude of the minimum value, which could indicate that, in those scenario sets where the net satisfaction was positive, there were many scenarios in which a small number of voters preferred the IRM outcome to the sincere outcome, with only a small number of scenarios where a large number of voters preferred the sincere outcome. For example, if we look at the set of scenarios with $m = 4$ and $n = 5$, the number of scenarios with the maximum net satisfaction of 0.2 is 769, while there are only 18 scenarios with the minimum net satisfaction of -0.6.

The results for the Borda rule and 2-approval utilities are very similar to those for the Borda rule and Borda utilities, with all averages close to 0. However in this case the net satisfaction was never above 0 and for $n = 5$ the minimum net satisfaction was -1, indicating that at least one scenario existed in which all of the voters preferred the sincere winner to the IRM winner. This is mostly due to the difference between the Borda utilities used to find the regrets, and the 2-approval utilities used to find the winner: the lowest net satisfaction often occurs in situations where two candidates tie or are close to tying under 2-approval, as both are ranked first or second by many candidates, but have very different Borda scores, because their rankings among those voters that place at least one of them below second are very different. If we look instead at the net satisfaction using 2-approval with 2-approval utilities rather than Borda utilities, then the averages are higher.

5 Conclusion and future work

Our work so far has established:

- IRM has a strong bias toward sincere outcomes;
- when the sincere winner is not elected by IRM, the IRM winner often has majority support over the sincere winner;
- under IRM, the overall loss in welfare owing to strategic behaviour is rather small, at least for scoring rules.

To the extent that IRM is believed to be a realistic solution concept, this substantially mitigates concerns about manipulation in voting games. Our results fit into a growing body of work on welfare effects of strategic voting. For example, [15] presents simulation experiments for sequential voting under plurality and antiplurality where the backward induction solution gave positive net satisfaction on average, while [14] shows that a refinement of

Nash equilibrium often leads to good outcomes under plurality. Earlier simulation results along these lines were obtained by Lehtinen [8, 9]. While socially bad outcomes can occur in the worst case with any solution concept (for example [2]), our preliminary investigations show that for solution concepts which presuppose a high level of information and reasoning about other players, such improvements in welfare often occur, whereas for simple heuristic attempts to manipulate, the overall outcome is generally worse than with sincere voting. Clearly, this issue deserves further study.

There is a difference in the behaviour of IRM when using a positively responsive rule such as a scoring rule, and a rule that is not, such as IRV. This should be explored further.

As pointed out by a referee, our mapping of utility functions to preference orders is not the only possibility. A more thorough investigation using ideas such as in [3, 1] would be interesting.

We have used only synthetic data. Checking the performance of IRM on real preference data, on data with only partial preferences, or in general studying how realistic a solution concept IRM is when voting games are played in practice, is another very interesting direction of research.

We thank numerous anonymous referees for their comments on various versions of this paper.

References

[1] Craig Boutilier et al. "Optimal social choice functions: a utilitarian view." *ACM Conference on Electronic Commerce* (2012), 197–214.

[2] Simina Brânzei et al. "How Bad Is Selfish Voting?" In: *AAAI.* Ed. by Marie desJardins and Michael L. Littman. AAAI Press, 2013.

[3] I Caragiannis and A D Procaccia. "Voting almost maximizes social welfare despite limited communication". *Artificial Intelligence* 175.9-10 (2011), 1655–1671.

[4] Jianying Cui and Xudong Luo. "A Unified Epistemic Analysis of Iterated Elimination Algorithms from Regret Viewpoint". In: *Logic, Rationality, and Interaction.* Ed. by Davide Grossi, Olivier Roy, and Huaxin Huang. Vol. 8196. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2013, 82–95.

[5] A. Dhillon and B. Lockwood. "When are plurality rule voting games dominance-solvable?" *Games and Economic Behavior* 46.1 (2004), 55–75.

[6] Emmanuel Filiot, Tristan Le Gall, and Jean-François Raskin. "Iterated Regret Minimization in Game Graphs". In: *MFCS.* Ed. by Petr Hlinený and Antonín Kucera. Vol. 6281. Lecture Notes in Computer Science. Springer, 2010, pp. 342–354.

[7] Joseph Y. Halpern and Rafael Pass. "Iterated regret minimization: A new solution concept". *Games and Economic Behavior* 74.1 (2012), 184–207.

[8] Aki Lehtinen. "The Borda rule is also intended for dishonest men". *Public Choice* 133.1 (Oct. 2007), 73–90.

[9] Aki Lehtinen. "The welfare consequences of strategic behaviour under approval and plurality voting". *European Journal of Political Economy* 24.3 (Sept. 2008), 688–704.

[10] Tyler Lu and Craig Boutilier. "Robust Approximation and Incremental Elicitation in Voting Protocols". In: *IJCAI.* Ed. by Toby Walsh. IJCAI/AAAI, 2011, pp. 287–293.

[11] Matthias Messner and Mattias K Polborn. "Strong and coalition-proof political equilibria under plurality and runoff rule". *International Journal of Game Theory* 35.2 (2007), 287–314.

[12] Leonard J Savage. "The theory of statistical decision". *Journal of the American Statistical Association* 46.253 (1951), 55–67.

[13] Murat R. Sertel and M. Remzi Sanver. "Strong equilibrium outcomes of voting games are the generalized Condorcet winners". *Social Choice and Welfare* 22.2 (2004), 331–347.

[14] David Robert Martin Thompson et al. "Empirical analysis of plurality election equilibria". In: *AAMAS.* Ed. by Maria L. Gini et al. IFAAMAS, 2013, pp. 391–398.

[15] Lirong Xia and Vincent Conitzer. "Stackelberg Voting Games: Computational Aspects and Paradoxes". In: *AAAI.* Ed. by Maria Fox and David Poole. AAAI Press, 2010.

Miranda Emery
Department of Computer Science, University of Auckland
Auckland, NEW ZEALAND
Email: `miranda.j.emery@gmail.com`

Mark C. Wilson
Department of Computer Science, University of Auckland
Auckland, NEW ZEALAND
Email: `mcw@cs.auckland.ac.nz`