A Scale Invariant Surface Curvature Estimator

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Abstract. In this paper we introduce a new scale invariant curvature measure, *similarity curvature*. We define a similarity curvature space which consists of the set of all possible similarity curvature values. An estimator for the similarity curvature of digital surface points is developed. Experiments and results applying similarity curvature to synthetic data are also presented.

1 Introduction

There exist numerous well known practical 3D shape applications in computer vision including 3D scan matching, alignment and merging [1], 3D object matching, and 3D object classification and recognition [2]. 3D object databases are an active research area.

It is generally useful to seek *invariant* properties when characterizing 3D objects. At a minimum, translation and rotation invariant characterization are desired as clearly neither of these transformations alters the essential shape property of an object. Surface curvature, a rotation and translation invariant property, meets this requirement [3].

Any characterization that is additionally *scaling invariant* enables determining the equivalence of shapes independent of size. Perhaps not so obvious, practically speaking, this scale invariance would also enable the use of uncalibrated measurement units in 3D digitization (e.g. scanning).

1.1 Curvature

A number of different curvature measures are defined in differential geometry. Curvature is well defined for continuously differentiable lines and surfaces [3]. Planar lines have only a single curvature measure whilst surfaces have a number of curvature measures, all of which are based on *normal curvature*.

On surfaces, two principle curvatures, κ_1 and κ_2, are defined, where κ_1 is the minimum normal curvature and κ_2 is the maximum normal curvature at a given point.

Historically, the *mean curvature* has then been defined as

$$H = (\kappa_1 + \kappa_2)/2$$

and the *Gaussian curvature* defined as

$$K = \kappa_1 \kappa_2$$

Additionally, the curvature measure *curvedness* has been defined as

$$C = \sqrt{(\kappa_1^2 + \kappa_2^2)/2}$$

None of the above curvature measures are scale invariant. Also note that, with digital data, there is inherent discontinuity and curvature can only be estimated [4].

1.2 Geometric Invariants

As previously noted, with existing surface curvature definitions we already have translation and rotation invariance. What we now seek is scaling invariance. Shape characterization based on moments has been studied since [5], with varying emphasis on invariance with respect to translation, rotation, reflection, or scaling.

Related work [6], among other things, generalizes and extends the invariance concepts contained in affine differential geometry. Affine invariance is stronger than what we seek in that it includes, for example, squash and stretch transformations.

A number of authors touch on scaling invariant properties in their exploration of multi-scale properties. For example, in [2], firstly surface feature points such as maximum curvature locations are identified. Then triples of feature points are combined using a geometric hashing algorithm in a way that is scaling invariant. Hash tables for various objects of interest are statistically compared to check for similarity matches between different objects.

2 A Scale Invariant Curvature Measure

In this paper, we present a scale invariant curvature measure that can be assigned *at every point on a surface*. We keep in mind that any definition of scale invariant surface curvature must be related to geometric similarity in which it is well known that 1) angles are preserved and 2) ratios of lengths are preserved.

2.1 Similarity Curvature

We begin with some definitions.

Definition 1. *The curvature ratio* κ_3 *is defined as*

$$\kappa_3 = \frac{\min(|\kappa_1|, |\kappa_2|)}{\max(|\kappa_1|, |\kappa_2|)}$$

In the case when κ_1 and κ_2 are both equal to zero, κ_3 is defined as being equal to zero. Note that $0 \le \kappa_3 \le 1$.

Definition 2. *The curvature measure R at a frontier point p is defined as*

$$R(p) = \begin{cases} (\kappa_3, 0) & \text{if signs of } \kappa_1 \text{ and } \kappa_2 \text{ are both positive,} \\ (-\kappa_3, 0) & \text{if signs of } \kappa_1 \text{ and } \kappa_2 \text{ are both negative,} \\ (0, \kappa_3) & \text{if signs of } \kappa_1 \text{ and } \kappa_2 \text{ differ, and } |\kappa_2| \ge |\kappa_1|, \\ (0, -\kappa_3) & \text{if signs of } \kappa_1 \text{ and } \kappa_2 \text{ differ, and } |\kappa_1| > |\kappa_2|. \end{cases}$$

Note that $R(p) \in \mathbb{R}^2$.

We define a *smooth compact 3D set* such that it is compact (i.e., connected, bounded and topologically closed) and curvature is defined at any point of its frontier (i.e., its surface is differentiable at any point).

Theorem 1. *The curvature measure R is (positive) scaling invariant, for any smooth 3D set.*

Proof. Consider a point on a surface and any associated normal curvature κ, as well as the resultant normal curvature κ' after scaling the surface by a factor s. Since, by definition, $\kappa = d\alpha/dl$, and scaling alters length but not angle, we have $\kappa' = \kappa/s$. (Note that the proof could also proceed by considering the effect that scaling has on the osculating circle associated with planar curvature.) Therefore, after scaling, both of the principle curvatures change by the same factor, and the ratio of the principle curvatures is unchanged. Also, neither the signs, nor the relative magnitudes of the principle curvatures are changed by scaling. □

Henceforth, we will refer to the curvature measure R as the *similarity curvature.*

2.2 The Similarity Curvature Space

Since the set of all possible values of the similarity curvature R is a subset of $\mathbb{R}^2$, it is natural to consider a two-dimensional plot representation. Also recall, from differential geometry, that all surface patches on continuous smooth surfaces are locally either elliptic, hyperbolic (saddle-like) or planar.

We introduce the similarity curvature plot template in Figure 1. The horizontal E-axis is for curvature values at locally elliptic surface points. The vertical H-axis is for curvature values at locally hyperbolic surface points. Plotted similarity curvature values will never be off the axes.

Note that the similarity curvature at every point on all spheres from the outside is constant and equal to $(1, 0)$. The similarity curvature on all spheres from the inside is $(-1, 0)$. The similarity curvature on every planar surface, every cylinder and every cone is constant[3] and equal to $(0, 0)$. Note that this is exactly where the Gaussian curvature is equal to zero.

[3] Excluding the cylinder and cone edges and the cone apex.

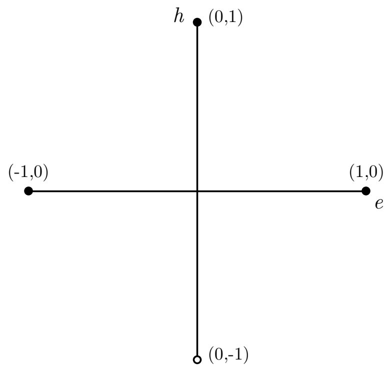

Fig. 1. EH-plot space for similarity curvature.

Continuous motion through points on a smooth surface, taking the similarity curvature at each point, results in a related continuous motion in the similarity plot space. We observe, for example, that it is not possible to traverse from similarity curvature (-1,0) to similarity curvature (1,0) without going through similarity curvature (0,0).

However, it is possible to traverse from (0,1) directly to (0,-1). This can be described by saying that the H axis *wraps around*. An example where this wrapping occurs will be given later in this paper.

2.3 Similarity Curvature Estimation

Similarity curvature is estimated using the following process. Firstly the mean and Gaussian curvatures are estimated. The principle curvatures are calculated from the mean and Gaussian curvatures as follows:

$$\kappa_1 = H - \sqrt{H^2 - K}$$

$$\kappa_2 = H + \sqrt{H^2 - K}$$

Then the (estimated) similarity curvature is calculated from the principle curvatures using the definition given earlier in this paper.

The mean and Gaussian curvatures are estimated as done by other authors [7]. Firstly, with reference to the left side of Figure 2, we consider a scan point and, say, six adjacent points. The points are thought to be connected by *edges*, and edges enclose, in this case, six *faces*. We also identify a central angle α_n associated with each face f_n, and each face f_n has area $\mathcal{A}(f_n)$.

The Gaussian curvature is estimated by

$$\tilde{K} = \frac{3(2\pi - \sum \alpha_n)}{\sum \mathcal{A}(f_n)}$$

On the right side of Figure 2, we identify a surface normal vector associated with each face from an edge-on view point. The angle between adjacent face

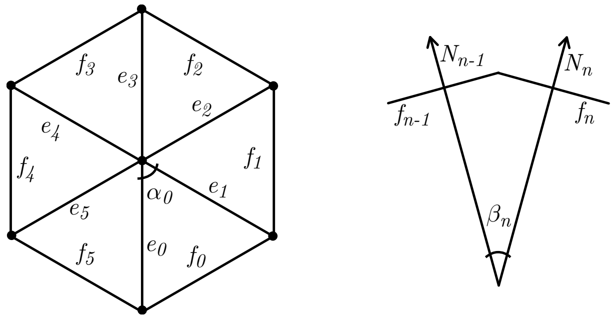

Fig. 2. Curvature estimators: point adjacency on the left and face normals on the right.

normals is designated as β. Angle β is positive if the faces form a convex surface (i.e., when viewed from the outside) and β is negative if the faces form a concave surface (i.e., when viewed from the outside).

The mean curvature is estimated by

$$\tilde{H} = \frac{3 \sum ||e_n|| \, \beta_n}{4 \sum \mathcal{A}(f_n)}$$

3 Similarity Curvature Experiments

Synthetic digital data has been created for a number of objects. Each object was digitized by orthogonally scanning from above using a hexagonal grid pattern. The hexagonal scan grid has a *pitch* dimension as shown in Figure 3. Note that, with this scanning method, only the portion of an object that faces towards the scanning source direction gets digitized.

Reference shapes included a sphere, cylinder, ellipsoid and torus. To evaluate the similarity curvature estimation, we considered each reference shape in turn with a scan pitch of 1, then a 10X scaling, and finally a 10X scan resolution. For the 10X scaling, all dimensions and the scan pitch were increased by a factor of 10. For the 10X resolution, the scan pitch was decreased by a factor of 10.

The resultant similarity curvature values have been accumulated for summary in associated E-axis and H-axis histograms.

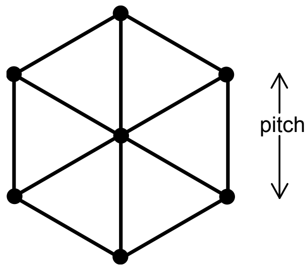

Fig. 3. Scan grid.

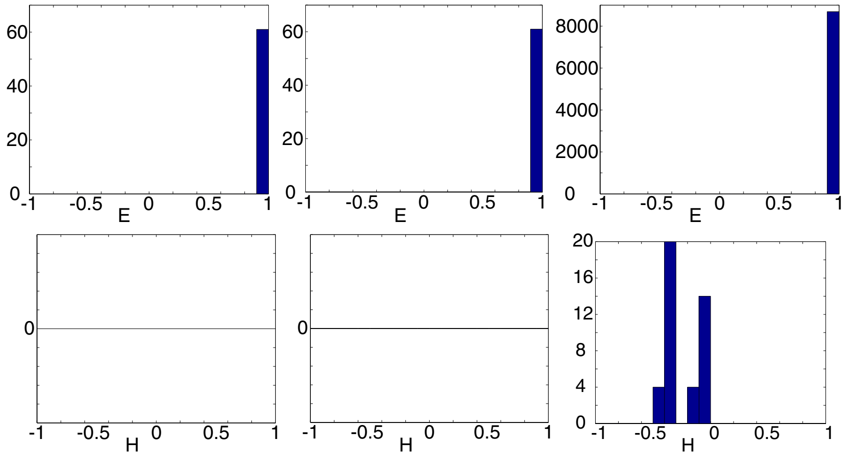

Fig. 4. Sphere EH-histogram: reference, 10x scale, 10x resolution.

Sphere results are shown in Figure 4. The reference sphere has a radius of 5. Observe that the similarity curvature has the constant value of 1 in the E histogram regardless of scale. There are a small number (approximately 0.5 percent) of noise values in the high resolution H histogram.

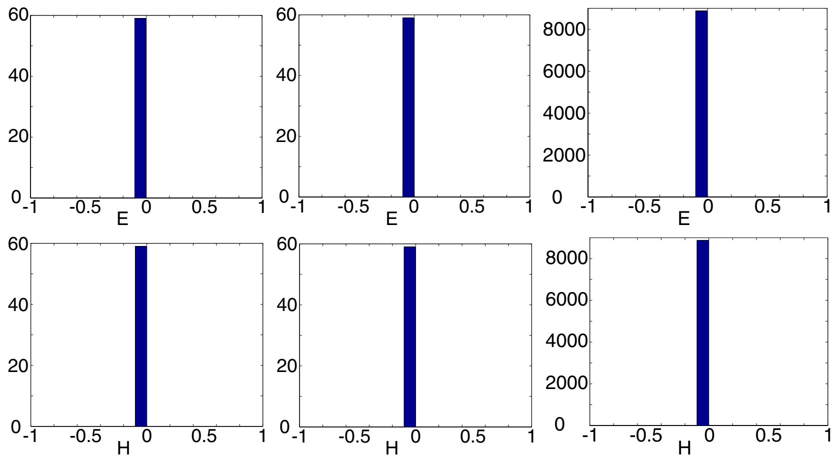

Fig. 5. Cylinder EH-histogram: reference, 10x scale, 10x resolution.

Cylinder results are shown in Figure 5. The reference cylinder has a radius of 5 and a height of 4. As expected, in all cases the E and H values are constant at zero.

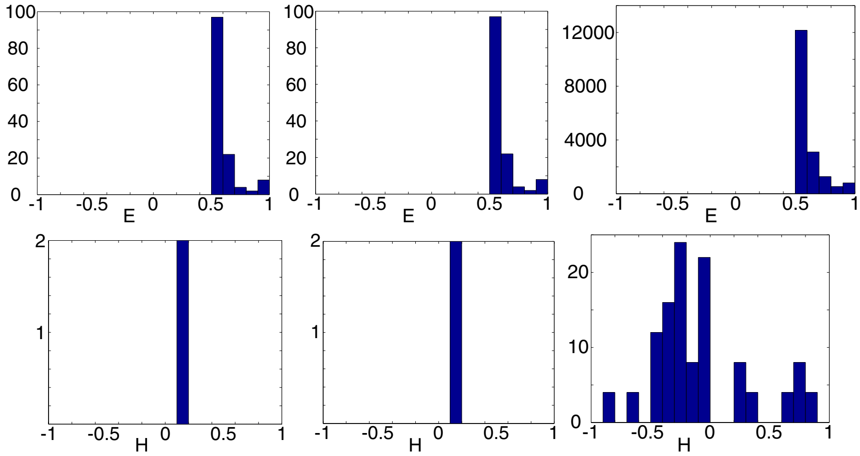

Fig. 6. Ellipsoid EH-histogram: reference, 10x scale, 10x resolution.

Ellipsoid results are shown in Figure 6. The reference ellipsoid has axes equal to 6 and 12. As expected, the E values are bounded by 0.5 and 1. Approximately 2 percent noise has accumulated in each of the H histograms.

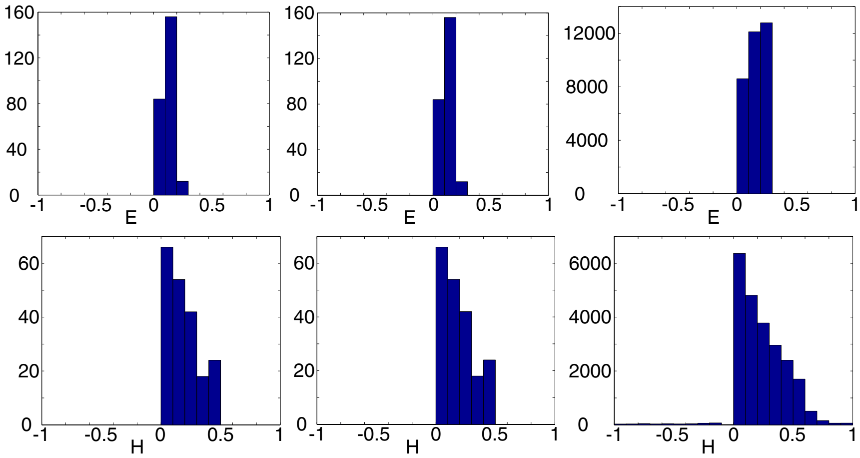

Fig. 7. Torus EH-histogram: reference, 10x scale, 10x resolution.

Torus results are shown in Figure 7. The reference torus has an inner radius of 6 and an outer radius of 14. As expected, based on the associated minimum and maximum curvatures, the E values are bounded by 0 and 0.29, and the H values are bounded by 0 and 0.67.

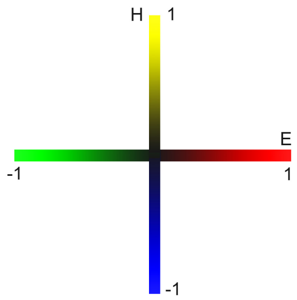

Fig. 8. Shading coded EH-plot axis. (For color see the online version of this LNCS publication.)

3.1 Shading Coded Similarity Curvature

It is possible to assign color coding to similarity curvature values. A color coding of EH-plot axis is shown in Figure 8 where negative E values are green, positive E values are red, negative H values are blue and positive H values are yellow. Shading coded values for similarity curvature can also be used to color each surface point on test objects.

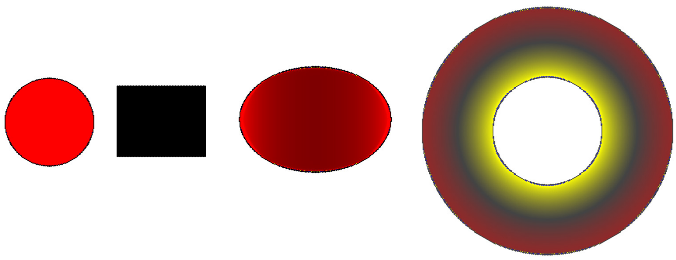

Fig. 9. Shading coded similarity curvature maps: sphere, cylinder, ellipsoid, torus.

Shading coded *curvature maps* have been introduced in previous work by the authors [8, 9]. A shading coded similarity curvature map for each of the test shapes is shown in Figure 9. The constant curvature of the sphere and the cylinder as well as the smooth transition through curvature values in the curvature maps of the ellipsoid and the torus are readily apparent.

Figure 10 shows the case of a torus having an inner radius of zero. Note the H-axis wrapping near the center of the torus. The bright yellow color transitions change abruptly to bright blue when the H-axis similarity curvature wraps around, changing sign.

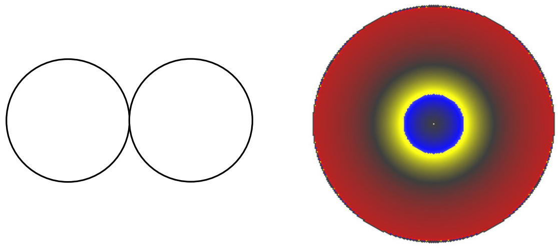

Fig. 10. Torus cross section on left and shading coded similarity curvature.

3.2 3D Object Detection

Similarity curvature can be used to identify and extract 3D shapes from within complex 3D scan scenes. We may wish to, for example, identify all spheres and spherical patches within a scene no matter what the sphere size or scan resolution.

We have constructed a test scan scene containing a surface with five each spherical, ellipsoid and toroid bumps as well as five pits having those same shapes. Some of the pits and bumps overlap. Several representations of the scene are shown in Figure 11. On the left is a shading coded depth map in which points closer to the scan source are white color shaded, and more distant points are shaded black. A color coded similarity curvature map is shown in the middle. Spherical bumps are bright red and spherical pits are bright green. Finally, all of the spherical bump surface patches have been identified and color coded as white in the image on the right.

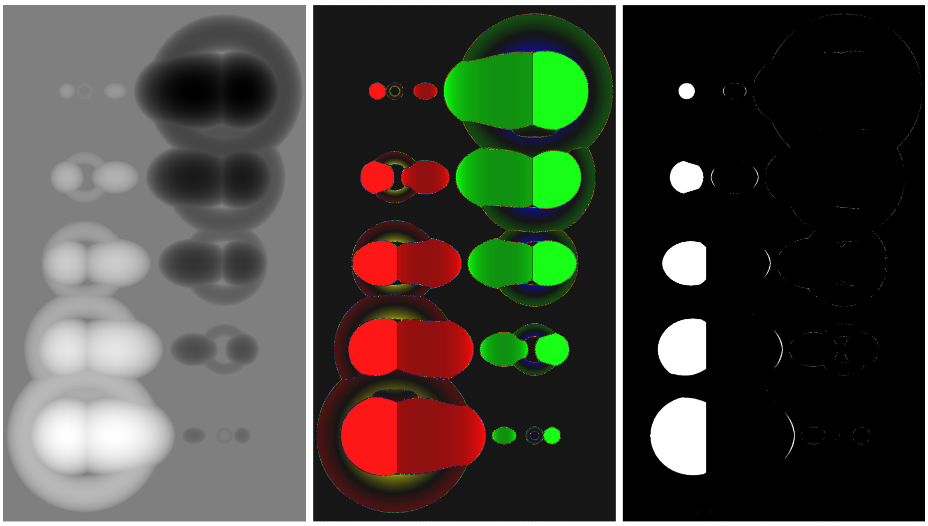

Fig. 11. Test scene depth map, curvature map, and extracted spherical patches.

4 Conclusion

Similarity curvature measure has been defined and an estimator has been introduced and tested. EH-plots as well as a color shaded coding have been presented. Experiments have demonstrated that similarity curvature can be used to characterize and identify simple synthetic digitized shapes. Further work is anticipated to include applying similarity curvature measure to real world scans and addressing the issue of noisy data.

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References

1. Levoy, M., Pulli, K., Curless, B., Rusinkiewicz, S., Koller, D., Pereira, L., Ginzton, M., Anderson, S., Davis, J., Ginsberg, J., Shade, J., Fulk, D.: The digital Michelangelo project: 3D scanning of large statues. In: Proc. SIGGRAPH. (2000) 131–144
2. Mokhtarian, F., Bober, M.: *Curvature Scale Space Representation: Theory, Applications, and MPEG-7 Sandardization.* Kluwer, Dordrecht (2003)
3. Davies, A., Samuels, P.: *An Introducion to Computational Geometry for Curves and Surfaces.* Oxford University Press, Oxford (1996)
4. Klette, R., Rosenfeld, A.: *Digital Geometry.* Morgan Kaufmann, San Francisco (2004)
5. Hu, M.: Visual problem recognition by moment invariants. IRE Trans. Inform. Theory **8** (1962) 179–187
6. Sapiro, G.: *Geometric Partial Differential Equations and Image Analysis.* Cambridge University Press, Cambridge (2001)
7. Alboul, L., van Damme, R.: Polyhedral metrics in surface reconstruction. In Mullineux, G., ed.: *The Mathematics of Surfaces VI*, Oxford, Clarendon Press (1996) 171–200
8. Rugis, J.: Surface curvature maps and Michelangelo's David. In McCane, B., ed.: Image and Vision Computing New Zealand. (2005) 218–222
9. Rugis, J., Klette, R.: Surface registration markers from range scan data. In Reulke, R., Eckardt, U., Flach, B., Knauer, U., Polthier, K., eds.: Proceedings, Combinatorial Image Analysis: 11th International Workshop, IWCIA. (2006) 430–444