The Creator versus Its Creation. A Mathematical Exercise*

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Abstract

We suggest two set-theoretical approaches to the God problem (existence, infinity, unicity, and unity), both based on H. Cartan's concept of filter. The first approach is alluded by Odifreddi's analysis of the structure of Gödel's ontological proof and considers God as a special, first being, essentially deviating from created beings. In the second approach, God is considered as an extrapolation (idealized representation) of the set of all (created) beings and it is related to the cofinite topology and to a previous model, by Marcus, of the lyrical language. We are permanently guided by Duns Scotus's basic requirements related to the existence, the unicity and the actual infinity of God. We finish by identifying a "God-topology" which is connected, but non-metrisable, showing the unity of God; partial, fractal, fragile relations cannot ascend to a genuine effigy.

1 Introduction

The aim of this paper is to discuss the problem of the mathematical modelling of God, in the rather general framework of the relations between a creator and its created work.[1] The creator is sometimes called *The First Being*, while the corresponding creation is represented by the set of all created beings. This convention was introduced by *Duns Scotus*, who has also discussed the existence,

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[1]In a cybernetic perspective, this problem was considered by N. Wiener. *God and Golem*, MIT Press, Cambridge, Mass., 1964; many authors have discussed this problem from a logic perspective, e.g. P. E. Oppenheimer, E. N. Zalta. On the logic of the ontological argument, in J. Tomberlin (ed.). *Philosophical Perspectives 5: The Philosophy of Religion*, Atascadero, Ridgeview, 1991, 509-529.

the (actual) infinity, and the unicity of the First Being.[2] We shall approach this problem in two ways. In the next section we introduce a set-theoretical model concerning the interaction First Being-created beings, where the former is defined by a maximality property with respect to the latter. In the final section we consider the idea of First Being viewed as a global representation of all created beings, an approach which is concordant with the attitude of some thinkers for whom God should be humanized, giving to each being (and, mainly, to each human being) the possibility to apprehend a portion of the First Being.[3] In this second perspective there is no sharp distinction between humans (beings, in general) and God, because God is to some extent in each of us and mainly in the interaction among beings.[4]

Both of the above strategies are based on the structure of a filter. The option for this structure is motivated by the fact that a filter is a way to choose the "biggest" among the subsets of a set. This is consistent with those representations of God as an entity which is "bigger", in some respects, than the usual beings. From this point of view, filters are opposed to classes of negligible sets, having a preference for "small" sets.

We finally construct a "God-topology", which is connected, but not metrisable. It models the unity of God.

2 The Creator as an Ultrafilter

In this section we present a set-theoretical model for the *First Being*, and we prove its "unicity" and infinity. This model has been suggested by Odifreddi's analysis[5] of Gödel's ontological proof.[6]

Let us denote by X a class of entities; $\mathbf{P}(X)$ refers to the set of all subsets of X. The couple $\mathbf{b} = (E, I)$ is called a *created being* if E is a subset of X and I is a collection of subsets of X such that the following two conditions are fulfilled:

1° For all Y in I, $E \subset Y$.

[2] Duns Scotus. *Philosophical Writings*, Nelson, Edinburgh, 1962; chronologically, the ontological proof was discussed 200 years before by St. Anselm (c. 1033-1109); cf. *Opera Omnia*, F. S. Schmitt (ed.), Edinburgh, 1946-1962, six volumes.

[3] Cf. J. Bishop. *Exploring Alternative Concepts of God*, manuscript, Auckland University, March 1994, 10 pp.

[4] "Then God said, 'Let us make man in our image and likeness to rule the fish in the sea, the birds of heaven, the cattle, all wild animals on earth, and all reptiles that crawl upon the earth.' So God created man in his own image; in the image of God he created him; male and female he created them.", cf. *The New English Bible, The Old Testament*, Oxford University Press, Cambridge University Press, 1970, Genesis I, 2, 26-28, p.2.

[5] Cf. P. Odifreddi. *La prova di Dio*, Preprint, Università di Torino, 1994, 8 pp. and P. Odifreddi. *Ultrafiltri, dittatori e dei*, Preprint, Università di Torino, 1994, 9 pp.

[6] Odifreddi's starting point is Gödel's axiomatic restatement of the ontological proof of the existence of God. J. Sobel. Gödel's ontological proof, in J. J. Thomson (ed.). *On Being and Saying*, MIT Press, 1987, 241-261 includes a detailed analysis of Gödel's proof. It also contains Gödel's original two pages note entitled "Ontologischer Beweis" and D. Scott three pages notes titled "Gödel's Ontological proof"; both notes are handwritten.

2° For every distinct superset I' of I there exists a subset Y in I' and an element e in E which is not in Y.

The elements of I are subsets of X, so they correspond to properties of entities in X. Condition 1) says that all elements in E satisfy all properties in I, while condition 2) is aimed to the converse relation: we put in I *all* properties of entities in E.

It is not difficult to see that the couple $\mathbf{b} = (E, I)$ is a created being if and only if the following two conditions are fulfilled:

a) For every x in X, x is in E if and only if x has every property p in I.

b) A property p is in I if and only if every x in E has the property p.

The set I of the created being $\mathbf{b} = (E, I)$ consists of all characteristic or essential features of the class of entities E. There are two opposite poles: $(\emptyset, \mathbf{P}(X))$ represents the *null* being in which we consider all possible properties (and, of course, the extension is empty), and the *universal* being $(X, \{X\})$ (no specific properties, all entities).[7] It is interesting that the maximality of the extension of an created being is somewhat "trivial" (as it gives rise only to the universal created being) in contrast with the maximality of the intension. We will come back to this problem later.

As expected, for all pairs of created beings $\mathbf{b}$, $\mathbf{b}'$

$$E \subset E' \text{ if and only if } I' \subset I,$$

i.e. the larger the extension of a created being is, the smaller its intension becomes, and conversely.

Two created beings $\mathbf{b} = (E, I)$, $\mathbf{b}' = (E', I')$ are called *isomorphic* if there is a bijective function of E to E'.

> *If the created beings* $\mathbf{b}$ *and* $\mathbf{b}'$ *are isomorphic, then there exists a natural bijective function from* I *onto* I'*.*

To prove this, we start by noting that for every Y in I there exists a unique $Z \subset X \setminus E$ such that[8]

$$Y = E \cup Z.$$

Let $u : E \longrightarrow E'$ be a bijection. Denote by $u^* : X \longrightarrow X$ the bijection such that $u^*(e) = u(e)$, for every $e \in E$. Finally, define

$$U : I \longrightarrow I' \text{ by } U(Y) = u(E) \cup u^*(Z),$$

where $Y = E \cup Z$ is the above decomposition of Y.

Next, we verify that U establishes a bijection from I onto I'. Indeed, U is *one-one*: if

$$U(Y_1) = u(E) \cup u^*(Z_1),\ U(Y_2) = u(E) \cup u^*(Z_2),$$

[7] It is easy to see that $\mathbf{b} = (\emptyset, I)$ is a created being if and only if $I = \mathbf{P}(X)$, and $\mathbf{b} = (X, I)$ is a created being if and only if $I = \{X\}$.

[8] Obviously, $E \cap Z = \emptyset$.

then $u^*(Z_1) = u^*(Z_2)$, that is $Z_1 = Z_2$, and therefore $Y_1 = Y_2$.

To show that U is *onto* let us consider Y' in I'. Again $Y' = E' \cup Z'$, with $E' \cap Z' = \emptyset$. Let $Y = E \cup (u^*)^{-1}(Z')$. In view of the relation $u(E) \cap u^*(Z') = \emptyset$ it follows that

$$U(Y) = u(E) \cup u^*((u^*)^{-1}(Z')) = E' \cup Z' = Y'.$$

So U is onto, therefore it establishes a bijection between I and I'.

It is interesting to see that the notion of created being has a nice mathematical (topological) structure. To see it, we recall the notion of filter introduced by H. Cartan.[9] A non-empty set $\mathbf{F} \subset \mathbf{P}(X)$ is a *filter* if the following two conditions are fulfilled:

1° If $X \in \mathbf{F}$ and $Y \in \mathbf{F}$, then $X \cap Y \in \mathbf{F}$.

2° If $X \in \mathbf{F}$ and $X \subset Y$, then $Y \in \mathbf{F}$.

A filter $\mathbf{F}$ is *proper* if $\mathbf{F} \neq \mathbf{P}(X)$, or, equivalently, $\emptyset$ is not in $\mathbf{F}$.

Non-null created beings can be characterized structurally as follows:

> *The couple $\mathbf{b} = (E, I)$ is a non-null created being if and only if I is a proper filter on X.*

Indeed, $I = \{Y \subset X | E \subset Y\}$ is the filter generated by E and $I \neq \mathbf{P}(X)$ as $\mathbf{b}$ is non-null.

According to Gödel, of every property and its negation exactly one is "positive" in a moral sense.[10] To model this idea we use a *choice function*, that is a function $f : \mathbf{P}(X) \to \{0, 1\}$ which satisfies the following condition: for all subsets $Y \subset X$,

$$f(Y) + f(X \setminus Y) = 1.$$

If $f(Y) = 1$, then we say that the property Y is *positive*.

A created being $\mathbf{b} = (E, I)$ is *positive* if there exists a choice function f such that all positive properties according to f are included in I, i.e. for every $Z \subset X$,

$$\text{if } f(Z) = 1, \text{ then } Z \in I.$$

Positive created beings have a *maximality* character which can be described as follows.

A filter $\mathbf{F}$ is called *maximal* (or *ultrafilter*) if for each filter $\mathbf{F}'$ with $\mathbf{F} \subset \mathbf{F}'$ one has $\mathbf{F}' = \mathbf{F}$; equivalently, for every $Z \subset X$, $Z \in \mathbf{F}$ or $X \setminus Z \in \mathbf{F}$.

Let $\mathbf{b} = (E, I)$ be a created being. Then the following statements are equivalent:

> *1) The created being $\mathbf{b}$ is positive.*
>
> *2) The set E has exactly one entity.*

[9] See, for instance, W. W. Comfort, S. Negrepontis. *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974.

[10] Like yin versus yang.

In case 1) or 2), the set I is an ultrafilter, but the converse implication is false.[11] To show that 2) implies 1) we assume that $E = \{e\}$ has only one element, and we define the choice function $f(Z) = 1$ if and only if $e \in Z$. As for every $Z \subset X$, $e \in Z$ or $e \notin Z$, it follows that $I = \{Z \subset X | f(Z) = 1\}$.

Finally we show that 1) implies 2). Suppose that $\mathbf{b} = (E, I)$ is a positive created being with respect to the choice function f. In case E has at least two distinct elements, say a, b, we consider the property $Y = \{a\}$. Both properties $Y, X \setminus Y$ are not in I: indeed, $b \notin Y$ and $a \notin X \setminus Y$. It follows that $f(Y) = f(X \setminus Y) = 0$, contradicting the fact that f is a choice function.[12]

As a consequence we get the following result:

All positive created beings are isomorphic.

For Scotus[13] the *First Being* "has the power enough to produce an infinite number all at once", so it is infinite.[14] The emphasis on "all at once" suggests the whole, the actual sense of infinity,[15] to a large extent a guarantee of the existence itself. Scotus supposes that the *First Being* is maximal and his proof of unicity and infinity (the later property follows from the former) essentially uses this assumption. His assertion according to which *The First Being has a distinct and necessary knowledge of all intelligibles* appears to correspond more to a "maximality principle" than to an unlimited omniscience power. The same idea appears in Gödel's definition: "x is God-like if it possesses all positive properties." Odifreddi pointed out that positive[16] properties involved in Gödel's ontological proof of the existence of a "God-like" being[17] form an ultrafilter.

All these conclusions can be derived in our model of created beings. To be precise we denote by fb the set $\{first\ being\}$ and put $\mathbf{FB} = (fb, [fb])$.

In view of the infinity of the ultrafilter $[fb]$ we have:

Every first being is infinite.

Moreover,

[11] J. L. Bell and A. B. Slomson. *Models and Ultraproducts. An Introduction*, North-Holland, Amsterdam, 1974, prove that the family of cofinite subsets of an infinite set is an example of an ultrafilter which is not principal; this example is relevant for the discussion in Section 3 as well.

[12] From the last proof it follows that not every created being is positive.

[13] John Duns Scotus (c. 1266-1308) was an outstanding scholastic theologian whose work had a profound influence on philosophical thought; his proof concerning the existence and unicity of God is a remarkable piece of metaphysical argumentation.

[14] Op. cit.

[15] With the exception of St. Augustine and a few others, the works of medieval thinkers were dominated by the negative connotation of the Greek word *apeiron* (meaning literally unbounded, but also infinite, indefinite, undefined, even in a pejorative sense, totally disordered). In particular, only the potential sense of the infinite was really considered. In Aristotle words: *"... being infinite is a privation, not a perfection but the absence of a limit...*(*Physics*, III, translated by R. Hope, University of Nebraska Press, 1961, 56.)

[16] In the moral aesthetic sense.

[17] In Odifreddi's terminology, "gods".

For first beings maximality is equivalent to unicity.

It remains to decide if it is possible to choose the unique *First Being* among all possible, equivalent *first beings*.[18] This problem cannot be solved algorithmically, nor even mathematically. The solution, to which many authors seem to subscribe, is based on **revelation, faith,** and **love**. *The act of revelation—which is not to be construed as an interference with natural processes, but which is the ingression of a new creative moment into the course of history—supplies the key to creation, which is itself an act of primeval revelation.*[19] According to Bishop (op. cit.), *"What knowledge we have of God has to be based on what has been revealed within the shared experience of the historical community faith." "Love itself can supply the resources which, of ourselves, we lack."* This solution appears to be consistent with the views of Poincaré and Gödel, as well with the *Randomness Hypothesis*, according to which the Universe is not deterministic but random.[20]

3 The Creator as Its Creation

We shall now change the perspective, by considering the set B of (created) beings as a primitive concept, assuming only the infinity of B. There are many reasons for this assumption. We only mention that the advantage of approximating a very large finite set, of imprecise, variable cardinality, by an infinite set is fundamental in some branches of knowledge, from mathematics and theoretical computer science to philosophy.[21]

On B we consider the family

$$\mathbf{F} = \{A| \text{ the complement of } A \text{ is finite}\}.$$

It is not difficult to recognize that $\mathbf{F}$ is again a filter (in B). Each created being $\mathbf{b}$, as an element of B, and each finite set of created beings (meaning any type of finite associations of created beings, following various possible professional, religious, national, political, or other types of criteria), as a finite subset of B, admits an infinite extension that can be only potentially acquired. In other words, for any created being $\mathbf{b}$, different from some fixed created being $\mathbf{c}$, we can assume the possibility of a moment when $\mathbf{b}$ interacts with $\mathbf{c}$, but by no means this can happen simultaneously with all created beings different from

[18] This set is itself infinite.

[19] Cf. F. A. Rothschild. *Between God and Man. An Interpretation of Judaism, From the Writings of Abraham J. Heschel*, The Free Press, New York, Collier-Macmillan, London, 1959, p.19.

[20] C. Calude, A. Salomaa. Algorithmically coding the universe, in G. Rozenberg, A. Salomaa (eds.) *Developments in Language Theory*, World Scientific, Singapore, 1994, 472-492.

[21] For instance, the field of grammatical inference, so important in the study of learning processes, is entirely based on the approximation of finite languages by some infinite, simpler languages.

c, because created beings have a finite life and their depth of interaction is also finite. The same is true when dealing with finite sets of beings. So, each individual created being (and each finite set of individuals) aspires to an infinite extension, to its relation to the First Being. This extension is only potential. In contrast with this, the *First Being* includes the infinite complementary part of any individual being, and of any finite set of created beings, hence it can cope with them simultaneously. So, the *First Being* possesses the exclusive privilege of actual infinity and the unicity with respect to this property.

The filter **F** gives the possibility to every created being to integrate itself in a God perspective, by virtue of its access to a potentially infinite extension. This includes the whole spiritual life. However, the filter **F** is not "greater than which nothing can be thought", as Duns Scotus required. So, we have to try to extend **F** to a process including an infinite sequence of transfinite cardinals of increasing size, in order to compensate the impossibility of a transfinite cardinal larger than any transfinite cardinal. A possibility would be to consider a filter $\mathbf{F}(1)$ similar to **F**, but instead to be applied to B it acts on the power set of B. Then we have to iterate the operator *"power set of"*, beyond any possible transfinite ordinal. God is represented by the whole transfinite sequence of filters obtained in this way. The first term of this sequence is the proper filter **F** of all cofinite subsets of B.[22] For each being **b** in B, the set $B \setminus \{\mathbf{b}\}$ represents the God-entension of **b**, i. e. the upper limit of its spiritual life. Each **b** captures a smaller or a larger part of its possible extension. At the same time, since the elements in B are (created) beings, it follows that the size of the integration of **b** in a God-perspective is given by its capacity to interact with other beings in B.[23] This fact corresponds to the way God is represented by some authors (see, Bishop, op. cit.).

It deserves to be noticed that the adopted representation of God leads to its existence, infinity and unicity. Moreover, the infinity becomes here the exclusive privilege of God, in contrast with the finiteness of any created being. More precisely, the potential infinity is available to any created being, while the actual infinity is the exclusive privilege of God. This corresponds to the requirements formulated by Duns Scotus (op. cit.).

Another approach to a humanized God finds its roots in a previous mathematical model of the lyrical language,[24] based on filters and topology.[25] Let us represent the set of human beings by the set of positive integers $\{1, 2, \ldots, n, \ldots\}$ and let us denote by $S(n)$ the set of all (poetic) meanings perceived by the individual n. Due to reasons largely explained in Marcus (op. cit.), the set $S(n)$ is

[22] See Bell and Slomson op. cit.

[23] "He is within and beyond all things and ideas.", cf. A. J. Heschel. *Man Is Not Alone: A Philosophy of Religion*, Farrar, Straus and Young, New York, 1951, 219-227.

[24] S. Marcus. Langage scientifique, structures rythmiques, langage lyrique, *Cahiers de Linguistique Théorique et Appliquée*, 5, 1968, 127-157; see also S. Marcus. *Mathematische Poetik*, Athenäum Verlag, Frankfurt/Main, 1973, ch. 4.

[25] A topology is a family of subsets (the open sets) of some fixed set, which includes the whole set and the empty set, and is closed under set union and finite intersection.

of the power of the continuum[26], the set S, defined as the union of all sets $S(n)$, $(n = 1, 2, \ldots)$, is *a fortiori* of the power of the continuum, but we can imagine a larger set S^* obtained by adding to S some meanings that are beyond the human perception.

We shall now construct a family $\mathcal{F}$ by putting together all "enough big" sets of meanings, i.e. $\mathcal{F}$ will include those sets of meanings in S which are so large, that they define the quasi-totality of all human poetic competences. Namely, a subset E of S will belong to $\mathcal{F}$ if E includes nearly all meanings perceived by the quasi-totality of human beings. This family is legitimated to represent God in its hypostasis of supreme lyrical understanding, viewed as the upper limit of the human poetic understanding.

It remains only to give a precise meaning to "quasi-totality" and to "nearly". By "quasi-totality of all human individuals" we understand all human individuals, excepting a finite part[27] of them. A subset E of S includes nearly all meanings in $S(n)$ if the set $S(n) \setminus E$ is at most countable.[28] In this case we say that *the set E captures nearly all meanings in $S(n)$*. Because every poetic sentence has an index of homonymy of the power of the continuum, even when its perception is restricted to a single individual, it follows that the set $S(n, x)$ of meanings of the sentence x perceived by the individual n is of the power of continuum. So, if E captures nearly all meanings in $S(n)$, then E captures nearly all meanings in $S(n, x)$, for any possible sentence x. Indeed, $S(n)$ is the set-theoretic union of the sets $S(n, x)$, for all possible finite strings x over a finite alphabet.

We may now formulate again, more accurately, the definition of the family $\mathcal{F}$. The subset E of S belongs to $\mathcal{F}$ if there exists a positive integer N such that for any n larger than N the difference $S(n) \setminus E$ is at most countable.

It is straightforward that $\mathcal{F}$ fulfills properties 1° and 2° in the definition of a filter.[29] For proving that $\mathcal{F}$ is proper it is enough to note that $S(n)$ is, for each n, of the power of the continuum.[30] There is no chance, however, that $\mathcal{F}$ is an ultrafilter, as it follows from classic facts in set theory.

It deserves to be noticed that the topology over S, obtained by adding to $\mathcal{F}$ the empty subset of S, leads to a connected, but not Hausdorff,[31] topological space.[32] So, the "God-topology" is essentially different from the topology of

[26] Even one poetic sentence restricted to the perception of a single individual develops to an uncountable set of meanings.

[27] Which can be empty.

[28] I.e. finite or infinite, but countable.

[29] The family of countable sets is hereditary and countably additive.

[30] We could replace this condition by the weaker one: $S(n)$ is not empty for infinitely many values of n.

[31] A topological space is connected if it cannot be partitioned into two non-empty open subsets each of which has no points in common with the closure of the other. A closed set is the complement of an open set; the closure of a set is the smallest closed set containing it. A topological space is a Hausdorff space in case every pair of distinct points have a pair of disjoint open neighborhoods.

[32] In general, any filter of sets (plus the empty set) is a topology, being closed under finite

the Euclidean metric space, which is characteristic for the way human beings perceive the Universe at its macroscopic scale.[33] This is another way to point out the fact that the Creator, represented by its infinite creation, is beyond the possibilities of any individual being or any finite set of beings.

Does this contradict the spectacular achievements of our time? Our daily life [34] apparently supports the idea that everything can be explained, that the Universe is rational and ordered, that reality is a simple affair which has only to be organized in order to be mastered. This is just an illusion! To give only one, but a strong example, we refer to the work by G. Chaitin.[35] He has studied the power of sets of axioms by measuring the information that they contain, and was able to prove a stronger version of Gödel's incompleteness result.[36] His result reveals some limitations of the human deductive capability.[37] In his own words:[38] *One normally thinks that everything that is true is true for some reason. I've found mathematical truths that are true for no reason at all. These mathematical truths are beyond the power of mathematical reasoning because they are accidental and random.* We are all surrounded by things which we apprehend, but cannot comprehend. The *reason* itself appears to be a mystery.

The connectedness of the "God-topology" shows its global coherence and unity: all representations of God are intimately related, and partial, fractal, fragile relations cannot ascend to a genuine effigy.

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intersections and arbitrary unions; and, if the filter is proper, then the topology is both connected and not Hausdorff, otherwise there would be two disjoint elements in the filter (in one case by disconnectness, in the other by separation of two elements), and the filter would be trivial.

[33]Even if we interpret the visual space as Non-Euclidean, as some authors argue, it still remains a metric space.

[34]"when the fake becomes indistinguishable from—even more authentic than—the original, when computers can create synthetic worlds that are more realistic than the real world, when technology scorns nature" (cf. B. Woolley. *Virtual Worlds*, Penguin, London, 1992, p.2.)

[35]*Information-Theoretic Incompleteness*, World Scientific, Singapore, 1992.

[36]Precisely, he has shown that an N-bit formal axiomatic system cannot enable one to exhibit any specific object with program-size complexity greater than $N + c$.

[37]See L. Brisson, F. W. Meyerstein. *Inventer l'Univers*, Les Belles Lettres, Paris, 1991, for a philosophical discussion of these issues.

[38]G. Chaitin. Responses to "Theoretical Mathematics", *Bull. Amer. Math. Soc.* 30(1994), 181-182.