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Those who most ignore, least escape.
David Hawkins

Strong Determinism vs. Computability*

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Abstract

Are minds subject to laws of physics? Are the laws of physics computable? Are conscious thought processes computable? Currently there is little agreement as to what are the right answers to these questions. Penrose ([41], p. 644) goes one step further and asserts that: *a radical new theory is indeed needed, and I am suggesting, moreover, that this theory, when it is found, will be of an essentially non-computational character.* The aim of this paper is three fold: 1) to examine the incompatibility between the hypothesis of strong determinism and computability, 2) to give new examples of uncomputable physical laws, and 3) to discuss the relevance of Gödel's Incompleteness Theorem in refuting the claim that an algorithmic theory—like strong AI—can provide an adequate theory of mind. Finally, we question the adequacy of the theory of computation to discuss physical laws and thought processes.

1 Introduction

Penrose [40] (see also [41]) has discussed a new point of view concerning the nature of physics that might underline conscious thought processes. He has argued that it might be the case that some physical laws are not computable, i.e. they cannot be properly simulated by computer; such laws can most probably arise on the "no-man's-land" between classical and quantum physics. Furthermore, conscious thinking is a non-algorithmic activity. He is opposing both strong AI (according to which the brain's action, and, consequently, conscious perceptions and intelligence, are manifestations of computer computations, Minsky [35, 36]), and Searle's [47] contrary viewpoint (although computation does not in itself evoke consciousness, a computer might nevertheless simulate the action of a brain mainly due to the fact that the human brain is a physical system behaving according to (computable) mathematical "laws").

The aim of this paper is to examine the incompatibility between the hypothesis of strong determinism and computability, to give new examples of uncomputable physical laws, and to discuss the relevance of Gödel's Incompleteness Theorem in refuting the claim that an algorithmic theory—like strong AI—can provide an adequate theory of mind. Our starting point is the following paragraph from Penrose [40] p.560:

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> It seems to me that if one has strong determinism, but *without* many worlds, then the mathematical scheme which governs the structure of the universe would probably *have* to be non-algorithmic. For otherwise one could in principle calculate what one was going to do next, and then one could 'decide' to do something different, which would be an effective contradiction between 'free will' and the strong determinism of the theory. By introducing non-computability into the theory one can evade this contradiction—though I have to confess that I feel somewhat uneasy about this type of resolution, and I anticipate something more subtle for the *actual* (non-algorithmic!) rules that govern the way that the world works!

2 From Boscovich to Gödel

Perfect determinism was considered earlier by Boscovich [4], Leibniz and Laplace (see Barrow [2]). The main argument is similar to the one used by Penrose: if all our laws, say, of motion, were in the form of equations which determine the future uniquely and completely from the present, then a "superbeing" having a perfect knowledge of the starting state would be able to predict the entire future. The puzzling consequence appears as soon as one tries to carry out this prediction!

Gödel was interested in this problem as well. According to notes taken by Rucker ([46], p.181) Gödel's point of view is the following:

> It should be possible to form a complete theory of human behaviour, i.e. to predict from the hereditary and environmental givens what a person will do. However, if a mischievous person learns of this theory, he can act in a way so as to negate it. Hence I conclude that such a theory exists, but that no mischievous person will learn it. In the same way, time-travel is possible, but no person will ever manage to kill his past self.

And he continues:

> There is no contradiction between free will and knowing in advance precisely what one will do. If one knows oneself completely then this *is* the situation. One does not deliberately do the opposite of what one wants.

3 Strong Determinism

According to Penrose ([40], p. 558-559) strong determinism

> is not just a matter of the future being determined by the past; the *entire history of the universe is fixed*, according to some precise mathematical scheme, *for all time*.

Thus strong determinism is a variant of Laplace's scenario,[1] according to which the stage is set at the beginning and everything follows "mechanistically" without the intervention of God, without the occurrence of "miracles" (cf. Frank [24]).

Strong determinism does not imply a computable Universe, as it says nothing about the computability of initial conditions or of physical laws.[2]

Let us discuss this in the context of the computer science. Any program p requiring some particular input s can be rewritten into a new program p' requiring no (the empty list $\emptyset$) input. This can for instance been realized by coding the input s of p as constants of p'. Likewise, any part of p' can be externalized as a subprogram s, whose code can then be identified with an input for the new program p. In this sense, the terms effective computation and initial value are interchangeable and the naming merely a matter of convention. Therefore, if strong determinism leaves unspecified the computability of initial values serving

[1] *"A thing cannot occur without a cause which produces it"*.

[2] Assuming the Church-Turing Thesis, this is equivalent to saying that the laws of nature correspond to recursive functions.

as input for recursive natural laws, it may as well leave unspecified the recursion theoretic status of natural laws.

All this sounds rather abstract and mathematical, but the emergence of chaotic physical motion has confronted the physics community with the theoretical question of whether or not to accept the classical (i.e., non-constructivist) continuum. As envisioned by Shaw [48] and Ford [23], along with many others, "classical chaos" emerges by the effectively computable "visualization" of the incompressible algorithmic information of the initial values. Thereby, the classical continuum serves as an "urn" containing (almost, i.e., with probability one) only (uncomputable) Martin-Löf/Chaitin/Solovay random elements. With probability one, the physical system "chooses" one random element of the continuum "urn" as its initial value. In this sense, chaotic dynamics expresses almost a tautology: put Martin-Löf/Chaitin/Solovay randomness in, get chaotic motion out. The non-tautologic feature is the "choice" of one element of the classical (i.e., non-constructivist) continuum. In order to be able to choose from non-denumerable many uncomputable objects, the axiom of choice has to be assumed. But then, one is confronted with "paradoxical" constructions utilizing this axiom (cf. Wagon [56, 49]). In particular, one could transform every given physical object into any other physical object (or class of objects) in three processing steps:

- decompose the original object into a finite number of pieces;
- apply isometric transformations such as rotations and translations to the pieces; and finally,
- rearrange them into the final form.

This might be the ultimate production belt: one can obtain an arbitrary number of identical copies from a single prototype! We mention this utopy here not because of immediate technological applicability but to point out the type of shock to which the physics community is going to be exposed if it pretends to keep the "skeleton in the closet of continuum physics". Indeed, all the following examples of strong determinism clashing with uncomputability and randomness originate in the assumption of the appropriateness of the classical continuum for physical modelling.

Quantum theory does not offer any real advancement over classical physics in this respect. It is a "half-way" theory, in between the continuum and the discrete. As Einstein put it [20],

> There are good reasons to assume that nature cannot be represented by a continuous field. From quantum theory it could be inferred with certainty that a finite system with finite energy can be *completely* described by a *finite* number of (quantum) numbers. This seems not in accordance with continuum theory and has to stipulate trials to describe reality by purely algebraic means. Nobody has any idea of how one can find the basis of such a theory.

Continuous hidden variable models of quantum mechanics such as Bohm's model [3] operate with pseudo-classical particles. The real-valued initial position of a Bohmean particle, for instance, is Martin-Löf/Chaitin/Solovay random with probability one. The particles move through computable quantum potentials. As in chaos theory, the random occurrence of single particle detections originates again in the assumption of the classical continuum. From this point of view, the Bohmean model of quantum mechanics is not a "mechanistic" theory, although its evolution laws might be recursive.

Everett's many-world interpretation of quantum mechanics [21] is not much of an advance either. It saves the strong determinism by abandoning the wave function collapse at the price of a Universe branching off into (sometimes uncountable) many Universes at any measurement or beam splitter equivalent. Currently, there is very little knowledge concerning the computational status of the wave function[3] or continuous observables. Implicitly, the underlying sets are the classical (i.e., non-constructive) continua.

4 Is Description Possible?

Can a system contain a description of itself? Of course, no finite system can contain itself as a proper part. What we mean by "description" here is an *algorithmic representation* of the system. Such an algorithmic

[3] See Pour-El and Richards [43], and the objections in Penrose [40], and Bridges [6].

representation could be interpretable as a "natural law" since it should allow the effective simulation of the system from within the system.

Von Neumann [55] was concerned with the question of self-description in the context of the self-reproduction of (universal) automata. His Cellular Automaton model was inspired by organic life-forms, and the description "blueprint" for self-reproduction was inspired by the DNA. Today, automaton self-reproduction is just one application of Kleene's fixed-point theorem [45, 39].

Von Neumann realized that there must be a difference between an "active" and a "passive" mode of self-description. The "passive" description is given to the system by some God-like external agent or oracle. It is then possible for a finite system to contain such a "passive" representation of itself within itself as a proper part. Based on this description, the system is capable of simulating itself.[4] Such a self-description in general cannot be obtained "actively" by self-inspection. The reason for this is computational complementarity [37, 49] and the recursive unsolvability of the rule inference problem [30, 49].

5 Is Prediction Possible?

Is there any incompatibility between the strong determinism and computability, as Penrose suggests? Is it indeed impossible for a person to "learn his own theory" (Gödel)?

Let us assume that we have both strong determinism and computable physical laws. For the remainder of this paper we fix a finite alphabet A and denote by A^* the set of all strings over A; $|x|$ is the length of the string x. A *(Chaitin) computer* C is a partial recursive function carrying strings (on A) into strings such that the domain of C is prefix-free, i.e. no admissible program can be a prefix of another admissible program. If C is a computer, then T_C denotes its time complexity, i.e. $T_C(x)$ is the running time of C on the entry x, if x is in the domain of C; $T_C(x)$ is undefined in the opposite case. One can prove Chaitin's Theorem (see, for instance, Chaitin [12, 13], Calude [8], Svozil [49]) stating the existence of a *universal computer* U such that for every computer C there exists a constant $sim(U,C)$—which depends upon U,C—such that in case $C(x)=y$, there exists[5] x' such that

$$U(x') = y, \tag{1}$$

$$|x'| \leq |x| + sim(U,C). \tag{2}$$

Assume, now, for the sake of a contradiction, that an "algorithmic prediction" is possible. Then the universal computer can simulate the predictor, so it can itself act as a predictor. What does this mean? The computer U can simulate every other computer (1), in a shorter time. Formally, to equation (1) we add

$$T_U(x') < T_C(x). \tag{3}$$

Now, let us examine the possibility that U is a predictor. For every string x in the domain of U let

$$t(x) = \min\{T_U(z) \mid z \in A^*, U(z) = U(x)\}, \tag{4}$$

i.e. $t(x)$ is the minimal running time necessary for U to produce $U(x)$.[6]

Next define the *temporal canonical program (input)* associated with x to be the first string (in quasi-lexicographical order) $x^{\#}$ satisfying the equation (4):

$$x^{\#} = \min\{z \in dom(U) \mid U(z) = U(x), T_U(z) = t(x)\}.$$

So,

$$U(x^{\#}) = U(x), \text{ and } T_U(x^{\#}) = t(x).$$

As the universal computer U is a predictor itself, and for itself, it follows from (3) that there exists a string x' such that $U(x') = U(x^{\#}) = U(x)$, and $T_U(x') < T_U(x^{\#}) = t(x)$, which is false. Therefore, every universal

[4] Certain prediction tasks cannot be speeded up, though; see the discussion below.

[5] And can be effectively constructed.

[6] Actually, $t(x)$ is not computable.

predictor is "too slow" for certain tasks, in particular, predicting "highly time-efficient" (or, alternatively, "highly time-consuming") actions of itself.[7]

The reason for the above phenomenon can be illustrated by showing the existence of "small-sized" computers requiring "very large" running times. To this aim we use Chaitin's version of the Busy Beaver function Σ. Denote by H Chaitin complexity (or, algorithmic information content), that is the function defined on (all) strings by the formula

$$H(x) = \min\{|y| \mid y \in A^*, U(y) = x\},$$

i.e. $H(x)$ is the length of the smallest program for the universal computer U to calculate x. For every natural m let us denote by $string(m)$ the mth string in quasi-lexicographical order, and let $\Sigma(n)$ be the largest natural number whose algorithmic information content is less than or equal to n, i.e.

$$\Sigma(n) = \max\{m \mid m \in I\!N, H(string(m)) \leq n\}.$$

Chaitin ([13], 80-82, 189) has shown that Σ grows larger than any recursive function, i.e. for every recursive function f, there exists a natural number N, which depends upon f, such that $\Sigma(n) \geq f(n)$, for all $n \geq N$: indeed, any program of length n either halts in time less than $\Sigma(n + O(1))$, or else it never halts.

As $H(string(\Sigma(n))) \leq n$, it follows that $U(y_n) = string(\Sigma(n))$, for some string y_n of length less than n. This program y_n takes, however, a huge amount of time to halt: there is a constant c such that for large enough n, $U(y_n)$ takes between $\Sigma(n - c)$ and $\Sigma(n + c)$ units of time to halt. To conclude, the equation (1) is compatible with (2) (Chaitin's Theorem), but incompatible with (3).

Computation is a physical process, inevitably bound to physical degrees of freedom; all known physical laws, in turn, are ultimately expressible by algorithms for information processing (i.e., they are computable). The above discussion revealed some mathematical limits; they can be completed with pure physical limits, as discovered by Mundici [38].[8] Due to the fact that every computer is subject to the irreversibility and uncertainty of time-energy, and maximality of the speed light, one can derive the following result: *The total time t and energy E spent for every computation consisting of n steps satisfy the inequality:*

$$t \geq n^2 \frac{h}{2\pi E},$$

where h is Planck constant. For instance, it follows that computations involving more than 10^{30} steps are infeasible.

This suggests that even in the case the Universe is deterministic and unique, and its underlying laws are algorithmic, an algorithmic prediction is impossible. It justifies also Gödel's claim according to which "no person will ever learn his theory" in spite of the fact that such a theory might exist.

6 Uncomputability and Randomness: Two Examples

Various physical problems lead to the question whether a function, in a certain a class, has a real root. Results due to Richardson [44], Caviness [11], Wang [57] (see also Matijasevič [34]) show that for a large class of well-defined functions such a problem is not algorithmically solvable. Da Costa and Doria [18] have proven some undecidability results in physics using this tool. A different approach, based on Specker's Theorem, was developed by Pour-El and Richardson [43]. In this chapter we shall build on the work of Richardson, Wang, and Chaitin to show that two problems in elementary physics are undecidable and display pure randomness.

6.1 Richardson-Wang and Chaitin Theorems

An *exponential Diophantine equation* is of the form

$$E_1(x_1, \ldots, x_m) = E_2(x_1, \ldots, x_m),$$

[7] For an early investigation of a forecast inspired by recursion theory see Popper [42].

[8] Gandy [27, 28] has put forward related arguments imposing limitations to mathematical knowledge by the finiteness of physical objects.

where E_1, E_2 are expressions constructed from variables and natural numbers, using addition, multiplication, and exponentiation. The equations which do not make use of exponentiation are called *Diophantine equations*. Fermat's famous equation

$$(p+1)^{s+3} + (q+1)^{s+3} = (r+1)^{s+3},$$

is an example of an exponential Diophantine equation. For every fixed s, the above equation is a Diophantine equation, for instance, the equation

$$(p+1)^3 + (q+1)^3 = (r+1)^3.$$

By a *family* of (exponential) Diophantine equations we understand an (exponential) Diophantine equation

$$E_1(a_1, \ldots, a_n, x_1, \ldots, x_m) = E_2(a_1, \ldots, a_n, x_1, \ldots, x_m), \tag{5}$$

in which the set of all variables $a_1, \ldots, a_n, x_1, \ldots, x_m$ is divided into two classes, *unknowns,* $x_1, \ldots, x_m$, and *parameters,* $a_1, \ldots, a_n$. A set $S \subset I\!N^n$ is called *(exponential) Diophantine* if there exists a family of (exponential) Diophantine equations (5) such that

$$S = \{(a_1, \ldots, a_n) \in I\!N^n | E_1(a_1, \ldots, a_n, x_1, \ldots, x_m) = E_2(a_1, \ldots, a_n, x_1, \ldots, x_m),$$

$$\text{for some naturals } x_1, \ldots, x_m\}.$$

Due to work of Davis, Matijasevič, Putnam, Robinson (see Matijasevič [34]) the following classes of sets were shown to coincide: 1) the class of recursively enumerable sets, 2) the class of exponential Diophantine sets, 3) the class of Diophantine sets.

By virtue of the existence of recursively enumerable sets which are not recursive (see, for instance, Calude [7]) we deduce that the problem of testing whether an arbitrary (exponential) Diophantine equation has a solution (in natural numbers) is recursively undecidable.[9] A universal (exponential) Diophantine set, i.e. a set which "codes" all (exponential) Diophantine sets is recursively enumerable, but not recursive.

In contrast with the case of (exponential) Diophantine equations—dealing with solutions in natural numbers—the problem of deciding the solvability of polynomial equations with integer coefficients in *real* unknowns is *decidable.* In the unary case this can be done by the well-known Sturm method; in the general case one have to use Tarski's method [53]. To get undecidability we have to allow the use of some other functions; an easy way to achieve this is to consider the addition, multiplication, composition and the sine function, all rationals and π.

For our aim it is convenient to reformulate Richardson [44] and Wang [57] results as follows. We define, for every natural $n \geq 1$, Δ_n to be the minimal (with respect to set-theoretical inclusion) family of expressions which contains all rationals and π, the variables $x_1, \ldots, x_n$, the functions $sin(x)$ and e^x, and which is closed under the operations of addition, multiplication, and composition.

The following predicates are recursively undecidable:

- For every $G(x_1) \in \Delta_1$, "there exists a real number r such that $G(r) = 0$".
- For every $G(x_1) \in \Delta_1$, the predicate "the integral $\int_{-\infty}^{+\infty} [(x^2+1)G^2(x)]^{-1} dx$ is convergent".

Following Chaitin [12, 13] we do not ask whether an arbitrary Diophantine equation has a solution, but rather whether it has an infinity of solutions. Of course, the new question is still undecidable. In the former case the answers to such questions are not independent[10], but in the later one the answers can be independent in case the equation is constructed properly. Actually Chaitin has effectively constructed such an exponential Diophantine equation (see his last Lisp construction in [14]) with the property that the number of solutions jumps from finite to infinite at *random* as a certain fixed parameter is varied. Actually,

[9] This solved in the negative Hilbert's Tenth Problem.

[10] The reason is simple: we can determine which equations have a solution if we know how many of them are solvable.

saying that the "number of solutions jumps from finite to infinite at *random*" is not a figure of speech, it is just a remarkable technical statement: if the parameter n takes the values $1, 2, \ldots$, and $\omega_n = 0$ in case the corresponding equation has finitely many solutions, and $\omega_n = 1$, in the opposite case, then the sequence $\omega_1\omega_2 \cdots \omega_i \cdots$ is random in Martin-Löf/Chaitin/Solovay sense; see Calude [8]. The real number number

$$\Omega = 0.\omega_1\omega_2 \cdots \omega_i \cdots$$

represents the halting probability of a universal computer. In case we assume the hypothesis of strong determinism, Ω has also a "physical" significance: it represents a constant of the Universe.[11] The number Ω is not invariant under changes of the underlying universal computer. However, all "constants" Ω share a number of fascinating properties (see, for instance, Calude [8]); these changes might be similar to changes of other "constants of Nature", as Newton's gravitational constant, the charge of an electron or the fine-structure constant, under certain circumstances (changing the number of dimensions of the space, for instance).

6.2 One-dimensional Heat Equation

Improper integrals, for example, Fourier and Laplace transforms, play a particularly important role in modelling physical phenomena (see, Courant, Hilbert [19], Ştefănescu [52]). Two examples involving the Laplace transform illustrate uncomputability and randomness.

Let us first consider the heat conduction on an infinite slab. It is described by the one-dimensional heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, x \in I\!R,\ t > 0, \\ u(x,0) = f(x), \\ u(x,t) \text{ is bounded.} \end{cases} \tag{6}$$

If $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$ are supposed to be continuous and bounded, then the solution of (6) may be obtained via the Laplace transform (see, Friedrichs [25]):[12]

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y)dy. \tag{7}$$

6.3 A Problem of Electrostatics

Let us consider the plane electrostatic problem[13] on $I\!R \times I\!R_+$ which satisfies the boundary potential condition

$$\Phi(x,0) = f(x).$$

If Φ is an electrostatic potential, then the electric field $\mathbf{E}$ is given by

$$\mathbf{E} = -\mathrm{grad}\ \Phi.$$

If $\mathbf{D}$ is a plane domain (i.e. an infinitely long cylinder with cross section $\mathbf{D}$) bounded by a surface $\mathbf{C}$ composed of several conductors[14] at different potentials, then Φ is is a solution of the system[15]

[11] There is something attractive about permanence.

[12] Notice that the solution of the problem (6) may be also obtained by means of the Fourier transform. It is possible that for some functions f the Laplace (or Fourier) transform does not exists, and still (7) verifies (6).

[13] A problem of electrostatics is plane if there is a distinguished direction such that all data are constant in this direction and the field to be determined is also constant in this direction; Friedrichs [25].

[14] The conductors are materials which do not exert any force on charged particles in their interior, but they do so at the boundary. In a state of equilibrium the charges contained in a conductor are distributed over the boundary.

[15] The same system can be derived from conduction of electricity on a conducting sheet covering the domain $\mathbf{D}$.

$$\begin{cases} \dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial^2\Phi}{\partial y^2} = 0, (x,y) \in \mathbf{D}, \\ \\ \Phi(x,0) = f(x). \end{cases} \tag{8}$$

The problem (8) can be solved via the formalism of differential forms.[16] The solution of (8) is given by

$$\Phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2+y^2}dt. \tag{9}$$

First we look at the solution of the one-dimensional heat equation (7). If $f(y) = (y^2+1)^{-1}$, then, for every fixed (x_0, t_0), the solution

$$u(x_0,t_0) = \frac{1}{2\sqrt{\pi t_0}}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x_0-y)^2}{4t_0}}}{y^2+1}dy$$

is finite.

Consider now the function $f(y) = e^{y^2}$. Let $t_0 > 1$ and $x_0 \in I\!R$ be fixed. Then

$$e^{-\frac{(x_0-y)^2}{4t_0}} f(y) > e^{y^2 - \frac{(x_0-y)^2}{4}} = e^{\frac{3}{4}y^2 + \frac{x_0 y}{2} - \frac{x_0^2}{4}}.$$

For fixed x_0, $\lim_{y\to\infty} \frac{3}{4}y^2 + \frac{x_0 y}{2} - \frac{x_0^2}{4} = \infty$, so the integral

$$\int_{-\infty}^{\infty} e^{-\frac{(x_0-y)^2}{4t_0}} f(y)dy$$

is divergent.

If $f(y) = (y^2+1)^{-1}H^{-2}(y)$ then, for every fixed (x_0,t_0), we get the solution

$$u(x_0,t_0) = \frac{1}{2\sqrt{\pi t_0}}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x_0-y)^2}{4t_0}}}{(y^2+1)H^2(y)}dy = \frac{1}{2\sqrt{\pi t_0}}\int_{-\infty}^{\infty} \frac{1}{(y^2+1)K^2(y)}dy.$$

In case H was in Δ_1, then K is in Δ_1 as well. So, the problem to test, for fixed (x_0, t_0), whether the solution $u(x_0,t_0)$ is finite or not for an arbitrary function $H \in \Delta_1$, is recursively undecidable.

Using Chaitin's construction we can exhibit a sequence of functions $H_i \in \Delta_1$ such that the induced sequence $c_1 c_2 \cdots c_i \cdots$, $c_i = 0$, if the corresponding solution is finite, $c_i = 1$, in the opposite case, is random. So, in the space of all solutions of (7) there are areas in which convergence and divergence alternate in a pure random way.

Similar results can be obtained for the solution of the electrostatic plane problem. For fixed x_0, y_0, $y_0 \neq 0$, the solution (9) can be represented as

$$\Phi(x_0,y_0) = \frac{1}{\pi y_0^2}\int_{-\infty}^{\infty} \frac{f(y_0 u + x_0)}{u^2+1}du. \tag{10}$$

If $f(x) = G(x)^{-2}$, where G is a function in Δ_1, then the problem of testing whether $\Phi(x_0,y_0)$ is finite or not is recursively undecidable. Again, we can effectively construct a sequence of solutions displaying pure randomness, i.e. for which the sequence of answers to the convergence problem is random.

[16]The local existence of a potential Φ is described by the equality $\mathbf{E} = -\mathrm{d}\Phi$; see Bamberg and Sternberg [1].

7 Incompleteness

In a remarkable paper entitled *Intelligent Machines*[17] ([54], 107-127) Turing investigates the possibility as to whether machines, i.e. computers, might show intelligent behaviour. He considers the argument that machines are inherently incapable of exhibiting human-like intelligent behaviour, because human mathematicians are capable of determining the truth or falsity of mathematical statements in a way that machines, as embodiments of formal systems that are subject to the limitations of Gödel's Incompleteness Theorem, cannot. Turing notes that Gödel's Incompleteness Theorem

> rests essentially on the condition that the machine must not make mistakes. But this is not a requirement for intelligence.

He is suggesting that machines might perhaps equal human mathematicians if they were equipped with a human-like capacity to make mistakes.

The analysis of predictability outlined in this paper is subject to Turing's objection regarding mistakes. Accordingly, we address the following question: Is Turing's argument irrefutable?

At a first sight, requiring the absence of mistakes might seem to be overly restrictive. But how can a mistake-making machine be constructed? Where should we place the border between "admissible" and "non-admissible" mistakes in order to preserve the "intelligibility" of our Universe. How can a mistake-making machine discover the regularities, common factors, recurrences, and implications, which tell us what things are and how are they going to be in the future? According to Barrow ([2] p. 269):

> the intelligibility of the world amounts to the fact that we find it to be algorithmically compressible. We can replace sequences of facts and observational data by abbreviated statements which contain the same information content. These abbreviations we often call "laws of Nature".

However, we know that a total compression of the Universe is not actually possible as the existence of chaotic processes points out (Chaitin [12, 13], Rucker [46], Svozil [49, 50, 51], Calude [8], Calude and Salomaa [10]). How can we describe seemingly random processes in nature and reconcile them with supposed order? How much can a given piece of information be compressed? Calude and Salomaa [10] have suggested that the Universe is actually globally *random*, and, consequently, locally *ordered*. The Universe, like any network-like structure can be seen both at local and global levels. Local properties require only a very nearsighted observer—and for this level, science is indeed very useful and successful—but global properties are much more difficult to "see", they need a sweeping vision. For instance, the overall shape of a spiderweb is a global property, while the average number of lines meeting a vertex is a local characteristic.

The relevance of Gödel's Incompleteness Theorem [32] argument has been questioned by different authors, especially by Boolos, Chalmers, Davis and Perlis (see [41]; it contains also Penrose's reply). In our opinion, Turing's critique—mentioned above—is the most substantial. It questions the status of Gödel's famous unprovable statement: is this unprovable statement—seen to be "true" by Penrose—esoteric, accidental? Does the incompleteness phenomenon have any relevance for a scientist's daily life? This is a rather delicate question. If we adopt a topological point of view (see Calude, Jürgensen, Zimand [9]), then incompleteness is a rather common, pervasive phenomenon: the set of true, but unprovable statements is topologically "very large", i.e. with respect to any reasonable topology the set of true and unprovable statements of a sufficiently rich, sound, and recursively axiomatizable theory is dense and in many cases even co-rare. It is important to notice that the above result holds true not only globally, but even for "fixed" problems. For instance, the halting problem: there exists a large set of true, but unprovable, statements stating that some Turing machine will never halt on a fixed entry.

The natural way to model "admissible mistakes" is to work with probabilistic Turing machines[18] instead of (ordinary) Turing machines. A probabilistic Turing machine has some distinguished states acting as

[17] This paper has attracted less interest than *Computing Machinery and Intelligence* ([54], 133-160); for instance, Penrose does not quote it at all.

[18] This type of machine is sometimes called a *Monte Carlo* algorithm.

"coin-tossing states" for which the finite control specifies $p \geq 2$ possible next states. The computation is deterministic except that in the distinguished states the machine uses the output of a random experiment to decide among the p possible next states. So, a probabilistic Turing machine can make mistakes; the output is not "truly correct", but "correct within a probability". Classical results due to De Leuuw, Moore, Shannon, and Shapiro [17] and Gill [29] show that the class of functions computed by probabilistic algorithms coincides with the class of recursive functions. The difference is only in complexity: if we do not insist on a *guarantee*, then sometimes it is possible to compute faster. All results pertaining incompleteness, previously discussed, remain valid, so it appears that Turing's objection cannot be supported anymore: this probabilistic space inherits the non-computability of the deterministic one.

8 Computability

Is the theory of computability (recursion theory)[19] an appropriate framework to discuss physical laws and thought processes? It is not unreasonable to suspect that the notion of computation will play a major role in future research in the natural sciences; however, the global picture is more complex than it appears on a first analysis.

Recursion theory is useful for proving the existence of uncomputable physical laws. If we are interested in "useful" physical laws, i.e. laws which can be effectively used for practical purposes, then the theory of computation might not be the appropriate tool. Indeed, it may happen that some function is computable, but it is very difficult to compute,[20] or even worse, that the computable function *is impossible* to compute at all. For instance, consider the Continuum Hypothesis[21] and the following function

$$f(n) = \begin{cases} 1, & \text{if the Continuum Hypothesis is true,} \\ 0, & \text{if the Continuum Hypothesis is false,} \end{cases}$$

suggested in Bridges [5]. According to classical logic, f is computable because there exists an algorithm that computes it, i.e. the algorithm that returns either one or zero, for all non-negative integers. Deep work due to Gödel [33] and Cohen[16] shows that neither the Continuum Hypothesis nor its negation can be proven within Zermelo-Fraenkel set theory augmented with the Axiom of Choice, the standard framework of classical mathematics, so we will never know which of the two algorithms—"print one", or "print zero"—is the right one. We conclude that the standard theory of computable functions does not match computational practice! The paradoxical nature of this example comes from the underlying logic of computability. To handle this problem we have to distinguish between *existence in principle* and *existence in practice*. A possible approach is to consider provable computable functions introduced by Fischer [22]. A computable function is called *provable* with respect to some formal system S which contains second order arithmetic if there exists an algorithm which computes it and which can be proven to be total in S. These functions are interesting because they are functions we usually work with in practice, e.g. in numerical analysis. What do we lose sacrificing all computable functions in favour of provable computable ones? Gordon [31] has proven that this class of functions is a complexity class, i.e. it can be computed with limited resources, say in time. Now, if we apply some results in Calude [7] we arrive at the conclusion that there is an essential difference between computable functions and provable computable functions: in a constructive sense, the former class is of second Baire category (i.e. large) while the later one is meagre (i.e. small). Informally this means that most computable functions are not provable computable; the difference between functions "computable in principle" and provable computable functions is significant.[22]

[19] A truly remarkable achievement of modern mathematics is the discovery of recursive (or, computable) functions, i.e. functions which can be computed by algorithms. Within the realm of this theory it is possible to prove the existence of functions that are not computable by any algorithm whatsoever. The theory of computability has not yet become part of mainstream physics, but it can serve perfectly well as a guiding principle to hitherto informal notions such as "determinism".

[20] Actually, for every computational measure, for instance, time or space, there exist arbitrarily difficult to compute functions; see Calude [7].

[21] There is no cardinal number strictly in between aleph-null, the cardinal of the set of natural numbers, and aleph-one, the cardinal of the set of reals.

[22] In this context it is interesting to note a result—obtained in 1964—which can be considered as "Chaitin (very first)

9 Conclusions

The paradox mentioned by Penrose is not real, because "real predictors" do not exist.[23] This is because every (universal) predictor is "too slow" for certain tasks, in particular for predicting actions of itself. Two more examples of uncomputability of physical laws are discussed. Turing's objection concerning Gödel's Incompleteness Theorem is confronted with the fact that, from a topological point of view, the incompleteness phenomenon is common and pervasive; this result is still true for probabilistic Turing machines, i.e. for machines allowed to make "reasonable" mistakes. Although we have refuted Penrose's argument that strong determinism and computability are logically incompatible, we have found independent reasons to support his conclusion concerning the non-computability of physical laws. Finally we are lead to the following question: is the theory of computation an appropriate framework to discuss physical laws and thought processes? We argue that for proving non-computability results the answer is affirmative; for more practical purposes, in which we are interested not only in discovering physical laws, but in using them to make predictions, the answer might be negative. Other aspects of the problem, e.g., the role of the observer and "approximation" in making predictions, will be treated in another paper.

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Incompleteness Theorem": *For any formal system there is a computable total function that goes to infinity more quickly than any provably computable total function in the formal system.* For the construction we take $F(n)$ to be n times the maximum of the values of the first n provably computable total functions for all arguments up to n; "first" means first in a recursive enumeration of all theorems in the formal system. This note was has kindly communicated to us in [15].

[23] Penrose himself seems to have anticipated this.

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