Liouville, Computable, Borel Normal and Martin-Löf Random Numbers

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To Solomon Marcus who loved Liouville Numbers

Figure 1: Discussing about Liouville numbers, September 2015

Four classes of numbers: $\mathcal{L}$, $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{M}$

A real number α is:

1. *Liouville* if it is irrational and for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that
$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k^k};$$
2. *computable* if it has a computable b-ary expansion for some $b \geq 2$;
3. *(Borel) normal* if for every base b, every word $w \in \{0, 1, \ldots, b-1\}^*$ appears in its b-ary expansion with the frequency $b^{-|w|}$;
4. *(Martin-Löf) random* in base b if its b-ary expansion $\mathbf{x} = x_1 x_2 \ldots x_n \ldots$ has the property: $H(x_1 x_2 \ldots x_n) \geq n - \mathrm{O}(1)$ (here H is the prefix complexity).

An irrational real number α is Liouville if for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k^k}.$$

Theorem. *Let $\alpha \in [0,1]$ be an irrational. Then, α is a Liouville number iff for every integer $k > 1$ there exists a base $b = b_{\alpha,k} \geq 2$ and two words*

$$v, w \in \{0,1,\ldots,b-1\}^*,\ |v| \leq |w|,\ |w| > 0,$$

such that the b-ary expansion $\mathbf{x}$ of α satisfies the equality

$$\mathbf{x} = v \cdot w^k \cdot \mathbf{x}',$$

for some sequence $\mathbf{x}'$.

How large are the classes $\mathcal{L}$, $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{M}$?

- $\mathcal{C}$ is the only countable class.
- $\mathcal{L}$ is a dense G_δ-set (hence co-meagre), measure zero set; it has Hausdorff dimension zero.
- $\mathcal{N}$ and $\mathcal{M}$ are constructive measure one, but constructively meagre in the Cantor topology.
- There exists a metric topology refining the Cantor topology in which $\mathcal{M}$ is co-meagre[1].

[1] C. S. Calude, S. Marcus, L. Staiger. A topological characterization of random sequences, *Inform. Process. Lett.* 88 (2003), 245–250.

The *irrationality exponent* of a real α is a measure of how "closely" α can be approximated by rationals:

$$\inf\left\{\mu \geq 0 \mid \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}},\, p, q \in \mathbb{Z}, q \neq 0 \text{ i.o.}\right\}.$$

Comment. Liouville numbers have infinite irrationality exponents.

Theorem. *Every random real has irrationality exponent 2.*

Corollary. $\mathcal{M} \cap \mathcal{L} = \emptyset$.

Relations between $\mathcal{L}$, $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{M}$

The following seven sets are empty:

$\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$,
$\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$,
$\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, and
$\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$.

The following eight sets are non-empty:

$\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$, $\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$,
$\mathcal{L} \cap \mathcal{C} \cap \mathcal{N} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$,
$\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$, $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$,
$\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$, $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \bar{\mathcal{M}}$.

It is an *open problem* whether there exist computable (so, not random), normal, non-Liouville numbers.

It is known that computable, non-Liouville numbers, normal to base 2, but *not normal exist.* For example, any Stoneham number

$$F(1/2) = \sum_{i=1}^{\infty} 2^{-k^i} \cdot k^{-i}$$

(where $k \in \mathbb{N}$ is odd , $k \geq 3$) is computable, normal in base 2, but not in base 6, and has irrationality exponent $\mu(F(1/2)) = k$, hence it is not Liouville.

Conjecture. $\bar{\mathcal{L}} \cap \mathcal{C} \cap \mathcal{N} \cap \bar{\mathcal{M}}$ is non-empty.