Is Independence an Exception?[1]

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Abstract: Gödel's Incompleteness Theorem asserts that any sufficiently rich, sound, and recursively axiomatizable theory is incomplete. We show that, in a quite general topological sense, incompleteness is a rather common phenomenon: With respect to any reasonable topology the set of true and unprovable statements of such a theory is dense and in many cases even co-rare.

Key words: independent statement, dense set, rare set, recursive equivalence.

1. Introduction

Gödel's Incompleteness Theorem asserts that any sufficiently rich, sound, and recursively axiomatizable theory is incomplete ([Gö1], [Gö2]). Variants of this profound result have been established that reveal many of its subtleties. A quite general account of this work, due to Gödel, Tarski, Rosser, Sheperdson, Smullyan, and others, is provided in Smullyan's recent book [Sm] on "Gödel's Incompleteness Theorems." While the proofs of this theorem and its variants are usually constructive, the list of concrete examples of true statements known to be independent is quite short. Such concrete examples were first exhibited in [Ha], providing an intriguing new view of a difficult complexity theoretic problem, and by [Pa], where the more classical case of a combinatorial problem was considered with respect to Peano arithmetic.

The intuitive reason for incompleteness can be phrased in several ways. In terms of information theory or descriptional complexity, there is an informational difference between the set of true statements and the set of proofs ([Be], [Ch1], [Ch2]). In more classical terms,

[1] This work was supported by the Natural Sciences and Engineering Council of Canada under Grant OGP0000243 and in part by the National Science Foundation under Grant NSF-CCR-8957604. Much of it was done during visits of the first author at The University of Western Ontario in the spring of 1990 and the fall of 1992 and of the second author at the University of Bucharest in the fall of 1990.

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it is a variant of the liar's paradox, that is, a consequence of diagonalization techniques. From both points of view, however, it would seem that independent statements are a rather unusual, if not abnormal, occurrence.

Indeed, this impression is recounted by Solovay ([Ko], p. 399): "The feeling was that Gödel's theorem was of interest only to logicians;" Smoryński ([Ko], p. 399) is quoted as stating: "It is fashionable to deride Gödel's theorem as artificial, as dependent on a linguistic trick."

Motivated by [Ha], in [Ca2] a sufficient condition is introduced for a theory to have unprovable true statements. This condition can be used to show that many common undecidability results have independent instances. For example, it is shown there that there are infinitely many independent instances of the Halting Problem. As a natural consequence of these considerations, we formulated the conjecture that *the set of independent statements is very large.*

Of course, such a statement needs to be made precise. Measuring largeness in terms of cardinalities does not make sense as all sets concerned are enumerable. In this paper, we measure largeness in terms of topology showing that for many reasonable topologies the set of these statements is not only dense, but even co-rare. We actually prove a stronger theorem, giving a suficient condition for a topology to have this property and also show that this condition is not all restrictive. Thus, in essence, we show that for any reasonable topology the set of unprovable and true statements is large with respect to that topology. In order to achieve generality, we express our results in recursion theoretic terms.

The consequences of our result still need to be determined—in the words of Chaitin ([Ch2], p. 148): "What is the meaning of Gödel for daily work in mathematics? ... How common is incompleteness and unprovability? Is it a very bizarre pathological case, or is it pervasive and quite common? Because if it is, perhaps we *should* be doing mathematics quite differently." By our results, unprovability is a common phenomenon.

In some sense, of course, Gödel's theorem indeed depends on a "linguistic trick," essentially based on the possibilities to assign names to problems. Our results cannot change this. Instead, they show that this linguistic trick is widely applicable and inevitable. Hence, they could lead one to conclude that the trick is not so exotic and artificial after all.

To avoid a potential misunderstanding, we stress once again that we study independence only in the sense of Gödel's Incompleteness Theorem. There are many other kinds of independent statements—the Parallel Postulate, for instance—which are not derived from that theorem and, thus, do not belong to that class. For the relevance of the question treated in this paper see the ample discussion in [Pe1] and [Pe2].

In deriving our results we take an approach common in the theory of computation, studying Gödel numbers rather than the problems they represent. Making this distinction between names of problems, that is, Gödel numbers and the problems themselves is very important and, indeed, necessary as it is undecidable whether two different given names represent the same problem. While stated in terms of Gödel numbers, our results do not rely on any specific properties of the given acceptable Gödel numbering, but obtain for any acceptable Gödel numbering.

Our paper is structured as follows. Some notation and basic notions are introduced or reviewed in Section 2. In Section 3, we prove two auxiliary results which imply that topologies of the kind considered in the sequel abound. Section 4 serves two purposes: to introduce a rigorous formulation of our main question, and to provide the recursion theoretic tools. The topological considerations are introduced in Section 5. The main results mentioned above are announced in Theorem 5.6, Theorem 5.7, and Corollary 5.8. Section 6 contains a few concluding remarks.

2. Basic Notions and Notation

In this section we introduce some notation and review some standard notions.

An *alphabet* is a finite non-empty set. For this paper, let X be an arbitrary but fixed alphabet. Then X^* denotes the set of *words* over X including the *empty word* ε. For a word $w \in X^*$, $|w|$ denotes the *length* of w.

On X^* we consider the *standard order* $\leq_{\text{stand}}$ which is defined as follows [Shy]. Let $\leq_X$ be an arbitrary, but fixed total order on the alphabet X. Then

$$x \leq_{\text{stand}} y \iff |x| < |y| \vee \big(|x| = |y| \wedge \exists u, v, w \in X^* \exists a, b \in X : (x = uav \wedge y = ubw \wedge a <_X b)\big)$$

for $x, y \in X^*$. This order coincides with the lexicographic ordering on strings of the same length.

Let P denote the set of all partial recursive functions of X^* into X^*. Without loss of generality, we assume that the minimization operator μ is taken with respect to the standard order. For $f \in P$ we write $f(x) = \infty$ when f is undefined at x. Let $(\varphi_x)_{x \in X^*}$ be an arbitrary, but fixed acceptable Gödel numbering of P. As usual, W_x denotes the domain of φ_x. The symbol $\leq_1$ denotes one-one reducibility. For further notation and basic facts in recursion theory see [Ro] and [Ca1].

3. Padding Lemmata

In this section we derive several auxiliary results that allow one to relate Gödel numberings to equivalence relations on X^*. They establish the effective existence of certain "compatible" pairs of acceptable Gödel numberings and recursive equivalence relations. The main purpose of these results is to show that topologies on X^* for which our results hold true abound, that is, that we are not discussing exceptional situations.

A binary relation R on X^* is *recursive* if and only if the binary predicate $(x, y) \in R$ is recursive. Let $\equiv$ be an equivalence relation on X^*. Then $[x]_\equiv$ denotes the equivalence class of x. The *index* of $\equiv$ is the number of equivalence classes. For a bijection $f : X^* \to X^*$, the relation $\equiv_f$ defined by $u \equiv_f v \iff f(u) \equiv f(v)$ is also an equivalence relation. A Gödel numbering $(\psi_x)_{x \in X^*}$ is *compatible* with $\equiv$ if, for all $u, v \in X^*$, $u \equiv v$ implies $\psi_u = \psi_v$. Note that, if $\equiv$ is compatible with an acceptable Gödel numbering, then $\equiv$ is of infinite index.

Example 3.1 The *length equivalence* $\equiv_{\text{length}}$ defined by $u \equiv_{\text{length}} v \iff |u| = |v|$ is a recursive equivalence relation of infinite index.

Lemma 3.2 *For a recursive equivalence relation of infinite index, there are infinitely many acceptable and compatible Gödel numberings.*

Proof: Let $\equiv$ be a recursive equivalence relation of infinite index on X^*. The relation $\equiv$ has a recursively enumerable *cross-section* $Q_\equiv$, that is, a recursively enumerable subset of X^* containing exactly one element of each equivalence class. Let $g : X^* \to X^*$ be a recursive and injective function with $g(X^*) = Q_\equiv$ and define $f : X^* \to X^*$ by $f(w) = \mu x[g(x) \equiv w]$. Then f is a recursive and surjective function satisfying $f(g(x)) = x$. Now define $\psi_x = \varphi_{f(x)}$ for all $x \in X^*$ where $(\varphi_x)_{x\in X^*}$ is an arbitrary acceptable Gödel numbering. Clearly, $(\psi_x)_{x\in X^*}$ is an acceptable Gödel numbering. Moreover, $u \equiv v$ implies $f(u) = f(v)$ and, therefore, $\psi_u = \psi_v$.

From $(\psi_x)_{x\in X^*}$ one derives infinitely many acceptable Gödel numberings that are compatible with $\equiv$. If $t : X^* \to X^*$ is a recursive surjection such that $\psi_{t(u)} = \psi_{t(v)}$ if $\psi_u = \psi_v$. Then $(\vartheta_x)_{x\in X^*}$ with $\vartheta_x = \psi_{t(x)}$ is an acceptable Gödel numbering which is compatible with $\equiv$. Moreover, there are infinitely many different functions t of this kind and each gives rise to a distinct Gödel numbering. □

To illustrate the last sentence in the proof of Lemma 3.2 consider $\psi_u = \lambda x.x$, $\psi_v = \lambda x.x + 1$ and $t(u) = v$, $t(v) = u$, $t(x) = x$ for all $x \neq u, v$. Clearly, t satisfies the above conditions and $\vartheta_u = \psi_{t(u)} = \psi_v \neq \psi_u$, that is, $\vartheta \neq \psi$.

Lemma 3.3 *Let $(\psi_x)_{x\in X^*}$ be an arbitrary acceptable Gödel numbering. Then there exist infinitely many recursive equivalence relations of infinite index on X^* with which $(\psi_x)_{x\in X^*}$ is compatible.*

Proof: We generate a sequence of equivalence relations $\equiv_i$ for $i = 0, 1, \ldots$. The relation $\equiv_i$ will differ from the equality in the following way: For a fixed word u it has i words different from u which are equivalent with u. Otherwise, the relation is just the equality.

Suppose $\equiv_i$ has been obtained, where $\equiv_0$ is the equality relation and where, for $i > 0$, distinct words $u_1, \ldots, u_i$ have been constructed such that $\psi_u = \psi_{u_1} = \cdots = \psi_{u_i}$ and $\equiv_i$ is the equality with $u \equiv_i u_1 \equiv_i \cdots \equiv_i u_i$ added.

We construct $\equiv_{i+1}$ from $\equiv_i$ recursively by adding one more word u_{i+1} such that $\psi_u = \psi_{u_{i+1}}$ and $u \equiv_{i+1} u_1 \equiv_{i+1} \cdots \equiv_{i+1} u_{i+1}$. This word can be obtained as the set $\{v \mid \psi_u = \psi_v\}$ contains an infinite recursively enumerable set.

Clearly, each such equivalence relation is compatible with the Gödel numbering and the relations are all distinct.

Moreover, each $\equiv_i$ is recursive. Consider $v, w \in X^*$ with $v \neq w$. If $v \neq u \neq w$ then $v \equiv_i w$ if and only if $v = w$; hence $v \not\equiv_i w$ in this case. Now suppose that $v = u$. Run the construction of $\equiv_i$ to obtain $u_1, \ldots, u_i$. This takes a finite number of steps. Now $v \equiv_i w$ if and only if $w \in \{u_1, \ldots, u_i\}$. □

Lemma 3.2 and Lemma 3.3 together imply that there are 'many' pairs $((\psi_x)_{x\in X^*}, \equiv)$ where $(\psi_x)_{x\in X^*}$ is an acceptable Gödel numbering and $\equiv$ is a compatible recursive equivalence relation. This will allow as to argue later that there is an abundance of topologies for which the set of true and unprovable statements of a theory is large.

We conclude this section with another useful example of a recursive equivalence different from the length equivalence.

Example 3.4 Choose and fix $a \in X$. For $x \in X^*$, let $n_a(x)$ be the number of occurrences of a in x. The *a-count equivalence* $\equiv_{n_a}$ is defined by $x \equiv_{n_a} y \iff n_a(x) = n_a(y)$. Suppose, $(\varphi_x)_{x\in X^*}$ is an acceptable Gödel numbering which compatible with $\equiv_{\mathrm{length}}$. For $x \in X^*$, let

$\psi_x = \varphi_{a^{n_a(x)}}$. Then, $(\psi_x)_{x \in X^*}$ is an acceptable Gödel numbering that is compatible with $\equiv_{n_a}$. The a-count equivalence is a recursive equivalence relation of infinite index such that, if $|X| > 1$, every equivalence class is infinite.

4. Independent Statements

Let $\mathbb{T}$ be the class of all theories T which are recursively enumerable, that is, such that the set of all theorems of T is recursively enumerable, consistent, sound, and sufficiently rich. Such a theory T consists of expressions some of which are *sentences* and among the latter some are *predicates.* Some sentences are *provable* (or *theorems*), and some are *true* (according to some reasonable criterion). Saying that T is recursively enumerable means that the set of theorems is recursively enumerable, assuming an appropriate acceptable Gödel numbering. Consistency means that there is no theorem the negation of which is also a theorem. Soundness means that every theorem is true. The theory is suficiently rich if it contains the (recursive) arithmetic. The theories of *Peano Arithmetic* and of *Set Theoretic Arithmetic* are examples of theories in $\mathbb{T}$ (see [Ro], pp. 96–98; see [Sm] for additional examples).

Let T be a theory in $\mathbb{T}$. A predicate H is *true* for x if $H(x)$ is true in T. A set $A \subseteq X^*$ is said to be *expressible* in T (or *definable in the language of* T) if there is a predicate H of T such that, for all $x \in X^*$, $H(x)$ is true if and only if $x \in A$. A sentence is said to be *independent of* T if it is true, but not provable in T. We write $T \vdash f$ to denote the fact that f is a theorem of T, that is, that there is a proof of f in T. The following basic result is well-known.

Proposition 4.1 *Let $A \subseteq X^*$ such that A is expressible in T, but not recursively enumerable. Then the sentence '$x \in A$' is independent of T for some $x \in A$, that is, $T \not\vdash$ '$x \in A$' for some $x \in A$.*

Indeed, the set $A_T = \{x \mid x \in X^*, T \vdash \text{'}x \in A\text{'}\}$ is recursively enumerable. If $T \vdash$ '$x \in A$' for all $x \in A$ then $A = A_T$, but A is not recursively enumerable by assumption.

There are many examples of sets A satisfying the assumptions of Proposition 4.1; these include the Halting Problem of Turing Machines, the Post Correspondence Problem, and many others.[5]

Recall that a set $A \subseteq X^*$ is said to be *productive* if there is a recursive function $f : X^* \to X^*$ such that for every $x \in X^*$ with $W_x \subseteq A$ one has $f(x) \in A \setminus W_x$. Given such a function f, we say that A is *productive via* f.

Note that Gödel's Incompleteness Theorem can be rephrased as follows: The set of (Gödel numbers of) true arithmetical sentences is productive. In other words, for every recursively enumerable set of true arithmetical sentences one can effectively find a true arithmetical sentence not in the given set (see [Ro]). It is the main goal of this paper to determine the "size" of such sets.

Lemma 4.2 *Let A be productive via f and expressible in T. Then there effectively exists an element $u \in A$ such that $T \not\vdash$ '$u \in A$'.*

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[5] Strictly speaking, these should be called the Non-Halting and Non-Correspondence Problems, respectively.

Proof: Let x be such that $W_x = \{y \mid y \in X^*,\ T \vdash `y \in A'\}$. Then $W_x \subseteq A$ and $f(x) \in A \setminus W_x$, that is, $u = f(x) \in A$ and $T \not\vdash `u \in A'$. The string x and, therefore, also the string u, can be constructed because the set of all theorems of T is recursively enumerable. $\square$

Example 4.3 The set $\overline{K} = \{x \mid x \in X^*,\ \varphi_x(x) = \infty\}$ is productive. A set $A \subseteq X^*$ is productive if and only if $\overline{K} \leq_1 A$. Most "natural" index classes are productive as they are complete for various levels of the arithmetic hierarchy. On the other hand, the set $\{x \mid x \in X^*,\ x \text{ is random}\}$ is not recursively enumerable, but not productive as it is immune (see [Ca1], p. 333).

For every theory $T \in \mathbb{T}$ and every set $A \subseteq X^*$ which is expressible in T but not recursively enumerable, let $I(A) = \{u \mid u \in A,\ T \not\vdash `u \in A'\}$. From the above results it follows that $I(A)$ is non-empty and, indeed, infinite. In the sequel, we answer the following question: *How large is $I(A)$?*

The main problem in addressing this question is to find a mathematical way of appreciating the size of $I(A)$ *in general,* that is, to ensure that degenerate, artificial, or non-representative cases do not distort the answer.

For instance, we have to avoid trivial situations like the following one: Consider a theory $T \in \mathbb{T}$ and a productive set $A \subseteq X^*$ which is expressible in T. One can find an element $u \in I(A)$ effectively and then construct the set $B = \{x \mid x \in X^*,\ g(x) \in A\}$ where $g(x) = u$ for all $x \in X^*$. Thus, if no further restrictive assumptions are made, B is expressible in T, $B = X^*$, and $T \not\vdash `x \in B'$ for all $x \in B$.

To avoid this rather artificial situation, we proceed as follows: We exhibit a class of sets A, which includes many natural ones, for which $I(A)$ is large according to a topological criterion with respect to many different topologies.

Recall that a set $A \subseteq X^*$ is said to be *saturated* with respect to an equivalence relation if A is a union of equivalence classes.

Theorem 4.4 *Let $A \subseteq X^*$ be expressible in T and assume that $\equiv$ is a recursive equivalence relation on X^* which saturates A. If A is not recursively enumerable then there exist infinitely many $x \in A$ with $[x]_\equiv \subseteq I(A)$. Moreover, if A is productive, then infinitely many such x can be effectively constructed and $\equiv$ is of infinite index.*

Proof: Let
$$B = \{z \mid z \in X^*,\ \exists u \in X^*\, (u \equiv z \wedge T \vdash `u \in A')\}.$$

Then B is recursively enumerable as T is recursively enumerable and $\equiv$ is recursive. Moreover, $B \subseteq A$ as T is sound and $\equiv$ saturates A. From the fact that A is not recursively enumerable it follows that $B \neq A$, that is, there is an element $x \in A \setminus B$ and, actually, the set $A \setminus B$ is infinite. We claim that $[x]_\equiv \subseteq I(A)$ for any such x.

Let $u \equiv x$. Then $u \in A$ as A is saturated by $\equiv$. If $T \vdash `u \in A'$ then $x \in B$, a contradiction!

If, in addition, A is productive then, by Lemma 4.2, infinitely many such elements x can be constructed. Moreover, as $\equiv$ saturates A, it is of infinite index. $\square$

Example 4.5 Let $\equiv$ be a recursive equivalence on X^* of infinite index and let (ψ_x) be a Gödel numbering which is compatible with $\equiv$ (see Lemma 3.2). For any $x_0 \in X^*$, the set $A = \{x \mid x \in X^*,\ \psi_x(x_0) = \infty\}$ is productive and saturated by $\equiv$. Actually, every index set which is not recursively enumerable is saturated by $\equiv$.

The following remark is an immediate consequence of the definitions. It will be used below to achieve translations between certain topologies on X^*.

Observation 4.6 *Let $\equiv$ be a recursive equivalence relation on X^*, let $A \subseteq X^*$ be saturated by $\equiv$, and let $f : X^* \to X^*$ be a recursive bijection. Then the relation $\equiv_f$ is a recursive equivalence relation which saturates the set $f^{-1}(A)$. Moreover, $\equiv_f$ is of infinite index if and only if $\equiv$ is of infinite index.*

Example 4.7 There are infinitely many recursive equivalence relations of infinite index which saturate $\overline{K}$. Indeed, let $\equiv$ be an arbitrary recursive equivalence relation of infinite index which is compatible with $(\varphi_x)_{x\in X^*}$. By Lemma 3.3 there are infinitely many such $\equiv$. It is well known that there is a recursive bijection $f : X^* \to X^*$ such that $f(\overline{K}) = A$ where A is the set defined in Example 4.5. By Observation 4.6, the relation $\equiv_f$ saturates the set $\overline{K}$. Moreover, if $\equiv^1$ and $\equiv^2$ are two such equivalence relations then $\equiv^1 \neq \equiv^2$ implies $\equiv^1_f \neq \equiv^2_f$.

5. Topological Arguments

Let τ be a topology on X^* and let $\mathbf{C}_\tau$ be its closure operator. A set $A \subseteq X^*$ is said to be *rare* with respect to τ if, for every $x \in X^*$ and every open neighbourhood N_x of x, one has $N_x \not\subseteq \mathbf{C}_\tau(A)$. A set subset of X^* is *dense* if its closure is equal to X^*. It is *co-rare* if its complement is rare. Intuitively, the properties of being rare, dense, and co-rare describe an increasing scale for the sizes of subsets of X^* with respect to the topology τ. Thus a dense set is "larger" than a rare one, and a co-rare set is "larger" than a dense set.

The topologies considered as examples in the sequel are generated by partial orders on X^* (see [Shy], [Jü]). A partial order $\leq$ on X^* defines a topology $\tau_\leq$ as follows: For any $u \in X^*$, let $N_u^\leq = \{v \mid v \in X^*,\ u \leq v\}$. Then the set $\{N_u^\leq \mid u \in X^*\}$ is the basic system of open neighbourhoods for $\tau_\leq$. The closure operator $\mathbf{C}_{\tau_\leq}$ is given by

$$\mathbf{C}_{\tau_\leq}(A) = \{u \mid u \in X^*,\ \exists v \in A : u \leq v\}$$

for $A \subseteq X^*$ (see [Ke], p. 57–58). If $\leq_a$ and $\leq_b$ are partial orders such that $\leq_a \subseteq \leq_b$ then any set which is dense with respect to $\tau_{\leq_a}$ is also dense with respect to $\tau_{\leq_b}$. The topologies to be considered will have the following property:

Top: *There is a recursive equivalence relation $\equiv$ on X^* such that, for every $x \in X^*$ and every open neighbourhood N_x of x, the set $\{y \mid y \in X^*,\ N_x \cap [y]_\equiv = \emptyset\}$ is finite.*

The condition Top excludes situations in which equivalence classes tend to form "clusters;" in such a case, clearly, no meaningful statement about the topological size of a set can be made. If, in particular, the equivalence relation is compatible with the given Gödel numbering, then Top excludes topologies in which equivalent Gödel numbers form such clusters. In this context, the Padding Lemmata, Lemma 3.2 and Lemma 3.3, guarantee that this can always be achieved, that is, for every topology satisfying TOP with respect to an arbitrary given recursive equivalence relation $\equiv$, one can construct infinitely many acceptable Gödel numberings that are compatible with $\equiv$.

Note that, if τ is a topology satisfying TOP with respect to the equality as equivalence relation, then every open set is co-finite.

Example 5.1 The *prefix order topology* is generated by the *prefix order* $\leq_{\mathrm{p}}$ defined by $x \leq_{\mathrm{p}} y \iff y \in xX^*$ for $x, y \in X^*$. It satisfies the condition TOP for the length equivalence.

Example 5.2 Let $\leq_X$ be the total order on the alphabet X introduced before. The *masking order* $\leq_{\mathrm{m}}$ is defined as follows. Let $x = x_1 \cdots x_k \in X^*$ and $y = y_1 \cdots y_l \in X^*$ with $x_1, \ldots, x_k, y_1, \ldots, y_l \in X$. Then $x \leq_{\mathrm{m}} y$ if and only if $k \leq l$ and $x_i \leq_X y_i$ for $i = 1, \ldots, k$. The relation $\leq$ generates the *masking order topology* which also satisfies the condition TOP for the length equivalence.

Example 5.3 With $\leq_X$ as before, the *lexicographic order* $\leq_{\mathrm{lex}}$ is defined by

$$x \leq_{\mathrm{lex}} y \iff x \leq_{\mathrm{p}} y \vee \exists u, v, w \in X^* \exists a, b \in X : (x = uav \wedge y = ubw \wedge a <_X b)$$

for $x, y \in X^*$. The relation $\leq_{\mathrm{lex}}$ generates the *lexicographic topology* which also satisfies the condition TOP for the length equivalence. As the lexicographic order is a total order, the open sets in the lexicographic topology are precisely the sets of the form $N_u^{\leq_{\mathrm{lex}}} = \{v \mid v \in X^*,\ u \leq_{\mathrm{lex}} v\}$ for $u \in X^*$. For $x \in X^*$, consider an open neighbourhood N_x of x, that is, $N_x = N_u^{\leq_{\mathrm{lex}}}$ for some $u \in X^*$ with $u \leq_{\mathrm{lex}} x$. Let $\equiv$ be an arbitrary equivalence relation on X^*. For $N_x \cap [y]_\equiv$ to be empty it is necessary and sufficient that $v <_{\mathrm{lex}} u$ for all $v \in [y]_\equiv$. The length equivalence does not satisfy this condition for any y.

The situation for the standard order on X^* is similar to that of the lexicographic order. Note, however, that the lexicographic and standard orders are quite different. In fact, with $\equiv$ the equality, the lexicographic topology does not satisfy TOP. On the other hand, the topology generated by the standard order satisfies TOP even with respect to the equality.

For further partial orders of interest in this context see [Shy] and [Jü] where the connection between partial orders on X^* and classes of codes is investigated.

Lemma 5.4 *Let $\leq$ be a recursive partial order on X^* and let $\equiv$ be a recursive equivalence relation such that the topology $\tau_\leq$ satisfies the condition* TOP *with respect to $\equiv$. Let $f : X^* \to X^*$ be a recursive bijection. Define the relation $\leq_f$ by $x \leq_f y \iff f(x) \leq f(y)$. Then the topology $\tau_{\leq_f}$ satisfies the condition* TOP *with respect to $\equiv_f$.*

Proof: For $x \in X^*$, let $N_x^{\leq_f}$ and $N_x^{\leq}$ be open neighbourhoods of x with respect to $\tau_{\leq_f}$ and $\tau_\leq$, respectively. Then

$$N_x^{\leq_f} \cap [z]_{\equiv_f} = f^{-1}\left(N_{f(x)}^{\leq} \cap [f(z)]_\equiv\right)$$

for all $y \in X^*$. $\square$

Example 5.5 The *suffix order topology* is generated by the *suffix order* $\leq_{\mathrm{s}}$ defined by $x \leq_{\mathrm{s}} y \iff y \in X^* x$ for $x, y \in X^*$. Consider the mapping $\mathrm{mir} : X^* \to X^*$ given by $\mathrm{mir}(\varepsilon) = \varepsilon$, $\mathrm{mir}(x) = x$ for $x \in X$, and $\mathrm{mir}(xy) = \mathrm{mir}(y)\,\mathrm{mir}(x)$ for $x, y \in X^*$. Then mir is the *mirror function* and $\leq_{\mathrm{s}} = (\leq_{\mathrm{p}})_{\mathrm{mir}}$. Therefore, by Lemma 5.4, the suffix order topology also satisfies the condition TOP with respect to the length equivalence.

We now derive our main result. It states, intuitively, that the sets of true, but unprovable statements are large with respect to all topologies satisfying the condition TOP. The examples above indicate that the class of such topologies is quite large and includes many natural ones.

Theorem 5.6 *Let $\equiv$ be a recursive equivalence relation on X^* and let $A \subseteq X^*$ be a set which is expressible in T, saturated by $\equiv$, and not recursively enumerable. Then the set $I(A)$ is dense in every topology τ on X^* which satisfies* TOP *with respect to $\equiv$.*

Proof: By Theorem 4.4 the set $S = \{x \mid x \in X^*,\ [x]_\equiv \subseteq I(A)\}$ is infinite. Consider an element $y \in X^*$ and an open neighbourhood N_y of y. By TOP, the set of elements $u \in X^*$ such that $N_y \cap [u]_\equiv = \emptyset$ is finite. Hence, there is an element $u \in S$ with $N_y \cap [u]_\equiv \neq \emptyset$, that is, $[u]_\equiv \subseteq I(A)$ and $N_y \cap [u]_\equiv \neq \emptyset$. Thus $I(A)$ is dense. □

The assumptions of Theorem 5.6 hold true for all the topologies considered so far. In these cases one takes the length equivalence as the equivalence relation. Hence, if $A \subseteq X^*$ is expressible in $T \in \mathbb{T}$, saturated by $\equiv_{\mathrm{length}}$, and not recursively enumerable then $I(A)$ is dense with respect to the prefix, masking, lexicographic, and suffix topologies as shown by Example 5.1, Example 5.2, Example 5.3, and Example 5.5.

A partial order $\leq$ on X^* is said to be *length preserving* if, for all $x, y \in X^*$, $x \leq y$ implies $|x| \leq |y|$. Of the partial orders introduced so far, only the lexicographic order is not length preserving. Further important examples of length preserving partial orders include the *infix order* $\leq_{\mathrm{i}}$ and the *embedding order* $\leq_{\mathrm{h}}$ [Shy]. The former is defined by $x \leq_{\mathrm{i}} y \iff y \in X^* x X^*$ for $x, y \in X^*$. For the latter, one has $x \leq_{\mathrm{h}}$ if and only if there exist $n \in \{1, 2, \ldots\}$ and $x_1, \ldots, x_n \in X^*$ such that $x = x_1 \cdots x_n$ and $y \in X^* x_1 X^* x_2 \cdots X^* x_n X^*$. The topologies defined by these two relations also satisfy the condition TOP.

Theorem 5.7 *Suppose that the topology τ is generated by a recursive and length preserving partial order and that it satisfies* TOP *with respect to a recursive equivalence relation $\equiv$. Let $A \subseteq X^*$ be a set which is expressible in T, saturated by $\equiv$, and not recursively enumerable. Then the set $I(A)$ is co-rare in τ.*

Proof: Let $\leq$ be the given recursive and length preserving partial order on X^*, and consider an element $x \in X^*$ and an open neighbourhood N_x of x with respect to the topology $\tau_\leq$. Assume that $N_x \subseteq \mathbf{C}_{\tau_\leq}(\overline{I(A)})$. Thus, for every y with $x \leq y$ there is a word u_y such that $y \leq u_y$ and $u_y \in \overline{I(A)}$ and, therefore, $|y| \leq |u_y|$. This means that the set $\{z \mid z \in X^*,\ N_x \cap [z]_\equiv = \emptyset\}$ is infinite, contradicting the condition TOP. □

Corollary 5.8 *Let $A \subseteq X^*$ be a set which is expressible in T, saturated by $\equiv_{\mathrm{length}}$, and not recursively enumerable. For any partial order $\leq \in \{\leq_{\mathrm{p}}, \leq_{\mathrm{m}}, \leq_{\mathrm{s}}, \leq_{\mathrm{i}}, \leq_{\mathrm{h}}\}$ the set $I(A)$ is co-rare with respect to the topology $\tau_\leq$.*

The statement of Corollary 5.8 applies to all of the important undecidable problems like the Halting Problem, Totality Problem, Emptiness Problem, Post Correspondence Problem in the theory of computation ([Ca1], [Roz]), to the Hamiltonian Problem in Hamiltonian mechanics [Co], and to many others. Thus, the set of independent instances of any of these problems is large, that is, co-rare, with respect to any topology generated by a recursive and length preserving partial order.

Our results hold true regardless of how the topology is generated and which equivalence relation is used. The topologies generated by partial orders and the length equivalence are only convenient examples.

6. Concluding Remarks

Our results prove the largeness of the set of true and unprovable sentences in a topological sense. They pertain to all productive sets and include, as special cases, various types of inseparability [Cas], [Yo]. By [Cas], examples of inseparable sets in the sense of Smullyan abound—in the Baire category sense; this reinforces the interpretation of "largeness" as proved above.

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