Dyson Statements that Are Likely to Be True but Unprovable

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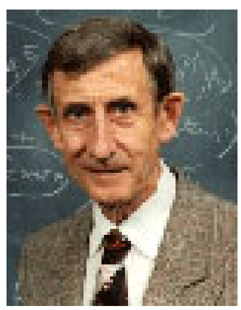
December 2, 2018

Gödel's Incompleteness Theorem states that every finitely-specified, sound, theory which is strong enough to include arithmetic cannot be both consistent and complete. In particular, if the theory is consistent, then it is incomplete, i.e. it contains statements that cannot be proved nor disproved. If we adopt a semantic criterion (a definition of true statements), then the theory contains true and unprovable statements. The set of is true and unprovable statements is large in both topological sense [4] and probabilistic sense [3] (see more in [1]). Are there examples of true and unprovable statements? Yes, but they are not simple [6].



Freeman Dyson

In [5], p. 86, Dyson stated:

> Thanks to Kurt Gödel, we know that there are true mathematical statements that cannot be proved. But I want a little more than this. I want a statement that is true, unprovable, and simple enough to be understood by people who are not mathematicians.

Dyson's proposal is the following. Consider the powers of two 2, 4, 8, 16, 32, 64, 128, ... and the powers of five 5, 25, 125, 625 ...

The mirror (reverse) of a number is the number formed with the same digits but written in opposite order. For example, the mirror of 12 is 21, the mirror of 131072 is 270131, the mirror of 5 is 5, etc.

Dyson's conjecture is:

> **It never happens that the reverse of a power of two is a power of five.**

Question 1. Write a simple program Π that never stops iff Dyson's conjecture is true. How small can you make Π? Use Π to test the conjecture up to powers smaller than a billion.

Here is Dyson's argument (not proof!) [5], 85–86 in favour of the plausibility of the conjecture:

> The digits in a big power of two seem to occur in a random way without any regular pattern. If it ever happened that the reverse of a power of two was a power of five, this would be an unlikely accident, and the chance of it happening grows rapidly smaller as the numbers grow bigger. If we assume that the digits occur at random, then the chance of the accident happening for any power of two greater than a billion is less than one in a billion. It is easy to check that it does not happen for powers of two smaller than a billion.
>
> [Here is a Mathematica code for testing Dyson's conjecture up to the power of two to a million completed in 526.6 seconds:
>
> ```
> n := 0; For[i = 1, i < 100000, i++, m = 2^i;
> n = n + Boole[
> IntegerPart[Log[5, FromDigits[Reverse[IntegerDigits[m]]]]] ==
> Log[5, FromDigits[Reverse[IntegerDigits[m]]]]]]; Print[n]
> ```
>
>]
>
> So the chance that it ever happens at all is less than one in a billion. That is why I believe the statement is true.

Question 2. Is it true that "The digits in a big power of two occur in a random way without any regular pattern" when "random" is taken as "algorithmic random" in the sense of Algorithmic Information Theory [1]? What about the digits of a large power of five?

Question 3. "If we assume that the digits occur at random, then the chance of the accident happening for any power of two greater than a billion is less than one in a billion." What is the probability used to measure the 'chance'?

Question 4. Is Dyson's conjecture true?

The conjecture appears not to be provable because [5], p.56:

> But the assumption that digits in a big power of two occur at random also implies that the statement is unprovable. Any proof of the statement would have to be based on some non-random property of the digits. The assumption of randomness means that the statement is true just because the odds are in its favor. It cannot be proved because there is no deep mathematical reason why it has to be true.

Question 5. Is Dyson's conjecture unprovable with respect to ZFC?

According to Dyson [5], 56:

> This argument does not work if we use powers of three instead of powers of five. In that case the statement is easy to prove because the reverse of a number divisible by three is also divisible by three. Divisibility by three happens to be a non-random property of the digits.

Question 6. What about other powers of prime numbers? Are they 'random'?

Dyson [5], 56:

> It is easy to find other examples of statements that are likely to be true but unprovable. The essential trick is to find an infinite sequence of events, each of which might happen by accident, but with a small total probability for even one of them happening. Then the statement that none of the events ever happens is probably true but cannot be proved.

Question 7. Does the above "trick" work? Why? Can you use it?

References

[1] C. S. Calude. *Information and Randomness: An Algorithmic Perspective*, 2nd Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.

[2] C. S. Calude. Incompleteness, complexity, randomness and beyond, *Minds and Machines: Journal for Artificial Intelligence, Philosophy and Cognitive Science* 12, 4 (2002), 503–517.

[3] C. S. Calude, H. Jürgensen. Is complexity a source of incompleteness? *Advances in Applied Mathematics* 35 (2005), 1–15.

[4] C. Calude, H. Jürgensen, M. Zimand. Is independence an exception ?, *Appl. Math. Comput.* 66 (1994), 63–76.

[5] F. Dyson. in J. Brockman. *What to Believe but Cannot Prove*, Pocket Books, London, 2005, 85–86. See also `http://www.edge.org/q2005/q05_9.html#dysonf`.

[6] J. Spencer. Large numbers and unprovable theorems, *American Mathematical Monthly*, 90, 10 (1983), 669–675.