Complexity and Incompleteness

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Chaitin's "heuristic principle", *the theorems of a finitely-specified theory cannot be significantly more complex than the theory itself* was proved for an appropriate measure of complexity in [1]. The measure δ is a computable variation of the program-size complexity H:

$$\delta(x) = H(x) - |x|.$$

The theorems of a finitely-specified, sound, consistent theory which is strong enough to include arithmetic have bounded δ-complexity, hence every sentence of the theory which is significantly more complex than the theory is unprovable. More precisely, according to Theorem 4.6 in [1], for any finitely-specified, sound, consistent theory strong enough to formalize arithmetic (like Zermelo-Fraenkel set theory with choice or Peano Arithmetic) and for any Gödel numbering g of its well-formed formulae, we can compute a bound N such that no sentence x with complexity $\delta_g(x) > N$ can be proved in the theory; this phenomenon is independent on the choice of the Gödel numbering.

Question 1. Find other natural measures of complexity for which Chaitin's "heuristic principle" holds true.

Previous results showing that incompleteness is not accidental, but ubiquitous [4] have been reinforced in [1] probabilistic terms: the probability that a true sentence of length n is provable in the theory tends to zero when n tends to infinity, while the probability that a sentence of length n is true is strictly positive.

Question 2. [Peter Cholak] Is it possible to show that there is a sequence of Gödel codings g_i ofsentences such that theprobability $p_{i,n}$ a sentece of length n is true goes to 0? Hence can you find a coding of sentences such that theprobability asentence is true is as close to zero as you want.

Sentences expressed by strings with large δ-complexity are unprovable.

Question 3. Given a theory as in the statement of Theorem 4.6 in [1], are there independent sentences x with low δ-complexity?

Even if such sentences do exist, in view of Theorem 5.2 in [1], the probability that a true sentence of length n with δ-complexity less than or equal to N is unprovable in the theory tends to zero when n tends to infinity.

Question 4. Implement any of the above proofs for a specific formal theory in an automated reasoning system (for example [12]) and compute an upper bound for the constant N.

Question 5. Revisit the relation between complexity, incompleteness and uncertainty studied in [5].

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