Computer Science 750 (2019)

Assignment 1

This assignment is worth 30 marks representing 7.5% of your total course grade.

Due date: before 18 August 2019 23.50 in Canvas

Name:

ID:

Questions

1. Construct a universal prefix-free Turing machine. You can assume that you have a computable enumeration of all prefix-free Turing machines $(M_i)_{i\in\mathbf{N}}$. **[10 marks]**

   Sample solution. We define $U(0^i1x) = M_i(x)$, $x \in \mathrm{dom}(M_i)$. We need to prove a) for every prefix-free Turing machine M there exists $c > 0$ such that for every input a there exists an input b with $|b| \leq |a| + c$ such that $U(b) = M(a)$ and b) U is a prefix-free. The condition a) is satisfied because by hypothesis there is an i such that $M = M_i$ so given a we construct $b = 0^i1a$ and the inequality is satisfied for $c = i + 1$. For b) let $u, v \in \mathrm{dom}(U)$ so there exist i, j, x, y such that $u = 0^i1x, v = 0^j1y$. If u is a prefix of v, then $i = j$, hence x is a prefix of y; as both x, y belong to the domain of $M_i = M_j$ which is a prefix-free Turing machine, $x = y$, so $u = v$.

2. Recall that $x^* = \min\{p \mid U(p) = x\}$. Prove that the set $\{x^* \mid x \in \{0,1\}^*\}$ is immune, that is, it contains no infinite c.e. subset. **[10 marks]**

   Sample solution. We proved in class the existence of a constant t such that for every x,

   $$H(x^*) \geq |x^*| - t. \tag{1}$$

   Assume by absurdity the existence of an infinite computable subset D of $\{x^* \mid x \in \{0,1\}^*\}$ and define the Turing machine $C(0^i1) = \min\{x \in D \mid |x| \geq t + 2^i\}$, $i \geq 1$ (C is p.c. as D is computable). The Turing machine C is prefix-free because $\mathrm{dom}(C) \subseteq \{0^i1 \mid i \geq 1\}$ and the larger set is prefix-free, and by definition of D, (1) and C we have:

   $$H(C(0^i1)) \geq |C(0^i1)| - t \geq t + 2^i - t = 2^i, \tag{2}$$

   for infinitely many $i \geq 1$.

   Using the universality of H we get a constant $c > 0$ such that for every x, $H(x) \leq H_C(x) + c$, so by (2)

   $$2^i \leq H(C(0^i1)) \leq H_C(C(0^i1)) + c \leq i + 1 + c,$$

   for infinitely many $i \geq 1$, a contradiction.

3. Prove that there is no p.c. function φ with infinite domain such that the program-size complexity $H(x) = \varphi(x)$ for all $x \in \mathrm{dom}(\varphi)$. Deduce that H is not computable. **[10 marks]**

   Assume by absurdity the existence of a p.c. function φ with infinite domain such that $H(x) = \varphi(x)$ for all $x \in \mathrm{dom}(\varphi)$; as $\mathrm{dom}(\varphi)$ is c.e. we can construct an infinite computable subset $B \subseteq \mathrm{dom}(\varphi)$ and define the Turing machine $C(0^i1) = \min\{x \in B \mid H(x) \geq 2^i\} = \min\{x \in B \mid \varphi(x) \geq 2^i\}$, $i \geq 1$. The Turing machine C is prefix-free because $\mathrm{dom}(C) \subseteq \{0^i1 \mid i \geq 1\}$ and the larger set is prefix-free, and by definition $H(C(0^i1)) \geq 2^i$, for infinitely many $i \geq 1$. Using the universality of H we get a constant $c > 0$ such that for every x, $H(x) \leq H_C(x)+c$, so

$$2^i \leq H(C(0^i1)) \leq H_C(C(0^i1)) + c = i + 1 + c,$$

for infinitely many $i \geq 1$, a contradiction. In particular there is no computable φ such that $H(x) = \varphi(x)$ for all x, so H is incomputable.
