AIT with Natural Complexity

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The project consists in developing a large part of AIT using the natural complexity ∇ [2, 3] instead of the prefix complexity H [4, 1].

All strings are binary and the set of strings is denoted by Σ^*. The length of x is denoted by $|x|$. The logarithms are binary too. Let $\mathbf{N} = \{1, 2, \ldots\}$ and let $\mathrm{bin} : \mathbf{N} \to \Sigma^*$ be the computable bijection which associates to every $n \geq 1$ its binary expansion without the leading 1,

n	n_2	$\mathrm{bin}(n)$	$\lvert\mathrm{bin}(n)\rvert$
1	1	λ	0
2	10	0	1
3	11	1	1
4	100	00	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

One works with prefix-free Turing machines M which processes strings into strings. The domain of M, $\mathrm{dom}(M)$, is the set of strings on which M halts (is defined).

The *natural complexity* of the string $x \in \Sigma^*$ (with respect to M) is $\nabla_M(x) = \min\{n \geq 1 \mid M(\mathrm{bin}(n)) = x\}$ (see [3]).

The Invariance Theorem states that we can effectively construct a machine U (called *universal*) such that for every machine M, there is a constant $\varepsilon > 0$ (depending upon U and M) such that $\nabla_U(x) \leq \varepsilon \cdot \nabla_M(x)$, for all strings x.

For example, if $U(0^i 1x) = M_i(x)$ (where (M_i) is an effective enumeration of all Turing machines), then $\nabla_U(x) \leq (2^{i+1} + 1) \cdot \nabla_{M_i}(x)$, because $0^i 1\mathrm{bin}(m) = \mathrm{bin}(2^{i+1+\lfloor \log(m) \rfloor} + m)$, for all $m \geq 1$.

In what follows we will fix a universal prefix-free Turing machine U and we will write ∇ instead of ∇_U. There are some advantages in working with the complexity ∇ instead of the classical complexity H (see [1]); for example, for every $N > 0$, the inequality $\#\{x \in \Sigma^* : \nabla(x) < N\} \leq N$ is obvious.

References

[1] C. S. Calude. *Information and Randomness: An Algorithmic Perspective*, 2nd Edition, Revised and Extended, Springer-Verlag, Berlin, 2002

[2] C. S. Calude, M. A. Stay. From Heisenberg to Gödel via Chaitin, *International Journal of Theoretical Physics* 44, 7 (2005), 1053–1065.

[3] C. S. Calude, M. A. Stay. Natural halting probabilities, partial randomness, and Zeta functions, *Information and Computation* 204 (2006), 1718–1739.

[4] G. J. Chaitin. *Algorithmic Information Theory*, Cambridge University Press, Cambridge, 1987. (third printing 1990)