THE UNIVERSITY OF AUCKLAND
Department of Computer Science
Private Bag 92019,
Auckland, New Zealand

415.320 Algorithmics

Assignment No 1

August 2000

Due date: 18 August 2000, time 4.00 pm, 320 box in the Computer Science Department main office.

1. [**10 points**] Prove the correctness of the following algorithm computing the nth Fibonacci number:

```
function Fibiter (n)
    i ← 1, j ← 0
    for k = 1 to n do
        j ← i + j
        i ← j − i
    return j.
```

2. [**30 points**] Consider the following algorithm accepting as input two positive integers, m, n:

```
function Manna (m, n)
    x1 ← m, x2 ← n
    x3 ← x2, x4 ← 0
    while x1 ≠ x2 do
        while x1 > x2, do
            x1 ← x1 − x2
            x4 ← x3 + x4
        while x1 < x2, do
            x2 ← x2 − x1
            x3 ← x3 + x4
    y ← x3 + x4, z = x1
    return y, z.
```

- What is the result produced by the algorithm for $m = 48, n = 56$?
- What is the function computed by the algorithm?
- Show that the relation
$$x_1x_3 + x_2x_4 = mn,$$
is an invariant of the algorithm.
- Prove the correctness of the algorithm.

3. **[10 points]** Prove that for every $c > 0$ there exists a positive integer N_c (depending upon c) such that $cn^2 < 2^n$, for all $n \geq N_c$. Find the minimum bound N_c for $c = 10$.

4. **[10 points]** Let $f : \mathbf{N} \to \mathbf{R}^+$ be a function. Which of the following implications is true:

   - $f(n) \in O(n) \Longrightarrow [f(n)]^2 \in O(n^2)$?
   - $[f(n)]^2 \in O(n^2) \Longrightarrow f(n) \in O(n)$?
   - $f(n) \in O(n) \Longrightarrow 2^{f(n)} \in O(2^n)$?
   - $2^{f(n)} \in O(2^n) \Longrightarrow f(n) \in O(n)$?

   Justify each answer with a proof or counter-example.

5. **[10 points]** Let $k > 0$. Prove that $O(n^k) = O((n+1)^k)$, but $O(n!) \neq O((n+1)!)$.

6. **[10 points]** Find the error in the following "proof" of the equality "$O(n) = O(n^2)$":

   Use the maximum rule to get

   $$O(n^2) = O(\underbrace{n + n + \cdots + n}_{n \text{ times}}) = O(\max(\underbrace{n, n, \ldots, n}_{n \text{ times}})) = O\,(n),$$

7. **[10 points]** Define the class $\Theta(f(n))$ and prove the equality $\Theta(n-1) + \Theta(n) = \Theta(n)$.

8. **[10 points]** Define the class $n^{O(1)}$ and construct a function $f : \mathbf{N} \to \mathbf{R}^{\geq 0}, f(n) \leq n$, for all n, that is not in $n^{O(1)}$. Justify your construction.