Supplemental Material to: "Performance Analysis for a Chaos-Based CDMA System in Wide-Band Channel"

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Chapter 1

Performance Analysis

1.1 Main Equation

Assume that the received signal is the superposition of the messages from N users. Let the chaotic sequence for the n-th user be

$$\mathbf{x}^{(n)} = \left[x_1^{(n)}, x_2^{(n)}, \ldots, x_{2\beta}^{(n)}\right]^\top,$$

where the operator $(\cdot)^\top$ denotes transposition. Without loss of generality, we assume that chaotic sequence of the user who received the signal is $\mathbf{x}^{(1)}$, and define

$$a_n = \left(\mathbf{x}^{(n)}\right)^\top \mathbf{x}^{(1)} = \left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(n)}. \tag{1.1}$$

Furthermore, we denote $A = a_1 + a_2 + \cdots + a_N$.

For writing the equations more compactly, we also introduce

$$\mathbf{x}_d^{(n)} = \left[x_0^{(n)}, x_1^{(n)}, \ldots, x_{2\beta-1}^{(n)}\right]^\top, \text{ for all } n \in \{1, 2, \ldots, N\}.$$

In the equation above, we use the convention that $x_0^{(n)} = x_{2\beta}^{(n)}$. Additionally, we define

$$b_n = \left(\mathbf{x}_d^{(n)}\right)^\top \mathbf{x}^{(1)} = \left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}_d^{(n)} \tag{1.2}$$

and $B = b_1 + b_2 + \cdots + b_N$.

For modeling the additive noise, we consider the zero-mean Gaussian random vector

$$\xi = [\xi_1, \xi_2, \ldots, \xi_{2\beta}]^\top, \tag{1.3}$$

whose covariance matrix equals $N_0\mathbf{I}$, where $N_0 > 0$ and $\mathbf{I}$ denotes the identity matrix of appropriate dimension. Let $C = \xi^\top \mathbf{x}^{(1)}$. The entries of ξ and those of $\mathbf{x}^{(n)}$, $\mathbf{x}_d^{(n)}$ are statistically independent for all $n \in \{1, 2, \ldots, N\}$.

Now we can re-write Eq. (8) from [1] by employing our own notation:

$$z = [\alpha_{00}s + \alpha_{01}(s-\tau)]A + \alpha_{01}\tau B + sC, \tag{1.4}$$

where α_{00} and α_{01} are statistically independent Rayleigh random variables which model the fading for the first path and the second path, respectively. We assume that the secondary path is delayed in time by τ. The design parameter s plays a key role as each chip is extended into s samples in order to analyze the performance of the system when the secondary path is delayed by only a portion of the chip interval. It is obvious that $0 \leq \tau \leq s$.

There have been previous attempts at evaluating the performance of the system by assuming that z is Gaussian [1]. One possible solution for improving the accuracy of BER is to use Gram-Charlier expansion for the probability density function (PDF) of z and to truncate it at the fourth moment (see [2, Section 6.17]). Hence, the supplementary terms in comparison with the crude Gaussian approximation will depend on skewness and kurtosis. If both skewness and kurtosis are close to zero, then we might expect a marginal improvement in accuracy, in comparison with the case when the Gaussian approximation is used. To clarify this aspect, we compute the skewness of each term within (1.4), under the following hypotheses [1]:

- The chaotic sequence for each user is generated by using the logistic map (see Section 3.1 for more details). The very first entry of each such chaotic sequence is drawn from the distribution given in (3.3) such that, for any two different users, the very first entries of their chaotic sequences are statistically independent.

- As already mentioned, the main path and the secondary path (where the delay τ occurs) are affected by Rayleigh fading. The fading factor remains constant over a transmitted bit interval. The parameter of the Rayleigh distribution used to model the fading on the main path is denoted b. For the secondary path, the parameter is $\tilde{b}$. So, we write $\alpha_{00} \sim \text{Rayleigh}(b)$ and $\alpha_{01} \sim \text{Rayleigh}(\tilde{b})$. We emphasize that α_{00} and α_{01} are statistically independent.

- For $n \in \{1, 2, \ldots, N\}$, the random variables α_{00} and α_{01} are statistically independent in rapport with the entries of the vectors $\mathbf{x}^{(n)}$ and $\mathbf{x}_d^{(n)}$. They are also statistically independent in rapport with the entries of the vector ξ.

In our calculations, the following results are useful [3, Chapter 35]:

$$E[\alpha_{00}] = b\left(\frac{\pi}{2}\right)^{1/2}, \tag{1.5}$$

$$E[\alpha_{00}^2] = 2b^2, \tag{1.6}$$

$$E[\alpha_{00}^3] = 3b^3 \left(\frac{\pi}{2}\right)^{1/2}, \tag{1.7}$$

where $E(\cdot)$ denotes the expectation operator. The moments of α_{01} can be obtained by replacing b with $\tilde{b}$ in the expressions above.

We introduce the supplementary definitions:

$$\delta_0 = \alpha_{00}s + \alpha_{01}(s - \tau), \tag{1.8}$$

$$\delta_1 = \alpha_{01}\tau. \tag{1.9}$$

Moreover, for an arbitrary random variable RV, we use the notation $\eta_3(\mathrm{RV})$ for its skewness. In particular, when $\mathrm{RV} \in \{A, B, (\delta_1 B)\}$ and N is fixed ($N > 2$), based on the calculations presented in Section 3.4, we have:

$$\eta_3(\mathrm{RV}) = \frac{1}{(2\beta)^{1/2}} \left[\mathcal{C}_N(\mathrm{RV}) + o(1)\right] \quad (2\beta \gg 1), \tag{1.10}$$

where $\mathcal{C}_N(A) = \dfrac{3(N - 3/4)}{(N - 1/2)^{3/2}}$, $\mathcal{C}_N(B) = \dfrac{3(N + 1/2)}{N^{3/2}}$ and $\mathcal{C}_N(\delta_1 B) = \dfrac{3\pi^{1/2}}{4}\mathcal{C}_N(B)$.

For $\tau \in \{0, s/2, s\}$, fixed N and $\beta \to \infty$, the skewness of the term $(\delta_0 A)$ is given by

$$\eta_3(\delta_0 A) = \mathcal{C}_{b,\tilde{b},\tau/s}[1 + o(1)], \tag{1.11}$$

where $\mathcal{C}_{b,\tilde{b},\tau/s} = \dfrac{2\pi^{1/2}(\pi - 3)}{(4 - \pi)^{3/2}} \dfrac{b^3 + (1 - \tau/s)^3\tilde{b}^3}{\left[b^2 + (1 - \tau/s)^2\tilde{b}^2\right]^{3/2}}$. The details for derivation of (1.11) can be found in Section 3.4. Additionally, one can verify without difficulties that $\eta_3(C) = 0$.

We have been able to compute the skewness for each term within (1.4), but the evaluation of $\eta_3(z)$ requires some more calculations. For applying the Gram-Charlier expansion, we should also evaluate the kurtosis of z. Because of the tedious calculations involved, the use of Gram-Charlier expansion does not provide a practical solution for the problem we want to solve.

There is also another important lesson which we learned from the computation of skewness for various terms. The result in (1.10) shows that the skewness of A, B, $(\delta_1 B)$ is almost zero when β is large. According to (1.11), the situation is different for $\delta_0 A$, whose skewness does not decrease to zero even if β is very large. This suggests that a better alternative to the Gaussian assumption for z is to make all the calculations under the hypothesis that A, B, C are Gaussian distributed. We will pursue this idea in the next section.

1.2 Theoretical Bit Error Rate

1.2.1 Main Idea

The analysis above suggests that the Gaussian assumption for z is inappropriate. However, from the same analysis we know that, for large β, the Gaussian distribution might be a good approximation for the conditional distribution of z given α_{00}, α_{01} and τ. This leads to the natural choice of computing first the conditional BER and then apply the law of total probability (for continuous distributions). This approach is in line with what has been already done for similar problems, in the case of narrow-band channels (see, for example, [4]). The details of the calculations are outlined below.

1.2.2 Conditional Bit Error Rate

We begin by computing

$$\begin{aligned}\mathrm{BER}(\alpha_{00},\alpha_{01},\tau) &= \frac{1}{2}\mathrm{erfc}\left(\frac{E[z|\alpha_{00},\alpha_{01},\tau]}{\sqrt{2\mathrm{Var}[z|\alpha_{00},\alpha_{01},\tau]}}\right)\\ &= \frac{1}{2}\mathrm{erfc}\left(\left\{\frac{2\mathrm{Var}[z|\alpha_{00},\alpha_{01},\tau]}{(E[z|\alpha_{00},\alpha_{01},\tau])^2}\right\}^{-1/2}\right), \end{aligned} \tag{1.12}$$

where $\mathrm{erfc}(\cdot)$ has the well-known expression [5, p. 48]:

$$\mathrm{erfc}(\psi) = \frac{2}{\sqrt{\pi}}\int_{\psi}^{\infty}\exp(-\omega^2)\mathrm{d}\omega.$$

We employ results which are proved in Chapter 3 (see Lemma 3.2.2, Lemma 3.2.4 and Lemma 3.3.1) in order to calculate

$$\begin{aligned}E[z|\alpha_{00},\alpha_{01},\tau] &= \delta_0 E[A] + \delta_1 E[B] + sE[C]\\ &= \frac{\delta_0}{2}(2\beta),\\ E[z^2|\alpha_{00},\alpha_{01},\tau] &= \delta_0^2 E[A^2] + \delta_1^2 E[B^2] + s^2 E[C^2] + 2\delta_0\delta_1 E[AB] \qquad (1.13)\\ &= \delta_0^2\left[\frac{N}{4}(2\beta) + \frac{1}{4}(2\beta)^2 - \frac{1}{8}(2\beta)\right]\\ &+ \delta_1^2\frac{N}{4}(2\beta) + s^2\frac{N_0}{2}(2\beta) + 2\delta_0\delta_1\frac{1}{8}(2\beta-2), \qquad (1.14)\end{aligned}$$

where δ_0 and δ_1 are given in (1.8) and (1.9), respectively.

The quantity of interest for us is

$$\frac{2\mathrm{Var}[z|\alpha_{00},\alpha_{01},\tau]}{(E[z|\alpha_{00},\alpha_{01},\tau])^2} = \left(\frac{\delta_1}{\delta_0}\right)^2\frac{2N}{2\beta} + \left(\frac{\delta_1}{\delta_0}\right)\frac{2(2\beta-2)}{(2\beta)^2} \tag{1.15}$$

$$+\left(\frac{1}{\delta_0}\right)^2 N_0 \frac{4s^2}{2\beta} \tag{1.16}$$

$$+\frac{2N-1}{2\beta}. \tag{1.17}$$

Under the assumption that $\alpha_{01} \neq 0$, $\tau \neq 0$ and $r = \alpha_{00}/\alpha_{01}$, we can rearrange some of the terms in the equation above:

$$\frac{\delta_1}{\delta_0} = \frac{1}{\frac{s}{\tau}(r+1)-1}, \tag{1.18}$$

$$\left(\frac{1}{\delta_0}\right)^2 N_0 \frac{4s^2}{2\beta} = \frac{N_0}{\beta\alpha_{00}^2}\frac{2}{\left[1+\frac{1}{r}\left(1-\frac{\tau}{s}\right)\right]^2}. \tag{1.19}$$

These identities lead to the following conclusions:

- Remark in (1.19) that the term $N_0(\beta\alpha_{00}^2)^{-1}$ can be written as $N_0(E_b\alpha_{00}^2)^{-1}$, where E_b is the bit energy and equals $2\beta E[x_i^2]$. We refer to [4, Eq. (14)] for the definition of the bit energy. The reader can also see in Chapter 3 (Eq. (3.5)) that $E[x_i^2] = 1/2$ for the logistic map. Combining the results from (1.12), (1.16) and (1.19), we can conclude that the BER decreases when the product $E_b\alpha_{00}^2$ raises. It is also interesting to note that the positive factor which multiplies $N_0(\beta\alpha_{00}^2)^{-1}$ in (1.19) is smaller than two for all possible values of r, τ and s.

- The performance is the same for all selections of s and τ for which the ratio s/τ has a certain value. An increase of s/τ guarantees a lower BER [see again (1.12), (1.15) and (1.19)].

- The increase of r has mixed effects in the sense that δ_1/δ_0 decreases, whereas the value of the expression in (1.19) grows.

1.2.3 Average Bit Error Rate

To gain more insight, we investigate separately the influence of the second propagation path and that of the additive Gaussian noise. Then we treat the general case.

Case #1: Effect of additive noise is neglected ($N_0 = 0$) If we ignore the term in (1.16), then $E_b/N_0 = \infty$ and the expression of conditional BER becomes

$$\text{BER}(r,\tau)|_{E_b/N_0=\infty} = \frac{1}{2}\text{erfc}\left(\zeta_{r,\tau}^{-1/2}\right),$$

where

$$\zeta_{r,\tau} = \frac{N/\beta}{\left[\frac{s}{\tau}(r+1)-1\right]^2} + \frac{(\beta-1)/\beta^2}{\frac{s}{\tau}(r+1)-1} + \frac{2N-1}{2\beta}.$$

Furthermore, we can calculate

$$\mathrm{BER}(\tau)|_{E_b/N_0=\infty} = \int_0^\infty \left[\mathrm{BER}(r,\tau)|_{E_b/N_0=\infty}\right] f(r)\mathrm{d}r, \tag{1.20}$$

where $f(r)$ denotes the PDF of r. Assuming that (i) $\alpha_{00} \sim \mathrm{Rayleigh}(b)$, (ii) $\alpha_{01} \sim \mathrm{Rayleigh}(\tilde{b})$ and (iii) α_{00} and α_{01} are statistically independent, we have the following expression for f(r) [6, Corollary 3.4]: $f(r) = \frac{2rb^2\tilde{b}^2}{\left(r^2\tilde{b}^2+b^2\right)^2}$, $0 < r < \infty$. We note in passing that the mode of $f(r)$ is given by $b/(3^{1/2}\tilde{b})$.

Case #2: $N_0 > 0$, α_{00} and α_{01} are linearly dependent Now we consider that

$$\alpha_{00}/\alpha_{01} = r_0, \tag{1.21}$$

where r_0 is fixed ($r_0 \geq 1$). This is a major deviation from the original set of assumptions listed in Section 1.1, but it will help us to gain more insight on the problem we analyze. This assumption leads to

$$\mathrm{BER}(\alpha_{00},\tau)|_{r=r_0} = \frac{1}{2}\mathrm{erfc}\left(\zeta_{\alpha_{00},\tau}^{-1/2}\right),$$

where

$$\begin{aligned}
\zeta_{\alpha_{00},\tau} &= \frac{N}{D_1^2\beta} + \frac{\beta-1}{D_1\beta^2} + \frac{2N_0}{\alpha_{00}^2\beta D_2^2} + \frac{2N-1}{2\beta}, \\
D_1 &= \frac{\delta_0}{\delta_1} = \frac{s}{\tau}(r_0+1)-1, \\
D_2 &= \frac{\delta_0}{\alpha_{00}s} = 1 + \frac{1}{r_0}\left(1-\frac{\tau}{s}\right).
\end{aligned} \tag{1.22}$$

It follows that

$$\mathrm{BER}(\tau)|_{r=r_0} = \int_0^\infty \left[\mathrm{BER}(\alpha_{00},\tau)|_{r=r_0}\right] f(\alpha_{00})\mathrm{d}\alpha_{00}, \tag{1.23}$$

where

$$f(\alpha_{00}) = (\alpha_{00}/b^2)\exp\left[-\alpha_{00}^2/(2b^2)\right], \quad 0 < \alpha_{00} < \infty, \tag{1.24}$$

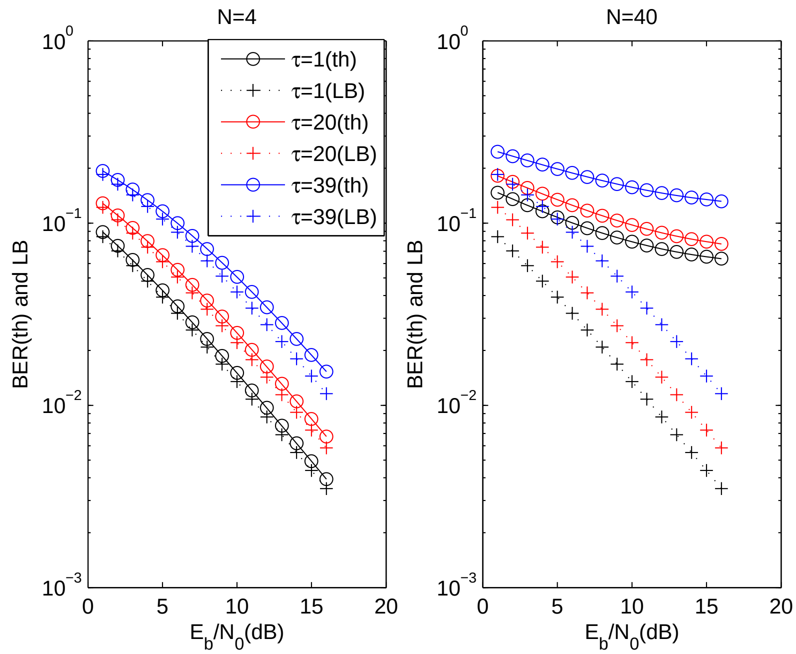

Figure 1.1: The values of BER(th) obtained when computing numerically the integral in (1.23) are compared with LB - the lower bound in (1.31). These results are for $N = 4$ users (left panel) and $N = 40$ users (right panel). The values of the time delay τ are given in the legend. Additionally, $s = 40$, $\beta = 50$ and $r_0 = 1.1$.

because $\alpha_{00} \sim \text{Rayleigh}(b)$. For writing the formula in (1.23) in a more convenient form, we firstly re-write $\zeta_{\alpha_{00},\tau}$ as $\zeta_{\gamma,\tau}$:

$$\zeta_{\gamma,\tau} = v + \frac{w}{\gamma}, \tag{1.25}$$

$$\gamma = \alpha_{00}^2,$$

$$v = \frac{N}{D_1^2\beta} + \frac{\beta - 1}{D_1\beta^2} + \frac{2N-1}{2\beta}, \tag{1.26}$$

$$w = \frac{2N_0}{\beta D_2^2}. \tag{1.27}$$

After some algebra, we get that the PDF of γ is $f(\gamma) = \exp(-\gamma/\bar{\gamma})/\bar{\gamma}$, where $0 < \gamma < \infty$ and $\bar{\gamma} = 2b^2$. So,

$$\text{BER}(\tau)|_{r=r_0} = \int_0^\infty \left[\text{BER}(\gamma,\tau)|_{r=r_0}\right] f(\gamma)\mathrm{d}\gamma$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty \operatorname{erfc}(\zeta_{\gamma,\tau}^{-1/2}) f(\gamma)\mathrm{d}\gamma \\
&= \int_0^\infty Q(\sqrt{2}\zeta_{\gamma,\tau}^{-1/2}) f(\gamma)\mathrm{d}\gamma && (1.28)\\
&= \frac{1}{\pi}\int_0^\infty \left[\int_0^{\pi/2} \exp\left(-\frac{\zeta_{\gamma,\tau}^{-1}}{\sin^2\theta}\right)\mathrm{d}\theta\right] f(\gamma)\mathrm{d}\gamma. && (1.29)
\end{aligned}
$$

In (1.28), we have used the well-known relationship between erfc$(\cdot)$ and the Gaussian Q-function. The reader can find more details in [7, p. 85], where is also presented the identity we employed in (1.29). However, the double integral in (1.29) cannot be easily computed, but it allows us to obtain a lower bound for $\mathrm{BER}(\tau)|_{r=r_0}$. It is enough to observe in (1.26) that $v > 0$, which leads to

$$
\begin{aligned}
\mathrm{BER}(\tau)|_{r=r_0} &\geq \frac{1}{2}\int_0^\infty \operatorname{erfc}\left(\sqrt{\gamma/w}\right) f(\gamma)\mathrm{d}\gamma \\
&= \frac{1}{\pi}\int_0^\infty \left[\int_0^{\pi/2} \exp\left(-\frac{\gamma/w}{\sin^2\theta}\right)\mathrm{d}\theta\right] f(\gamma)\mathrm{d}\gamma \\
&= \frac{1}{\pi}\int_0^{\pi/2} \left[\int_0^\infty \exp\left(-\frac{\gamma/w}{\sin^2\theta}\right)\frac{\exp(-\gamma/\bar{\gamma})}{\bar{\gamma}}\mathrm{d}\gamma\right]\mathrm{d}\theta && (1.30)\\
&= \frac{1}{\pi}\int_0^{\pi/2} \left(1+\frac{\bar{\gamma}}{w\sin^2\theta}\right)^{-1}\mathrm{d}\theta \\
&= \frac{1}{2}\left(1-\frac{1}{\sqrt{1+w/\bar{\gamma}}}\right). && (1.31)
\end{aligned}
$$

In (1.30), we used the fact that w is positive (see the definition in (1.27)), whilst in (1.31) we applied the identity from [7, Eq. (5.6)].

From (1.27), we know that $w = (E_b/N_0)^{-1} \times (2/D_2^2)$. This shows clearly the influence of the ratio E_b/N_0 on the lower bound we obtained in (1.31). The same identity shows that the lower bound also depends on r_0, τ and s (see again the definition in (1.22)). However, the lower bound does not depend on the number of users N because, in the derivation of (1.31), the v-term from (1.25) was ignored. For example, if we take $N = 4$ or $N = 40$, the lower bound in (1.31) is the same (given that all other settings are the same). For illustration, we plot in Fig. 1.1 the values of the lower bound when $\beta = 50$, $r_0 = 1.1$, $s = 40$, $\tau \in \{1, 20, 39\}$ and $E_b/N_0 \in \{1\mathrm{dB}, 2\mathrm{dB}, \ldots, 16\mathrm{dB}\}$. In the same figure, we show the values of the integral in (1.23), which are numerically computed for $N = 4$ and $N = 40$, respectively. Remark that the lower bound is a good approximation of the integral when $N = 4$. It is not surprising that the approximation becomes much worse when N is large. This suggests that the lower bound might be used to approximate (1.23) only when N is small.

Case #3: $N_0 > 0$, α_{00} and α_{01} are statistically independent With the convention that $\zeta_{\alpha_{00},\alpha_{01},\tau}$ is given by the expression in (1.15)-(1.17), we get

$$\mathrm{BER}(\alpha_{00}, \alpha_{01}, \tau) = \frac{1}{2}\mathrm{erfc}\left(\zeta_{\alpha_{00},\alpha_{01},\tau}^{-1/2}\right), \tag{1.32}$$

$$\mathrm{BER}(\tau) = \int_0^\infty \int_0^\infty \mathrm{BER}(\alpha_{00}, \alpha_{01}, \tau) f(\alpha_{00}) f(\alpha_{01}) \mathrm{d}\alpha_{00} \mathrm{d}\alpha_{01}, \tag{1.33}$$

$$\tag{1.34}$$

where $f(\alpha_{00})$ is the same as in (1.24). For the distribution of α_{01}, we make the assumption that is Rayleigh$(\tilde{b})$ and consequently we have $f(\alpha_{01}) = (\alpha_{01}/\tilde{b}^2)\exp\left[-\alpha_{01}^2/(2\tilde{b}^2)\right]$ for $0 < \alpha_{01} < \infty$.

Case #4: $N_0 > 0$, α_{00} and α_{01} are statistically independent, τ-delay is random In most of the practical applications, the value of τ is not known a priori. This is why we propose to evaluate how the fact that τ is random impacts on the performance. The model used in [1] assumes that τ is sampled from an Exponential distribution (with parameter λ), but it was pointed out in the same reference that τ should be rather modeled as a discrete random variable than as a continuous one. We introduce a new model for τ, which is discrete. The novel model is described by resorting to the following algorithm:

1. Let the random variable τ_c have the Exponential distribution with parameter λ ($\lambda > 0$). Hence, the PDF of τ_c is given by

$$f(\tau_c) = \begin{cases} \lambda \ell^{\tau_c} & \text{if } \tau_c \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell = \exp(-\lambda)$.

2. Quantize τ_c as follows: $\tau_q = \lfloor \tau_c \rfloor$, where $\lfloor \cdot \rfloor$ denotes the largest integer not greater than the real number in the argument. It is clear that

$$\Pr(\tau_q = i) = (1 - \ell)\ell^i \text{ for all } i \in \{0, 1, 2, \ldots\}.$$

3. Take $\tau = \tau_q \pmod{s}$. Simple calculations lead to

$$\Pr(\tau = j) = \frac{1 - \ell}{1 - \ell^s}\ell^j \text{ for all } j \in \{0, 1, \ldots, s - 1\}. \tag{1.35}$$

We note that

$$\begin{aligned}
E[\tau] &= \frac{1-\ell}{1-\ell^s}\sum_{j=0}^{s-1} j\ell^j \\
&= \frac{(1-\ell)\ell}{1-\ell^s}\sum_{j=1}^{s-1} j\ell^{j-1} \\
&= \frac{\ell}{1-\ell^s}\frac{(s-1)\ell^s - s\ell^{s-1}+1}{1-\ell}, \qquad (1.36) \\
E[\tau^2] &= \frac{1-\ell}{1-\ell^s}\sum_{j=0}^{s-1} j^2\ell^j \\
&= \frac{1-\ell}{1-\ell^s}\sum_{j=1}^{s-1} j^2\ell^j \\
&= \frac{1-\ell}{1-\ell^s}\left[\sum_{j=1}^{s-1} j(j-1)\ell^j + \sum_{j=1}^{s-1} j\ell^j\right] \\
&= E[\tau] + \frac{(1-\ell)\ell^2}{1-\ell^s}\sum_{j=2}^{s-1} j(j-1)\ell^{j-2} \\
&= E[\tau] + \frac{\ell^2}{1-\ell^s}\left[\frac{2(1-\ell^s)}{(1-\ell)^2} - \frac{2\ell^{s-1}s}{1-\ell} - \ell^{s-2}s(s-1)\right]. \qquad (1.37)
\end{aligned}$$

At the same time, it is well-known that the moments of τ_c are [3, Chapter 14]:

$$E[\tau_c] = \frac{1}{\lambda}, \qquad (1.38)$$

$$E[\tau_c^2] = \frac{2}{\lambda^2}. \qquad (1.39)$$

In Fig. 1.2, we compare the moments computed with (1.36)-(1.37) with those given by (1.38)-(1.39) for the case when $s = 40$ and $1/\lambda \in \{1, 2, \ldots, 20\}$. Remark in figure that the moments of τ and τ_c are almost the same when the mean of τ_c is small, but they tend to be different when the mean of τ_c increases.

The formulae above will be employed in our future calculations. Most importantly, the model we introduce for τ leads to the following expression for BER:

$$\text{BER} = \sum_{j=0}^{s-1} \Pr(\tau = j)\text{BER}(j), \qquad (1.40)$$

where $\Pr(\tau = j)$ is given in (1.35) and $\text{BER}(j)$ is evaluated by using (1.32)-(1.33).

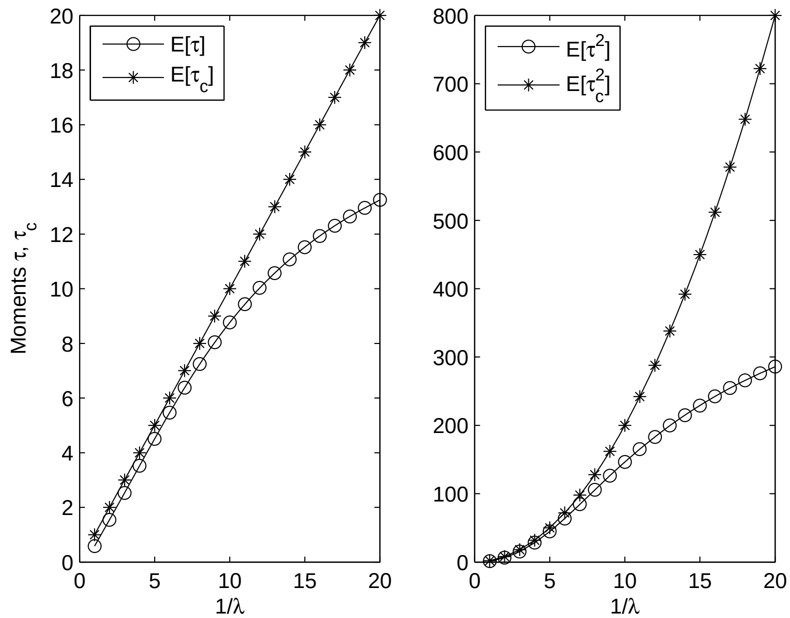

Figure 1.2: First- and second-order moments of τ and τ_c computed with formulae (1.36)-(1.37) and (1.38)-(1.39), respectively. In calculations, we use $s = 40$ and the value of $1/\lambda$ ranges from 1 to $s/2$.

1.2.4 Relationship Between the Values of the Transmitted Bits and the Conditional BER

All formulae for BER which we have derived so far are based on the expression of the conditional BER in (1.12). However, for obtaining (1.12), we assumed that the bits transmitted by all users have value "$+1$". This might give the impression that all our results are valid only for a particular case. Next we prove their generality.

Let $\gamma_i^{(n)}$ be the i-th bit transmitted by the n-th user. Obviously, $\gamma_i^{(n)} \in \{-1, +1\}$ for all i and n. With the convention that $i > 1$, we denote z_i the random variable used by the first user in order to decide if the value of the i-th received bit is either "+1" or "-1". Hence, we have:

$$z_i = \delta_0 \sum_{n=1}^{N} \gamma_i^{(n)} a_n + \delta_1 \sum_{n=1}^{N} \gamma_i^{(n)} b_n + \delta_1 x_1^{(1)} \sum_{n=1}^{N} \left[\gamma_{i-1}^{(n)} - \gamma_i^{(n)}\right] x_{2\beta}^{(n)} + sC, \qquad (1.41)$$

where all symbols have the same significance as before, except $\gamma_i^{(n)}$ and $\gamma_{i-1}^{(n)}$ which are newly introduced. The key results are given in the following lemma.

Lemma 1.2.1. *For all* $\gamma_{i-1}^{(1)}, \ldots, \gamma_{i-1}^{(N)}, \gamma_i^{(1)}, \ldots, \gamma_i^{(N)} \in \{-1, +1\}$*, the following identities hold true:*

$$E\left[z_i|\alpha_{00}, \alpha_{01}, \tau\right] = \gamma_i^{(1)} E\left[z|\alpha_{00}, \alpha_{01}, \tau\right], \tag{1.42}$$

$$E\left[z_i^2|\alpha_{00}, \alpha_{01}, \tau\right] = E\left[z^2|\alpha_{00}, \alpha_{01}, \tau\right], \tag{1.43}$$

where z *is defined in (1.4) and the expressions of its first- and second-order moments are given in (1.13)-(1.14).*

Proof is deferred to Section 3.5.

We can see from Lemma 1.2.1 that the conditional mean of z_i does not depend on the sign of the previous bit transmitted for the first user nor on the bits transmitted for other users. Additionally, the values of the bits have no influence on $E\left[z_i^2|\alpha_{00}, \alpha_{01}, \tau\right]$. All that remains is to assume that is equally likely for $\gamma_i^{(1)}$ to be either "+1" or "-1", and to perform the usual calculations [4, 5]:

$$\begin{aligned}
\mathrm{BER}(\alpha_{00}, \alpha_{01}, \tau) &= \frac{1}{2} \Pr\left(z_i < 0|\alpha_{00}, \alpha_{01}, \tau, \gamma_i^{(1)} = +1\right) \\
&\quad + \frac{1}{2} \Pr\left(z_i > 0|\alpha_{00}, \alpha_{01}, \tau, \gamma_i^{(1)} = -1\right) \\
&= \frac{1}{2}\mathrm{erfc}\left(\frac{E[z_i|\alpha_{00}, \alpha_{01}, \tau, \gamma_i^{(1)} = +1]}{\sqrt{2\mathrm{Var}[z|\alpha_{00}, \alpha_{01}, \tau, \gamma_i^{(1)} = +1]}}\right) \\
&= \frac{1}{2}\mathrm{erfc}\left(\frac{E[z|\alpha_{00}, \alpha_{01}, \tau]}{\sqrt{2\mathrm{Var}[z|\alpha_{00}, \alpha_{01}, \tau]}}\right).
\end{aligned} \tag{1.44}$$

The equality in (1.44) is based on Lemma 1.2.1 and shows clearly that all the formulae we derived for BER are valid for all possible values of the transmitted bits.

Next we focus on the computation of BER when z in (1.4) is assumed to be Gaussian distributed.

1.3 Computation of Bit Error Rate by Using the Gaussian Approximation for z

For sake of comparison, we compute approximate BER's by applying the method from [1], which assumes the distribution of z to be Gaussian. We write down the calculations for the four cases considered in the previous section.

Case #1 : The approximate BER, which we denote $\check{\text{BER}}(\tau)|_{E_b/N_0=\infty}$, is given by

$$\check{\text{BER}}(\tau)|_{E_b/N_0=\infty} = \frac{1}{2}\text{erfc}\left(\check{\zeta}_\tau^{-1/2}\right), \tag{1.45}$$

where

$$\check{\zeta}_\tau = \frac{2\text{Var}[z]}{(E[z])^2} = \frac{2E[z^2]}{(E[z])^2} - 2. \tag{1.46}$$

As $C = 0$, we employ Lemma 3.2.2, Lemma 3.2.4 and equations (1.4), (1.5), (1.6) for the following calculations:

$$\begin{aligned}
E[z] &= \{E[\alpha_{00}]s + E[\alpha_{01}](s-\tau)\} E[A] + E[\alpha_{01}]\tau E[B] \\
&= (\pi/2)^{1/2}\,[bs + \tilde{b}(s-\tau)]\beta, && (1.47)\\
E[z^2] &= \left\{s^2 E[\alpha_{00}^2] + (s-\tau)^2 E[\alpha_{01}^2] + 2s(s-\tau)E[\alpha_{00}]E[\alpha_{01}]\right\} E[A^2] \\
&\quad + \tau^2 E[\alpha_{01}^2]E[B^2] + 2\tau\left\{sE[\alpha_{00}]E[\alpha_{01}] + (s-\tau)E[\alpha_{01}^2]\right\} E[AB] \\
&= \left[2\tilde{b}^2(s-\tau)^2 + 2b^2s^2 + \pi b\tilde{b}s(s-\tau)\right] \times \left[(N\beta)/2 - \beta/4 + \beta^2\right] && (1.48)\\
&\quad + 2\tau(\beta/4 - 1/4) \times \left[2\tilde{b}^2(s-\tau) + (b\tilde{b}s\pi)/2\right] + N\beta\tilde{b}^2\tau^2. && (1.49)
\end{aligned}$$

Case #2 : Using the notation from Section 1.2.3, we re-write the expression of z as

$$z = \alpha_{00}\,[s + (s-\tau)/r_0]\,A + \alpha_{00}(\tau/r_0)B + sC.$$

This leads to the following results:

$$\begin{aligned}
E[z] &= E[\alpha_{00}]\,[s + (s-\tau)/r_0]\,E[A] \\
&= (\pi/2)^{1/2}\,b\,[s + (s-\tau)/r_0]\,\beta, \\
E[z^2] &= E[\alpha_{00}^2]\,[s + (s-\tau)/r_0]^2\,E[A^2] + E[\alpha_{00}^2](\tau/r_0)^2E[B^2] + s^2E[C^2] \\
&\quad + 2E[\alpha_{00}^2]\,[s + (s-\tau)/r_0]\,(\tau/r_0)E[AB].
\end{aligned} \tag{1.50}$$

All that remains is to plug-in the expressions for the moments of the random variables involved, to calculate $\check{\zeta}_\tau$ like in (1.46) and then to use the erfc($\cdot$) function for computing $\check{\text{BER}}(\tau)|_{r=r_0}$.

Case #3 : It is easy to see that, in this case, the expression of $E[z]$ coincides with the one in (1.47). Similarly, $E[z^2]$ can be calculated as the summation of (1.48)-(1.49) with $s^2E[C^2]$. Note that $s^2E[C^2] = s^2N_0\beta$ (see Lemma 3.3.1).

Case #4 : It is straightforward to write down the following identities:

$$
\begin{aligned}
E[z] &= E[\delta_0]E[A], \\
E[\delta_0] &= -E[\tau]E[\alpha_{01}] + s\{E[\alpha_{00}] + E[\alpha_{01}]\}, \\
E[z^2] &= E[\delta_0^2]E[A^2] + E[\delta_1^2]E[B^2] + s^2E[C^2] + 2E[\delta_0\delta_1]E[AB], \\
E[\delta_0^2] &= E[\tau^2]E[\alpha_{01}^2] - 2sE[\tau]\{E[\alpha_{01}^2] + E[\alpha_{00}]E[\alpha_{01}]\} \\
&\quad + s^2\{E[\alpha_{00}^2] + E[\alpha_{01}^2] + 2E[\alpha_{00}]E[\alpha_{01}]\}, \\
E[\delta_1^2] &= E[\tau^2]E[\alpha_{01}^2], \\
E[\delta_0\delta_1] &= -E[\tau^2]E[\alpha_{01}^2] + E[\tau]s\{E[\alpha_{00}]E[\alpha_{01}] + E[\alpha_{01}^2]\}.
\end{aligned}
$$

The moments of z can be then evaluated with the help of results from Lemma 3.2.2, Lemma 3.2.4, Lemma 3.3.1 and equations (1.5), (1.6). For the first- and second-order moments of τ we apply (1.36) and (1.37).

In the next chapter, we resort to numerical examples for a better understanding of the results obtained so far.

Chapter 2

Numerical Examples

2.1 Experimental Settings

In this chapter, we compare the theoretical results from the previous chapter with those obtained from simulations. We mention from the very beginning that each experimental result shown in Figs. 2.1-2.8 is produced by simulating the transmission of 10^6 bits. In all cases, the spread factor is $2\beta = 100$ and each chip is extended into $s = 40$ samples. The parameter b of the Rayleigh distribution from which we sample α_{00} is taken to be $\sqrt{2}/2$. This choice guarantees that the expected power of fading on the main channel is one: $E[\alpha_{00}^2] = 1$ (see (1.6)). Selection of other parameters is explained below, for each considered case. The nomenclature for the four cases we analyze is the same as in Section 1.2.3 and Section 1.3.

2.2 Experimental Results

Case #1 : In addition to $b = \sqrt{2}/2$, we have also to set, in this case, the parameter $\tilde{b}$ for the Rayleigh distribution of α_{01}. So, we take $\tilde{b} = 0.9b$. As the additive Gaussian noise is not considered ($E_b/N_0 \to \infty$), we are mainly concerned with the degradation of performance when the number of users increases. This is why we plot BER versus N in Fig. 2.1. The values of BER are computed as follows (in parentheses we indicate the acronyms used in the legend of the figure): (th) numerical integration of (1.20); (app) Gaussian approximation in (1.45); (exp) simulation of 10^6 bits. Remark in the same figure that $\tau \in \{1, s/2, s-1\}$, where $s = 40$. Disregarding how BER is computed, BER increases when the ratio τ/s raises and N is kept fixed. We also remark that, for a given value of τ/s, BER grows when N becomes larger. These results are not surprising and they are in line with the analysis from Section 1.2. Remark the agreement between the values of BER(th) and BER(exp). However, for a given set of experimental parameters, BER(app) is much larger than both BER(th) and BER(exp), which shows clearly that in the absence of additive Gaussian noise, the Gaussian assumption for z leads to a poor approximation of BER.

Case #2 : In contrast to Case #1, we now take E_b/N_0 to be relatively small, namely $E_b/N_0 = 2$dB. Then we choose $r_0 = 1.1$ (see (1.21)) and keep all other settings as in Fig. 2.1. The theoretical and empirical values of BER are shown in Fig. 2.2. Observe the nearly linear dependence between BER and the number of users.

In the second experiment conducted for Case #2, we maintain all the settings as in the first one, except that $N = 4$ and E_b/N_0 is varied between 1dB and 8dB. The results are plotted in Fig. 2.3, where we can see the improvement in performance when E_b/N_0 grows. In figure can be observed how inaccurate the Gaussian approximation is when E_b/N_0 is relatively large. For instance, the BER computed with Gaussian approximation when $\tau = 1$ and $E_b/N_0 = 8$db is not only larger than the empirical BER obtained for the same experimental settings, but is also larger than the empirical BER corresponding to $\tau = 20$ and $E_b/N_0 = 8$db.

In the last experiment for Case #2, the focus is on r_0. We fix $N = 4$, $E_b/N_0 = 2$dB and let r_0 to take values from $\{1, 1.2, \ldots, 2\}$. According to Fig. 2.4, the increase of r_0 slightly lowers the BER if $\tau = s - 1$. On the contrary, BER is monotonically increasing with r_0 when $\tau \in \{1, s/2\}$. For understanding this behavior, we use the definitions in Section 1.2.3 in order to compute $\zeta_{\alpha_{00},\tau}$. For example, when $\tau = 39$, we have $s/\tau \approx 1$, which implies $D_1 \approx r_0$ and $D_2 \approx 1$. It follows that

$$\begin{aligned}\zeta_{\alpha_{00},39} &\approx \frac{N}{r_0^2\beta} + \frac{\beta-1}{r_0\beta^2} + ct(r_0)\\ &\approx \frac{0.08}{r_0^2} + \frac{0.02}{r_0} + ct(r_0),\end{aligned}$$

where $N = 4$, $\beta = 50$ and $ct(r_0)$ represents those terms which do not depend on r_0. Hence, if r_0 grows, then $\zeta_{\alpha_{00},39}$ becomes smaller and BER$(39)|_{r=r_0}$ decreases (see (1.23)). For $\tau = 1$, we have $D_1 \approx s(r_0 + 1)$ and $D_2 \approx 1 + 1/r_0$. So,

$$\begin{aligned}\zeta_{\alpha_{00},1} &\approx \frac{N}{s^2(r_0+1)^2\beta} + \frac{\beta-1}{s(r_0+1)\beta^2} + \frac{1}{(1+1/r_0)^2}\frac{1}{\alpha_{00}^2}\frac{2N_0}{\beta} + ct(r_0)\\ &\approx \frac{5\times10^{-5}}{(r_0+1)^2} + \frac{5\times10^{-4}}{r_0+1} + \frac{1}{\alpha_{00}^2}\frac{1.26}{(1+1/r_0)^2} + ct(r_0).\end{aligned}$$

As we know that $E[\alpha_{00}^2] = 1$, it means that the dominant term in the equation above is the one that contains the factor $1/\alpha_{00}^2$. This leads to the conclusion that $\zeta_{\alpha_{00},1}$ is monotonically increasing with r_0 and explains the behavior observed in Fig. 2.4.

Case $\#3$: Similarly to Case #1, we take $\tilde{b} = 0.9b$ for the plots in Figs. 2.5-2.6. In Fig. 2.5, $E_b/N_0 = 2$dB and the number of users ranges from 4 to 24. It is interesting that the difference between BER(th) and BER(exp) increases when N raises, but at least for $\tau = 1$ and $\tau = 20$, BER(th) − BER(exp) is clearly smaller than BER(app) − BER(exp).

In the particular case of Fig. 2.6, the number of users is small ($N = 4$) and we remark the decrease of BER when E_b is kept fixed and N_0 is lowered. As already observed in other graphs within this section, BER(th) and BER(exp) are almost the same. Additionally, the Gaussian approximation BER(app) almost coincides with BER(th) when E_b/N_0 is small. The smaller N_0 is, the worse the Gaussian approximation is, and this trend confirms what we have already noticed for Case #1 and Case #2.

Another interesting aspect is that in Case #2, the ratio α_{00}/α_{01} is fixed to 1.1, whilst in Case #3 the ratio of the means of distributions from which α_{00} and α_{01} are drawn is about 1.1. This explains the similarities between Fig. 2.2 & Fig. 2.5 and Fig. 2.3 & Fig. 2.6.

Case $\#4$: The key point is the randomness of the delay τ. As we already know from Section 1.2.3, τ is a random variable obtained by quantizing τ_c. Bearing in mind that, in our settings, the maximum possible value of τ is $s - 1 = 39$, we conduct experiments for the situation when τ is sampled from an Exponential distribution with mean 5, as well as for the case when the mean of the Exponential distribution is 20. All other experimental settings are described in the caption of Fig. 2.7, where we show how the BER depends on the number of users (N). We are mainly interested in comparing the theoretical BER given in (1.40) with the empirical results obtained from simulations. Under the assumption that z is Gaussian distributed, the approximate BER is calculated by using the expressions of $E[\tau]$ and $E[\tau^2]$ from (1.36)-(1.37) in the formulae outlined in Section 1.3. For comparison with results reported previously (see [1]), we also compute another approximate BER which is obtained by employing the formulae in (1.38) and (1.39) for $E[\tau]$ and $E[\tau^2]$, respectively. As we can see in Fig. 2.7, the two Gaussian approximations can differ significantly if $1/\lambda$ (the mean of the Exponential distribution from which τ_c is sampled) is large. In the same figure, one can observe the difference between BER computed with (1.40) and the Gaussian approximations. The fact that the two Gaussian approximations are over-pessimistic can be also observed in Fig. 2.7, where the number of users is fixed and the ratio E_b/N_0 is increased from 1dB to 8dB.

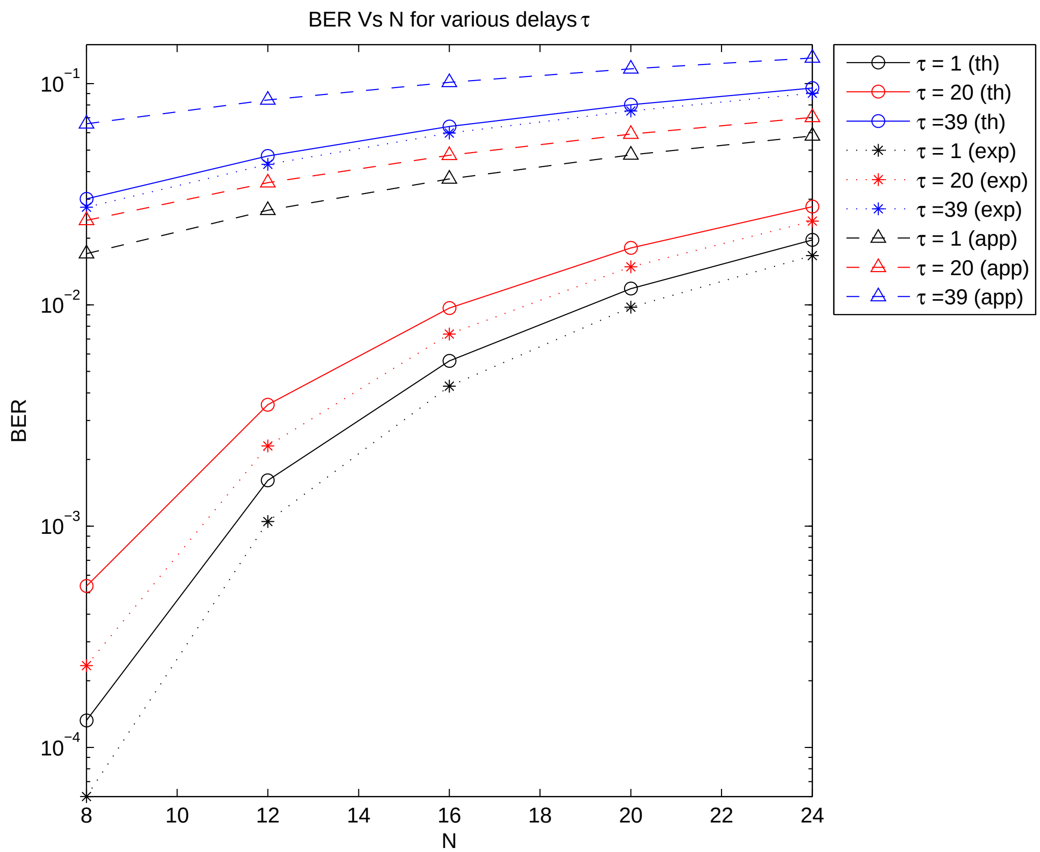

Figure 2.1: Case #1 : BER(exp) [computed empirically from 10^6 simulated bits] is compared with BER(th) [calculated by numerical integration with formulae from Section 1.2.3] and BER(app) [calculated with the Gaussian approximation from Section 1.3] when the number of users N increases from 8 to 24. Other settings: $\beta = 50$, $s = 40$, $b = \sqrt{2}/2$, $\tilde{b} = 0.9b$ and the values of τ are listed in the legend of the figure.

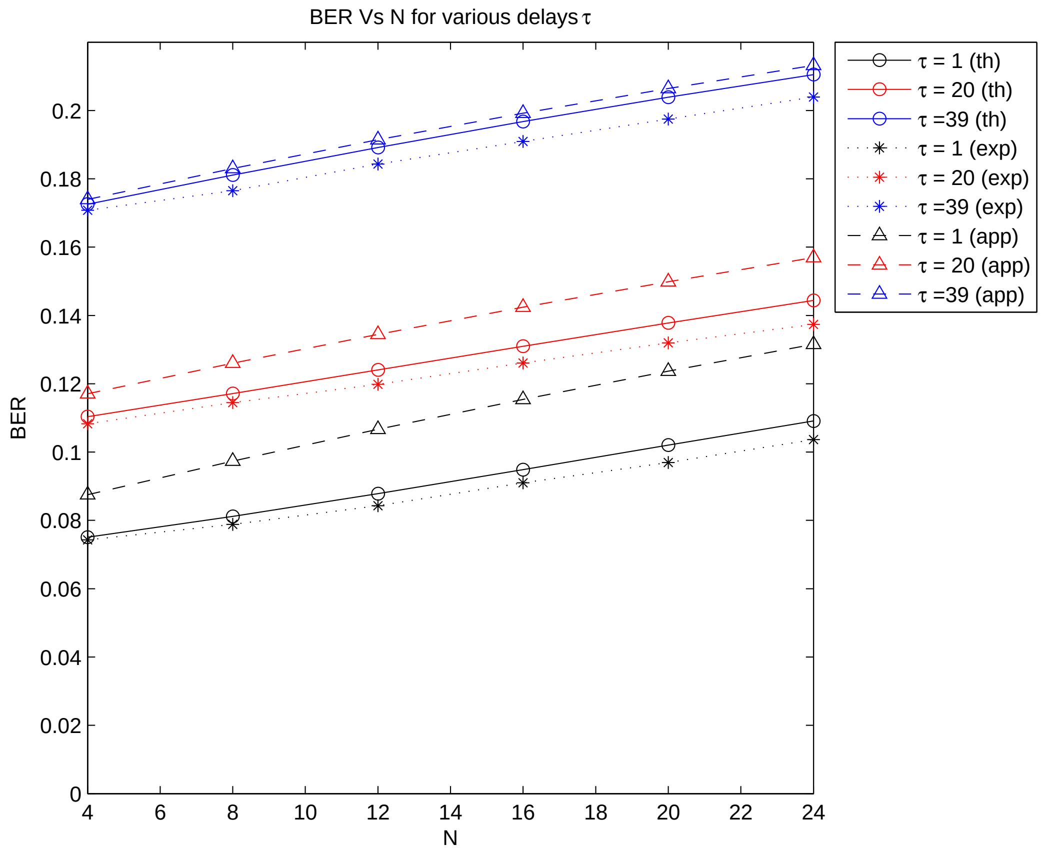

Figure 2.2: Case #2 : Here there are two major differences in rapport with the experiment whose results are shown in Fig. 2.1: (i) $\alpha_{01} = \alpha_{00}/r_0$, where $r_0 = 1.1$ and (ii) $E_b/N_0 = 2$dB.

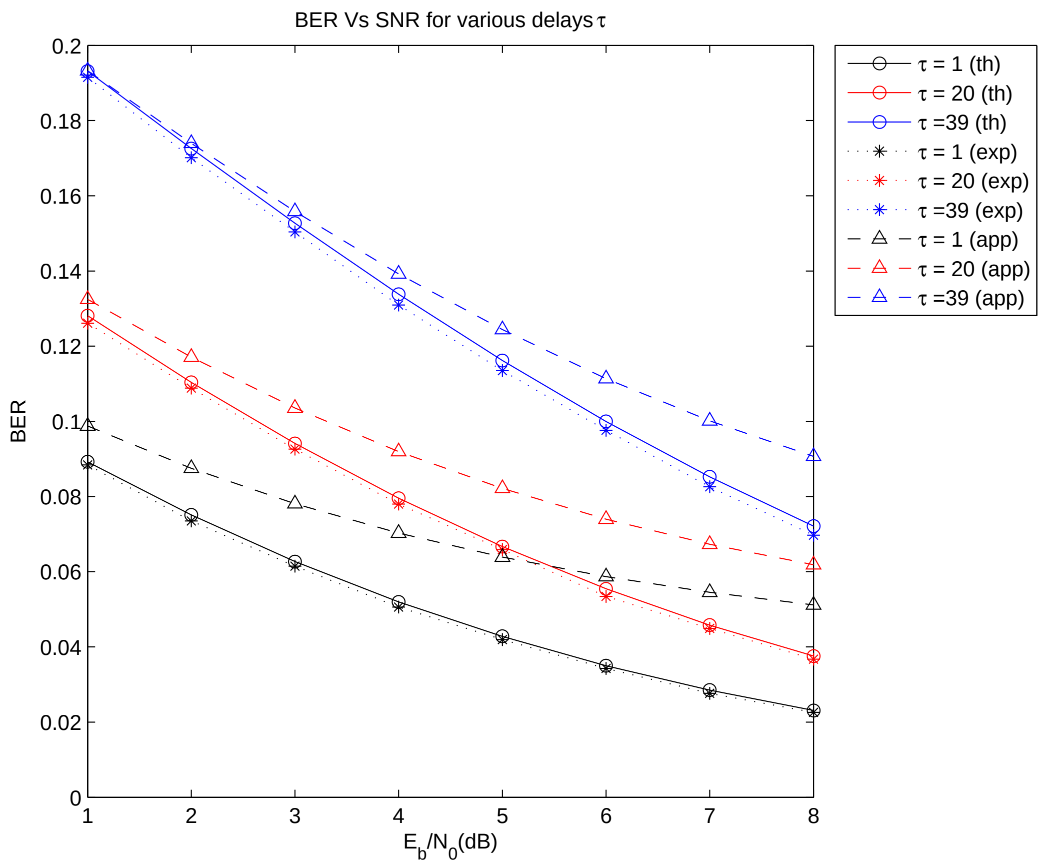

Figure 2.3: Case #2 : Same settings as in Fig. 2.2, apart from the fact that N is fixed to value 4 and E_b/N_0 is increased from 1dB to 8dB.

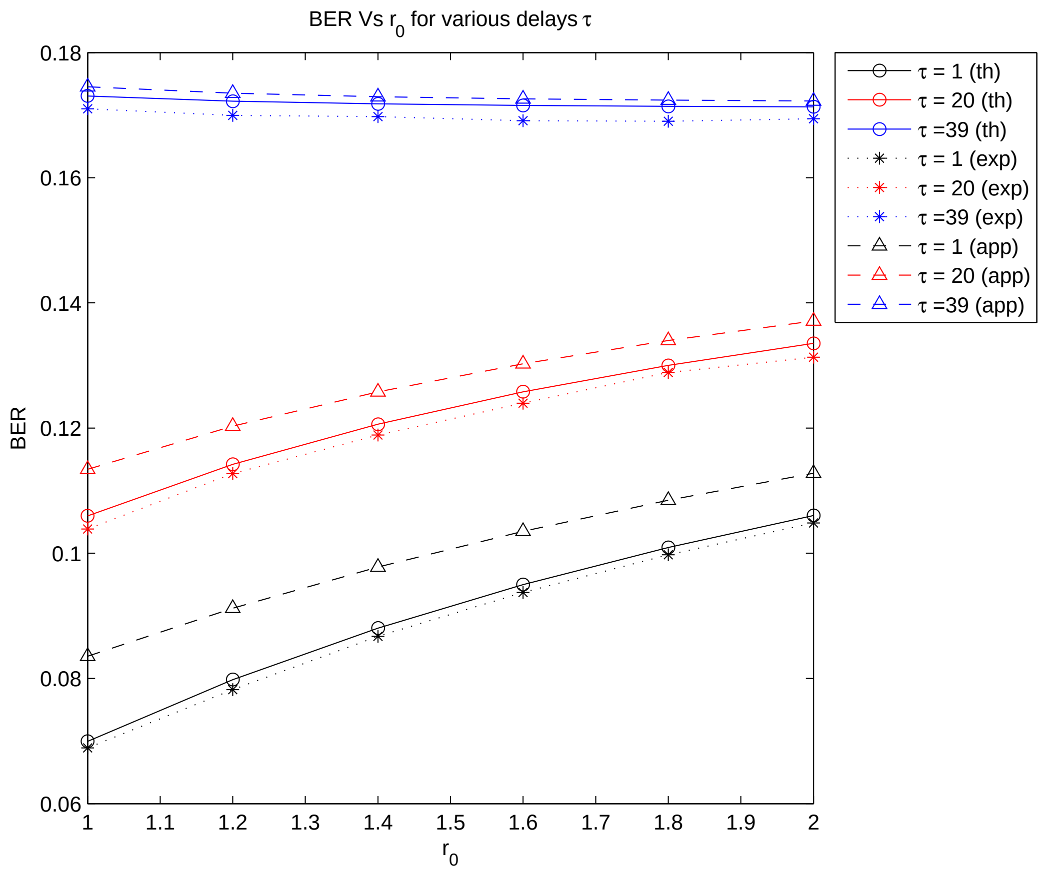

Figure 2.4: Case #2 : In order to investigate the impact of r_0 on the performance of the CDMA-system, we alter the settings from Fig. 2.2 such that $N = 4$, $E_b/N_0 = 2$dB and $r_0 \in \{1, 1.2, \ldots, 2\}$.

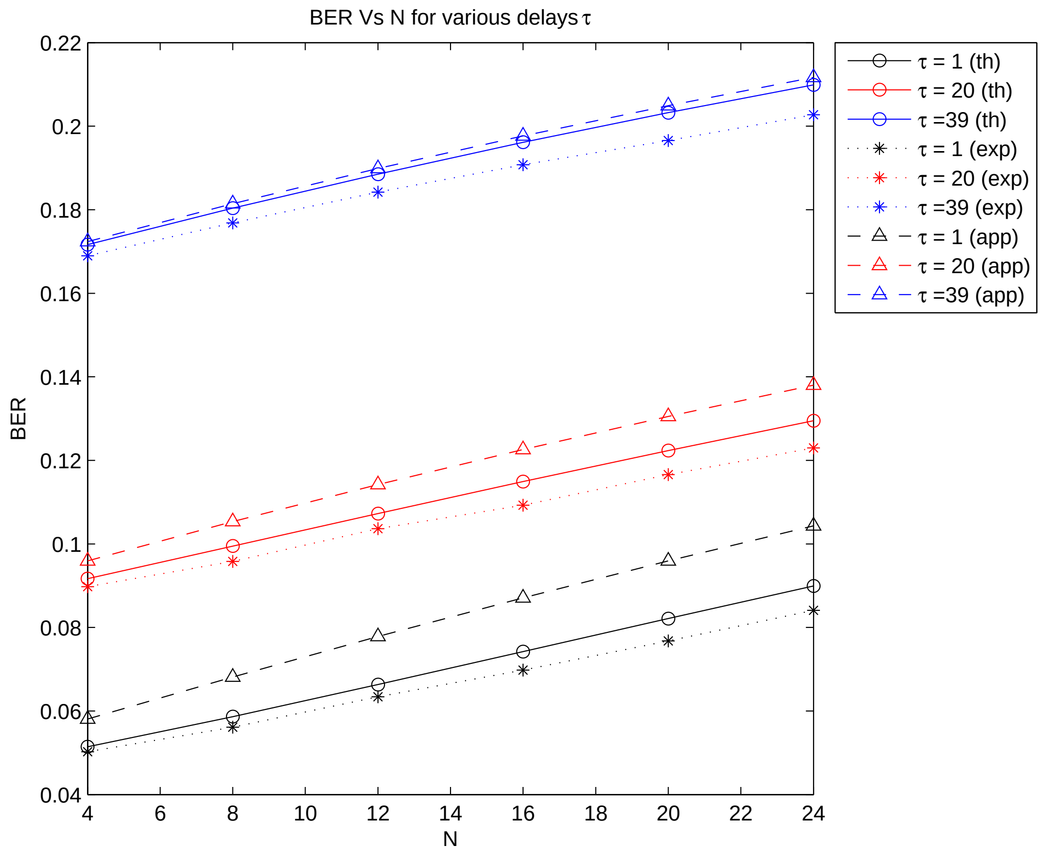

Figure 2.5: Case #3: The only difference between this experiment and the one in Fig. 2.1 is that N_0 is not zero, but is chosen such that $E_b/N_0 = 2$dB.

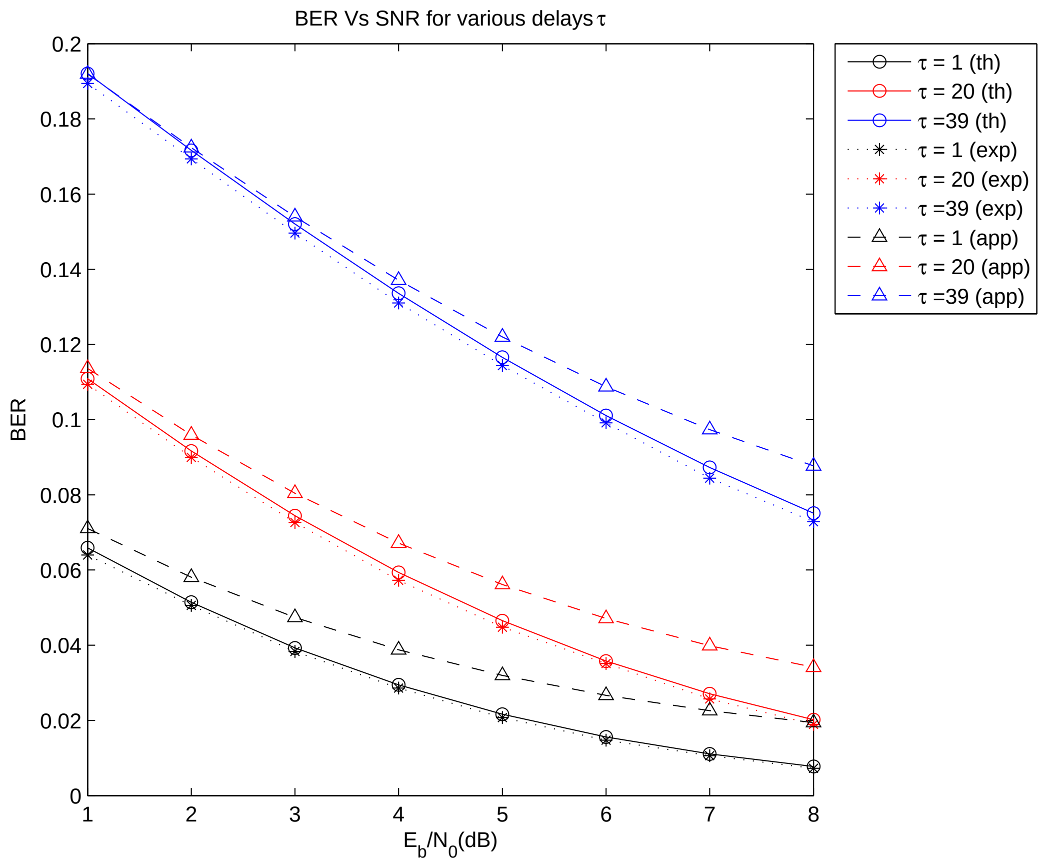

Figure 2.6: Case #3: The number of users is $N = 4$ and the variance of the additive Gaussian noise (N_0) is varied in order to see the effect of E_b/N_0 on BER(th) [see Section 1.2.3], BER(app) [see Section 1.3] and BER(exp) [computed empirically]. All other settings are the same as in Fig. 2.1.

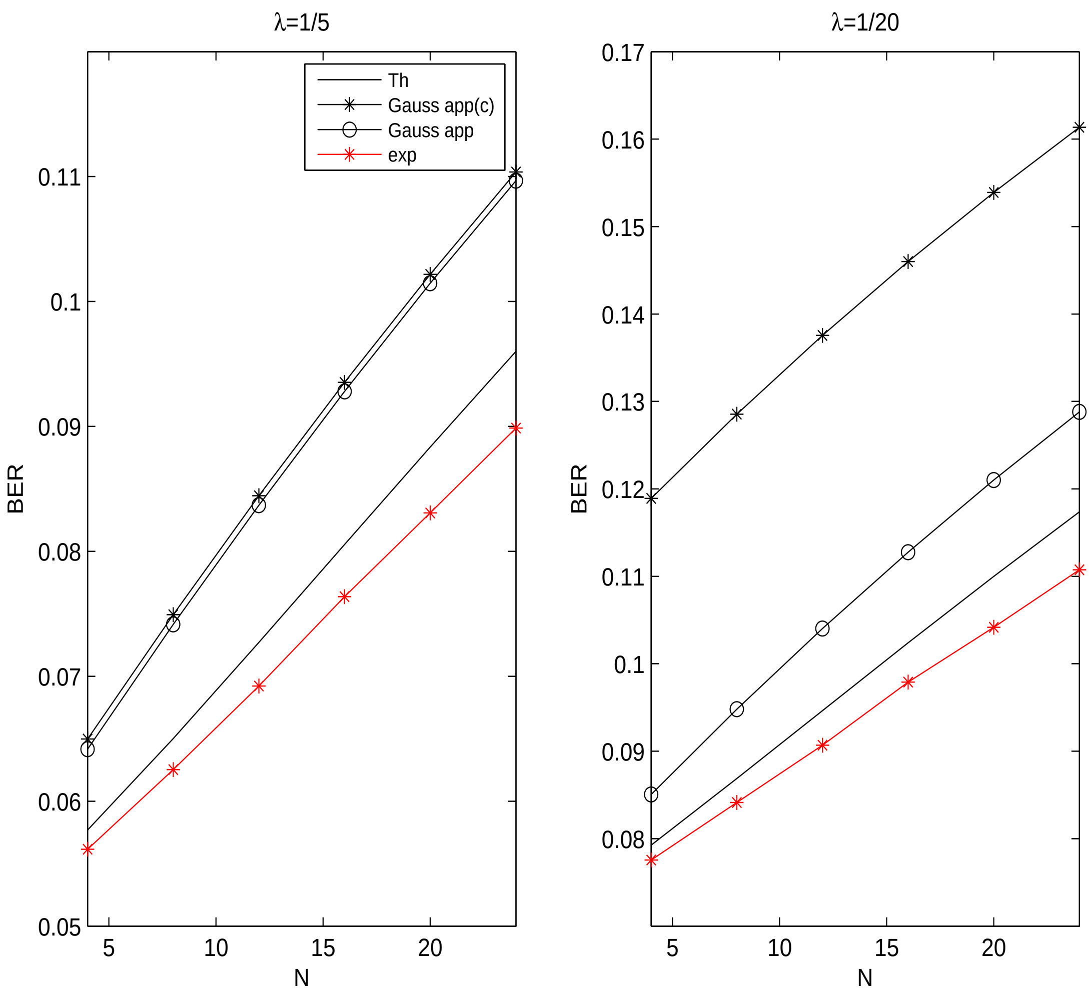

Figure 2.7: Case #4 : Comparison of BER(Th), which represents the BER given in (1.40), with two Gaussian approximations. The first approximation [Gauss app(c)] employs the formulae in (1.38) and (1.39), whereas the second one [Gauss app] uses (1.36) and (1.37). Additionally, $b = \sqrt{2}/2$, $\tilde{b} = 0.9b$, $2\beta = 100$, $s = 40$ and $E_b/N_0 =$ 2dB. The parameter of the Exponential distribution from which τ_c is drawn is either 1/5 (left panel) or 1/20 (right panel). For each value of N (number of users) shown in the plots, BER(exp) is computed by simulating the transmission of 10^6 bits.

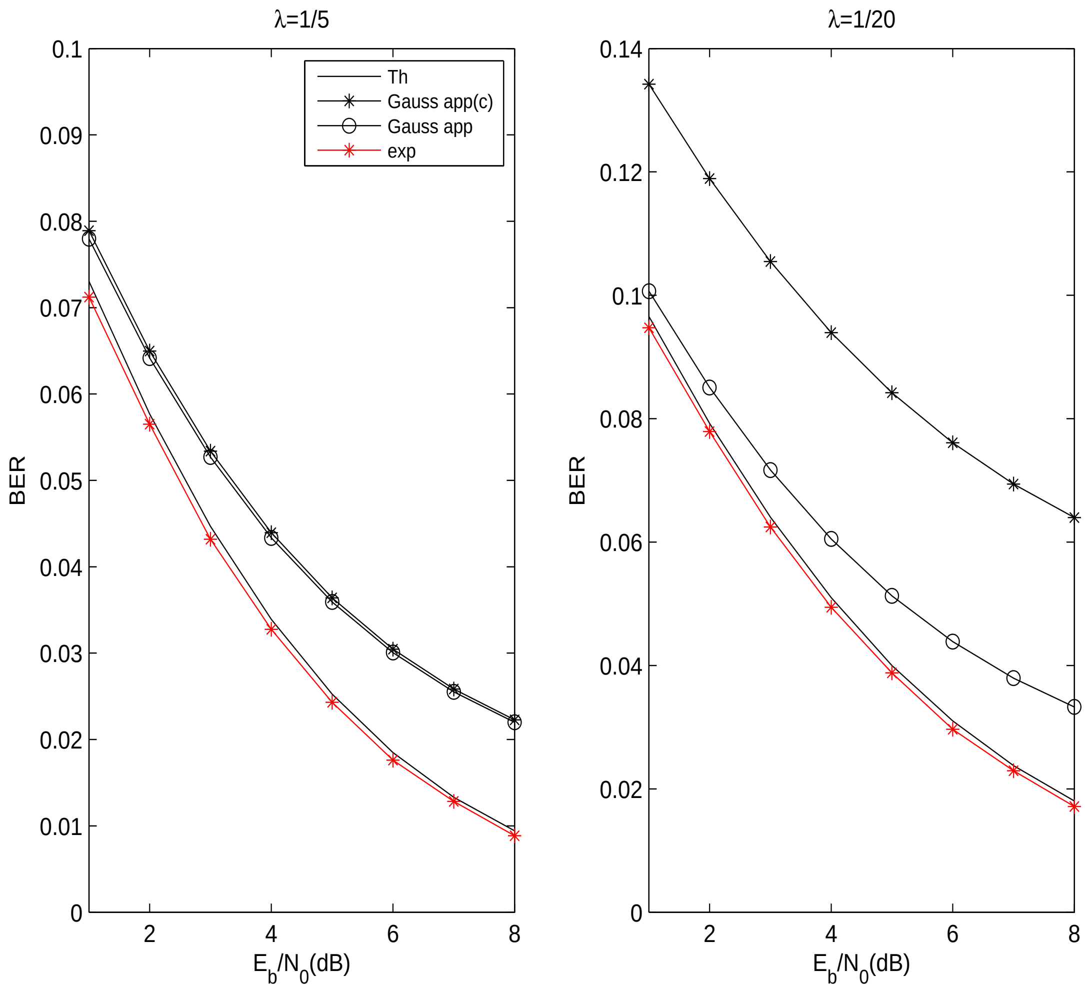

Figure 2.8: Case #4 : For this figure, all settings are the same as in Fig. 2.7, except that the number of users is fixed to $N = 4$ and the variance of the additive Gaussian noise N_0 is selected such that E_b/N_0 takes the values shown on the abscissas of the plots.

Chapter 3

Statistical Properties

3.1 Statistics for Sequences of Random Variables Generated by Logistic Map

In this section, we investigate the statistical properties of sequences of random variables $\{x_i\}$, which are generated as follows:

$$x_{i+1} = g(x_i), \tag{3.1}$$
$$g(x_i) = 1 - 2x_i^2. \tag{3.2}$$

We reproduce below some of the properties of $\{x_i\}$ which can be found in [5]. For instance, it is well-known the expression of the PDF for a random variable x generated as described in (3.1)-(3.2):

$$\rho(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Moreover, the following identities hold true (see Appendix 3B and Appendix 7B in [5]):

$$E[x_i^q] = 0, \quad 0 < q \ (q \text{ odd}), \tag{3.4}$$
$$E[x_i^2] = 1/2, \tag{3.5}$$
$$E[x_i^2 x_{i+m}^2] = 1/4, \quad 0 < m, \tag{3.6}$$
$$E[x_i x_{i+m}] = 0, \quad 0 < m, \tag{3.7}$$
$$E[x_i x_{i+m} x_{i+p} x_{i+r}] = 0, \quad 0 < m < p < r, \tag{3.8}$$
$$E[x_i^2 x_{i+m}] = -1/4, \quad m = 1, \tag{3.9}$$

$$E[x_i^2 x_{i+m}] = 0, \qquad m \neq 1, \tag{3.10}$$

where $E[\cdot]$ denotes the expectation operator. As our aim is to extend these results, we firstly prove the next lemma.

Lemma 3.1.1. *(a) If $q > 0$ is even, then we have*

$$\int_0^\pi \cos^q(\varphi)\mathrm{d}\varphi = \frac{q-1}{q}\frac{q-3}{q-2}\cdots\frac{3}{4}\frac{\pi}{2}. \tag{3.11}$$

(b) For $\gamma, \delta > 0$ even, we have

$$\int_0^\pi \cos(\gamma\varphi)\cos(\delta\varphi)\mathrm{d}\varphi = \begin{cases} \pi/2 & \text{if } \gamma = \delta \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Proof. (a) It is easy to observe that

$$\int_0^\pi \cos^2(\varphi)\mathrm{d}\varphi = \int_0^\pi \frac{1+\cos(2\varphi)}{2}\mathrm{d}\varphi = \frac{\pi}{2}. \tag{3.13}$$

For $q \geq 2$, we have the following identities

$$\begin{aligned}
\int_0^\pi \cos^q(\varphi)\mathrm{d}\varphi &= \int_0^\pi \cos^{q-1}(\varphi)\cos(\varphi)\mathrm{d}\varphi \\
&= \left[\cos^{q-1}(\varphi)\sin(\varphi)\right]_0^\pi - \int_0^\pi \cos^{q-2}(\varphi)(q-1)[-\sin^2(\varphi)]\mathrm{d}\varphi \\
&= (q-1)\int_0^\pi \cos^{q-2}(\varphi)[1-\cos^2(\varphi)]\mathrm{d}\varphi \\
&= (q-1)\int_0^\pi \cos^{q-2}(\varphi) - (q-1)\int_0^\pi \cos^q(\varphi)\mathrm{d}\varphi,
\end{aligned}$$

which lead to

$$\int_0^\pi \cos^q(\varphi)\mathrm{d}\varphi = \frac{q-1}{q}\int_0^\pi \cos^{q-2}(\varphi)\mathrm{d}\varphi. \tag{3.14}$$

The result in (3.11) is a straightforward consequence of (3.13) and (3.14).

(b) If $\gamma = \delta$, then the evaluation of the integral can be done like in (3.13). However, for $\gamma \neq \delta$, we get:

$$\begin{aligned}
\int_0^\pi \cos(\gamma\varphi)\cos(\delta\varphi)d\varphi &= \frac{1}{2}\int_0^\pi \{\cos[(\gamma+\delta)\varphi] + \cos[(\gamma-\delta)\varphi]\}d\varphi \\
&= \frac{1}{2}\left[\frac{\sin[(\gamma+\delta)\varphi]}{\gamma+\delta}\right]_0^\pi + \frac{1}{2}\left[\frac{\sin[(\gamma-\delta)]\varphi}{\gamma-\delta}\right]_0^\pi \\
&= 0.
\end{aligned}$$

□

Proposition 3.1.1. *If x_i is a random variable from a sequence generated by the logistic map, then the following identities hold true:*

$$
\begin{aligned}
E[x_i^4] &= 3/8, && && (3.15)\\
E[x_i^6] &= 5/16, && && (3.16)\\
E[x_i^4 x_{i+1}^2] &= 7/32, && && (3.17)\\
E[x_i^4 x_{i+m}^2] &= 3/16, && 1 < m, && (3.18)\\
E[x_i^2 x_{i+m}^4] &= 3/16, && 1 < m, && (3.19)\\
E[x_i^2 x_{i+m}^2 x_{i+p}^2] &= 1/8, && 0 < m < p, && (3.20)\\
E[x_i x_{i+m} x_{i+p}] &= 0, && 0 < m < p, && (3.21)\\
E[x_i x_{i+1}^2 x_{i+2}] &= 0, && && (3.22)\\
E[x_i^2 x_{i+1} x_{i+m}] &= 0, && 2 < m, && (3.23)\\
E[x_i^2 x_{i+1} x_{i+2}] &= 1/8, && && (3.24)\\
E[x_i x_{i+m} x_{i+m+1}^2] &= 0, && 0 < m, && (3.25)\\
E[x_i x_{i+1} x_{i+m}^2] &= 0, && 0 < m, && (3.26)\\
E[x_i^2 x_{i+m-1} x_{i+m}] &= 0, && 2 < m, && (3.27)\\
E[x_i^3 x_{i+1}^3] &= 0, && && (3.28)\\
E[x_i^2 x_{i+1}^3 x_{i+2}] &= 1/8, && && (3.29)\\
E[x_{i-1} x_i^3 x_{i+1}^2] &= 0, && && (3.30)\\
E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] &= 0, && i < j < k, && (3.31)\\
E[x_i^2 x_{i+1}^2 x_{i+2} x_{i+3}] &= 1/16, && && (3.32)\\
E[x_i^2 x_{i+1}^2 x_{i+m} x_{i+m+1}] &= 0, && 2 < m && (3.33)\\
E[x_i^3 x_{i+1}] &= 0. && && (3.34)
\end{aligned}
$$

Proof. Let $q > 0$ be an even integer. Then,

$$E[x_i^q] = \int_{-1}^{1} \frac{x^q}{\pi\sqrt{1-x^2}} \mathrm{d}x.$$

With the change of variable $x = \cos(\varphi)$, $\mathrm{d}x = -\sin(\varphi)\mathrm{d}\varphi$, we get

$$
\begin{aligned}
E[x_i^q] &= \frac{1}{\pi} \int_{\pi}^{0} \cos^q(\varphi) \frac{1}{\sin(\varphi)} [-\sin(\varphi)] \mathrm{d}\varphi \\
&= \frac{1}{\pi} \int_{0}^{\pi} \cos^q(\varphi) \mathrm{d}\varphi,
\end{aligned}
$$

and the identities in (3.15)-(3.16) are immediately obtained from (3.11).

For the calculation of $E[x_i^4 x_{i+m}^2]$, we employ the definition of $g(\cdot)$ from (3.2) and apply the same change of variable as before. We adopt the notational convention from [5] that, for any $m > 0$, $g^{(m)}(x) = g(g^{(m-1)}(x))$ and $g^{(1)}(x) = g(x)$. So,

$$\begin{aligned} E[x_i^4 x_{i+m}^2] &= \int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} x^4 \left[g^{(m)}(x)\right]^2 \mathrm{d}x \\ &= \frac{1}{\pi}\int_{\pi}^{0} \frac{1}{\sin(\varphi)} \cos^4(\varphi)\left[-\cos(2^m\varphi)\right]^2 \left[-\sin(\varphi)\right]\mathrm{d}\varphi \quad \text{[see Eq. (3.61) in [5]]} \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos^4(\varphi)\cos^2(2^m\varphi)\mathrm{d}\varphi. \end{aligned} \tag{3.35}$$

After some elementary calculations, we get

$$\begin{aligned} \cos^4(\varphi) &= \left[\cos^2(\varphi)\right]^2 \\ &= \left[\frac{1+\cos(2\varphi)}{2}\right]^2 \\ &= \frac{1}{4}\left[1 + 2\cos(2\varphi) + \cos^2(2\varphi)\right] \\ &= \frac{1}{4}\left[1 + 2\cos(2\varphi) + \frac{1+\cos(4\varphi)}{2}\right] \\ &= \frac{1}{4} + \frac{1}{2}\cos(2\varphi) + \frac{1}{8} + \frac{1}{8}\cos(4\varphi) \\ &= \frac{3}{8} + \frac{1}{2}\cos(2\varphi) + \frac{1}{8}\cos(4\varphi), \end{aligned} \tag{3.36}$$

$$\cos^2(2^m\varphi) = \frac{1}{2} + \frac{1}{2}\cos(2^{m+1}\varphi). \tag{3.37}$$

Combining (3.12) with (3.35)-(3.37), we obtain (3.17) and (3.18).

Applying the same techniques as above, we have for $m > 1$ that

$$E[x_i^2 x_{i+m}^4] = \frac{1}{\pi}\int_0^{\pi} \cos^2(\varphi)\cos^4(2^m\varphi)\mathrm{d}\varphi = \frac{3}{16},$$

which proves the identity in (3.19).

In order to prove (3.20), we note that

$$\begin{aligned} E[x_i^2 x_{i+m}^2 x_{i+p}^2] &= \int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} x^2 \left[g^{(m)}(x)\right]^2 \left[g^{(p)}(x)\right]^2 \mathrm{d}x \\ &= \frac{1}{\pi}\int_{\pi}^{0} \frac{1}{\sin\varphi}\cos^2(\varphi)\cos^2\left(2^m\varphi\right)\cos^2\left(2^p\varphi\right)(-\sin\varphi)\mathrm{d}\varphi \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{\pi} \cos^2(\varphi)\cos^2\left(2^m\varphi\right)\cos^2\left(2^p\varphi\right)\mathrm{d}\varphi \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{1+\cos 2\varphi}{2}\cdot\frac{1+\cos(2^{m+1}\varphi)}{2}\cdot\frac{1+\cos(2^{p+1}\varphi)}{2}\mathrm{d}\varphi \\
&= \frac{1}{8} \qquad \text{[see (3.12)].}
\end{aligned}
$$

For proving (3.21), it is enough to apply the definition of the expectation and to use the fact that $\psi(x) = \rho(x)xg^{(m)}(x)g^{(p)}(x)$ is an odd function. The function $\psi(x)$ has this property because both $\rho(x)$ and $g(x)$ are even functions. The proof for (3.22), (3.25), (3.26), (3.28), (3.30), (3.31) and (3.34) is similar to that for (3.21).

For (3.23) and (3.24), we note that

$$
\begin{aligned}
E[x_i^2 x_{i+1} x_{i+m}] &= \frac{1}{\pi}\int_0^{\pi} \cos^2(\varphi)[-\cos(2\varphi)][-\cos(2^m\varphi)]\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{\pi} [1+\cos(2\varphi)]\cos(2\varphi)\cos(2^m\varphi)\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos^2(2\varphi)\cos(2^m\varphi)\mathrm{d}\varphi \qquad \text{[see (3.12)]} \\
&= \frac{1}{4\pi}\int_0^{\pi} [1+\cos(4\varphi)]\cos(2^m\varphi)\mathrm{d}\varphi.
\end{aligned}
$$

Using the result in (3.12), we get

$$
E[x_i^2 x_{i+1} x_{i+m}] = \begin{cases} 1/8 & \text{if } m = 2 \\ 0 & \text{if } m > 2. \end{cases}
$$

Similarly, for (3.27), we have:

$$
\begin{aligned}
E[x_i^2 x_{i+m-1} x_{i+m}] &= \frac{1}{\pi}\int_0^{\pi} \cos^2(\varphi)[-\cos(2^{m-1}\varphi)][-\cos(2^m\varphi)]\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{\pi} [1+\cos(2\varphi)]\cos(2^{m-1}\varphi)\cos(2^m\varphi)\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos(2\varphi)\cos(2^{m-1}\varphi)\cos(2^m\varphi)\mathrm{d}\varphi \qquad \text{[see (3.12)]} \\
&= \frac{1}{4\pi}\int_0^{\pi} \cos(2\varphi)[\cos(3\times 2^{m-1}\varphi)+\cos(2^{m-1}\varphi)]\mathrm{d}\varphi.
\end{aligned}
$$

It follows from (3.12) that

$$
E[x_i^2 x_{i+m-1} x_{i+m}] = \begin{cases} 1/8 & \text{if } m = 2 \\ 0 & \text{if } m > 2. \end{cases}
$$

Remark that proving the equality in (3.29) reduces to calculate $E[x_i^2 x_{i+1}^2 x_{i+m} x_{i+m+1}]$ for the particular case when $m = 1$. However, $E[x_i^2 x_{i+1}^2 x_{i+m} x_{i+m+1}]$ appears also in (3.32) and (3.33). In order to check all these identities, we assume that $m \geq 1$ in the following calculations:

$$\begin{aligned}
&E[x_i^2 x_{i+1}^2 x_{i+m} x_{i+m+1}] \\
&= \frac{1}{\pi} \int_0^\pi \cos^2(\varphi)[-\cos(2\varphi)]^2[-\cos(2^m\varphi)][-\cos(2^{m+1}\varphi)]\mathrm{d}\varphi \\
&= \frac{1}{8\pi} \int_0^\pi [1+\cos(2\varphi)][1+\cos(4\varphi)][\cos(3\times 2^m\varphi)+\cos(2^m\varphi)]\mathrm{d}\varphi \\
&= \frac{1}{16\pi} \int_0^\pi [2+3\cos(2\varphi)+2\cos(4\varphi)+\cos(6\varphi)][\cos(3\times 2^m\varphi)+\cos(2^m\varphi)]\mathrm{d}\varphi \\
&= \frac{1}{16\pi} \int_0^\pi \cos(6\varphi)\cos(3\times 2^m\varphi)\mathrm{d}\varphi + \frac{1}{16\pi} \int_0^\pi [3\cos(2\varphi)+2\cos(4\varphi)]\cos(2^m\varphi)\mathrm{d}\varphi.
\end{aligned}$$

Now we only need to apply (3.12) in order to get (3.29), (3.32) and (3.33).

□

3.2 Chaos-Based CDMA System: Statistical Properties of the Received Signal

3.2.1 Calculations for the A-Term

Lemma 3.2.1. *When $N > 2$, with the definition from (1.1), we have:*

$$E[a_1] = \frac{1}{2}(2\beta), \tag{3.38}$$

$$E[a_1^2] = \frac{1}{4}(2\beta)^2 + \frac{1}{8}(2\beta), \tag{3.39}$$

$$E[a_1^3] = \frac{1}{8}(2\beta)^3 + \frac{3}{16}(2\beta)^2 + \frac{3}{32}(2\beta) - \frac{3}{32}, \tag{3.40}$$

$$E[a_2^2] = \frac{1}{4}(2\beta), \tag{3.41}$$

$$E[a_1 a_2^2] = \frac{1}{8}(2\beta)^2 + \frac{1}{16}(2\beta), \tag{3.42}$$

$$E[a_2^3] = \frac{3}{16}(2\beta) - \frac{3}{16}. \tag{3.43}$$

Proof. For simplicity, we drop the superscript for the entries of the vector $\mathbf{x}^{(1)}$ and write $\mathbf{x}^{(1)} = [x_1, x_2, \ldots, x_{2\beta}]^\top$. It follows that

$$
\begin{aligned}
E[a_1] &= \sum_{i=1}^{2\beta} E[x_i^2] = \frac{1}{2}(2\beta) \qquad \text{[see (3.5)]},\\
E[a_1^2] &= E\left[\left(\sum_{i=1}^{2\beta} x_i^2\right)^2\right]\\
&= (2\beta)E[x_1^4] + (2\beta)(2\beta-1)E[x_1^2x_2^2]\\
&= \frac{3}{8}(2\beta) + \frac{1}{4}(2\beta)(2\beta-1) \qquad \text{[see (3.6), (3.15)]},\\
&= \frac{1}{4}(2\beta)^2 + \frac{1}{8}(2\beta),\\
E[a_1^3] &= E\left[\left(\sum_{i=1}^{2\beta} x_i^2\right)^3\right]\\
&= \sum_{i=1}^{2\beta} E[x_i^6] + 3\sum_{i=1}^{2\beta-1} E[x_i^4x_{i+1}^2]\\
&+ 3\sum_{\substack{1\le i,j\le 2\beta\\ j\neq i, j\neq i+1}} E[x_i^4x_j^2] + 6\sum_{\substack{1\le i,j,k\le 2\beta\\ i\neq j, j\neq k, k\neq i}} E[x_i^2x_j^2x_k^2]\\
&= 2\beta E[x_1^6] + 3(2\beta-1)E[x_1^4x_2^2] + 3(2\beta-1)^2E[x_1^4x_3^2]\\
&+ 2\beta(2\beta-1)(2\beta-2)E[x_1^2x_2^2x_3^2]\\
&= (2\beta)\frac{5}{16} + 3(2\beta-1)\frac{7}{32} + 3(2\beta-1)^2\frac{3}{16} \qquad \text{[see (3.16) − (3.18)]}\\
&+ (2\beta)(2\beta-1)(2\beta-2)\frac{1}{8} \qquad \text{[see (3.20)]}\\
&= \frac{1}{8}(2\beta)^3 + \frac{3}{16}(2\beta)^2 + \frac{3}{32}(2\beta) - \frac{3}{32}.
\end{aligned}
$$

In this proof, for the calculations which involve a_2, we adopt the convention that $\mathbf{x}^{(2)} = [y_1, y_2, \ldots, y_{2\beta}]^\top$. So,

$$
\begin{aligned}
E[a_2^2] &= E\left[\left(\sum_{i=1}^{2\beta} x_iy_i\right)^2\right]\\
&= 2\beta E[x_1^2y_1^2] + 2\beta(2\beta-1)E[x_1y_1x_2y_2]\\
&= 2\beta E[x_1^2]E[y_1^2] + 2\beta(2\beta-1)E[x_1x_2]E[y_1y_2]
\end{aligned}
$$

$$= \frac{1}{4}(2\beta). \qquad [\text{see } (3.5), (3.7)]$$

Similarly, we get

$$\begin{aligned}
E[a_1 a_2^2] &= E\left[\left(\sum_{k=1}^{2\beta} x_k^2\right)\left(\sum_{i=1}^{2\beta} x_i y_i\right)^2\right] \\
&= E\left[\left(\sum_{k=1}^{2\beta} x_k^2\right)\left(\sum_{i=1}^{2\beta} x_i^2 y_i^2 + \sum_{\substack{1\le i,j\le 2\beta \\ i\ne j}} x_i x_j y_i y_j\right)\right] \\
&= \sum_{\substack{1\le i,k\le 2\beta \\ i\ne k}} E[x_k^2 x_i^2 y_i^2] + \sum_{i=1}^{2\beta} E[x_i^4 y_i^2] \\
&+ \sum_{\substack{1\le i,j,k\le 2\beta \\ i\ne j, k\ne i, k\ne j}} E[x_k^2 x_i x_j y_i y_j] + \sum_{\substack{1\le i,j\le 2\beta \\ i\ne j}} E[x_i^3 x_j y_i y_j] \\
&= 2\beta(2\beta-1)E[x_1^2 x_2^2]E[y_2^2] + 2\beta E[x_1^4]E[y_1^2] \\
&+ 2\beta(2\beta-1)(2\beta-2)E[x_1^2 x_2 x_3]E[y_2 y_3] + 2\beta(2\beta-1)E[x_1^3 x_2]E[y_1 y_2] \\
&= 2\beta(2\beta-1)\frac{1}{4}\frac{1}{2} + 2\beta\frac{1}{2}\frac{3}{8} \qquad [\text{see } (3.5)-(3.7), (3.15)] \\
&= \frac{1}{8}(2\beta)^2 + \frac{1}{16}(2\beta).
\end{aligned}$$

$$\begin{aligned}
E[a_2^3] &= E\left[\left(\sum_{i=1}^{2\beta} x_i y_i\right)^3\right] \\
&= \sum_{i=1}^{2\beta} E[x_i^3 y_i^3] + 3\sum_{i=1}^{2\beta-1} E[x_i^2 y_i^2 x_{i+1} y_{i+1}] + 3\sum_{\substack{1\le i,j\le 2\beta \\ j\ne i, j\ne i+1}} E[x_i^2 y_i^2 x_j y_j] \\
&+ 6\sum_{\substack{1\le i,j,k\le 2\beta \\ i\ne j, j\ne k, k\ne i}} E[x_i y_i x_j y_j x_k y_k] \\
&= \sum_{i=1}^{2\beta} E[x_i^3]E[y_i^3] + 3\sum_{i=1}^{2\beta-1} E[x_i^2 x_{i+1}]E[y_i^2 y_{i+1}]
\end{aligned}$$

$$+3\sum_{\substack{1\le i,j\le 2\beta\\ j\neq i, j\neq i+1}} E[x_i^2x_j]E[y_i^2y_j] + 6\sum_{\substack{1\le i,j,k\le 2\beta\\ i\neq j, j\neq k, k\neq i}}^{2\beta} E[x_ix_jx_k]E[y_iy_jy_k]$$
$$= 2\beta E[x_1^3]E[y_1^3] + 3(2\beta-1)E[x_1^2x_2]E[y_1^2y_2] + 3(2\beta-1)^2E[x_1^2x_3]E[y_1^2y_3]$$
$$+ 2\beta(2\beta-1)(2\beta-2)E[x_1x_2x_3]E[y_1y_2y_3]$$
$$= 3(2\beta-1)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) \qquad \text{[see (3.4), (3.9), (3.10), (3.21)]}$$
$$= \frac{3}{16}(2\beta) - \frac{3}{16}.$$

□

After these preparations, we prove the following lemma.

Lemma 3.2.2. *For $N > 2$, we have:*

$$E[A] = \frac{1}{2}(2\beta),$$
$$E[A^2] = \frac{1}{4}N(2\beta) + \frac{1}{4}(2\beta)^2 - \frac{1}{8}(2\beta),$$
$$E[A^3] = \frac{3}{8}(2\beta)^2N + \frac{3}{8}(2\beta)N - \frac{3}{16}N + \frac{1}{8}(2\beta)^3 - \frac{3}{16}(2\beta)^2 - \frac{9}{32}(2\beta) + \frac{3}{32}.$$

Proof. Most of the calculations are straightforward. We use the fact that the chaotic sequences assigned to two different users are statistically independent and $E[\mathbf{x}^{(1)}] = E[\mathbf{x}^{(2)}] = \cdots = E[\mathbf{x}^{(N)}] = \mathbf{0}$:

$$E[A] = E[a_1] + (N-1)E[a_2]$$
$$= \frac{1}{2}(2\beta) + (N-1)E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(2)}\right] \qquad \text{[see (3.38)]}$$
$$= \frac{1}{2}(2\beta) + (N-1)E\left[\left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(2)}\right]$$
$$= \frac{1}{2}(2\beta),$$
$$E[A^2] = E[a_1^2] + (N-1)E[a_2^2] + 2(N-1)E[a_1a_2]$$
$$+ (N-1)(N-2)E[a_2a_3]$$
$$= \frac{1}{4}(2\beta)^2 + \frac{1}{8}(2\beta) + \frac{N-1}{4}(2\beta) \qquad \text{[see (3.39), (3.41)]}$$
$$+ 2(N-1)E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(2)}\right]$$

$$+ (N-1)(N-2)E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(2)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(3)}\right]$$
$$= \frac{1}{4}N(2\beta) + \frac{1}{4}(2\beta)^2 - \frac{1}{8}(2\beta).$$

The calculations for $E[A^3]$ are slightly more complicated. Applying the well-known multinomial theorem, we get

$$A^3 = a_1^3 + 3\sum_{n=2}^{N} a_1^2 a_n + 3\sum_{n=2}^{N} a_1 a_n^2$$
$$+ \sum_{n=2}^{N} a_n^3 + 3\sum_{\substack{2\le m,n\le N\\ m\ne n}} a_m^2 a_n + 6\sum_{\substack{1\le m,n,p\le N\\ m\ne n, n\ne p, p\ne m}} a_m a_n a_p,$$

which leads to

$$E[A^3] = E[a_1^3] + 3(N-1)E[a_1^2 a_2] + 3(N-1)E[a_1 a_2^2]$$
$$+ (N-1)E[a_2^3] + 3(N-1)(N-2)E[a_2^2 a_3]$$
$$+ N(N-1)(N-2)E[a_1 a_2 a_3].$$

The expression above can be further simplified by using the fact that the chaotic sequences assigned to two different users are statistically idependent:

$$E[a_1^2 a_2] = E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(2)}\right] = 0,$$
$$E[a_2^2 a_3] = E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(2)} \left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(2)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(3)}\right] = 0,$$
$$E[a_1 a_2 a_3] = E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(2)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}^{(3)}\right] = 0.$$

It follows that

$$E[A^3] = E[a_1^3] + 3(N-1)E[a_1 a_2^2] + (N-1)E[a_2^3]$$
$$= \frac{1}{8}(2\beta)^3 + \frac{3}{16}(2\beta)^2 + \frac{3}{32}(2\beta) - \frac{3}{32} \qquad \text{[see (3.40)]}$$
$$+ 3(N-1)\left[\frac{1}{8}(2\beta)^2 + \frac{1}{16}(2\beta)\right] \qquad \text{[see (3.42)]}$$
$$+ (N-1)\left[\frac{3}{16}(2\beta) - \frac{3}{16}\right] \qquad \text{[see (3.43)]}$$
$$= \frac{3}{8}(2\beta)^2 N + \frac{3}{8}(2\beta)N - \frac{3}{16}N + \frac{1}{8}(2\beta)^3 - \frac{3}{16}(2\beta)^2 - \frac{9}{32}(2\beta) + \frac{3}{32}.$$

□

3.2.2 Calculations for the B-Term

Lemma 3.2.3. *Applying the definition from (1.2) to the case when $N > 2$, we get:*

$$E[b_i] = 0, \qquad 1 \le i \le N, \tag{3.44}$$

$$E[b_i^2] = \frac{1}{4}(2\beta), \qquad 1 \le i \le N, \tag{3.45}$$

$$E[b_i b_j] = 0, \qquad 1 \le i < j \le N. \tag{3.46}$$

$$E[b_1 b_2^2] = \frac{1}{16}(2\beta - 2), \tag{3.47}$$

$$E[b_1^3] = \frac{9}{16}(2\beta) - \frac{3}{4}, \tag{3.48}$$

$$E[b_2^3] = \frac{3}{16}(2\beta) - \frac{3}{8}. \tag{3.49}$$

Proof. For simplicity, we write $\mathbf{x}^{(1)} = [x_1, x_2, \ldots, x_{2\beta}]^\top$ and $\mathbf{x}_d^{(2)} = [y_0, y_1, \ldots, y_{2\beta-1}]^\top$. Hence, we have:

$$E[b_1] = \sum_{i=0}^{2\beta-1} E[x_i x_{i+1}] = 0 \qquad \text{[see (3.7)]},$$

$$E[b_2] = \sum_{i=0}^{2\beta-1} E[y_i] E[x_{i+1}] = 0 \qquad \text{[see (3.4)]},$$

$$\begin{aligned}
E[b_1^2] &= E\left[\left(\sum_{i=0}^{2\beta-1} x_i x_{i+1}\right)^2\right] \\
&= \sum_{i=0}^{2\beta-1} E[x_i^2 x_{i+1}^2] + \sum_{\substack{0 \le i,j \le 2\beta-1 \\ i \ne j}} E[x_i x_{i+1} x_j x_{j+1}] \\
&= \frac{1}{4}(2\beta) + \sum_{\substack{0 \le i,j \le 2\beta-1 \\ i \ne j}} E[x_i x_{i+1} x_j x_{j+1}] \qquad \text{[see (3.6)]} \\
&= \frac{1}{4}(2\beta).
\end{aligned} \tag{3.50}$$

The identity in (3.50) can be verified by assuming that $i < j$ and applying the previous results to the following cases:

$0 \le i < 2\beta - 2$	$i + 1 < j$	Eq. (3.8)
$0 < i \le 2\beta - 2$	$i + 1 = j$	Eq. (3.22)

$i = 0$	$j = 1$	Eq. (3.23)
$i = 0$	$j = 2\beta - 1$	Eq. (3.25)

Moreover, we have

$$\begin{aligned} E[b_2^2] &= E\left[\left(\sum_{i=0}^{2\beta-1} y_i x_{i+1}\right)^2\right] \\ &= \sum_{i=0}^{2\beta-1} E[y_i^2]E[x_{i+1}^2] + \sum_{\substack{0 \le i,j \le 2\beta-1 \\ i \ne j}} E[y_i y_j]E[x_{i+1}x_{j+1}] \\ &= \frac{1}{4}(2\beta) \qquad \text{[see (3.5) and (3.7)].} \end{aligned}$$

Since the proof of (3.46) is straightforward, we do not present it here. However, the proof of (3.47) is slightly more complicated:

$$\begin{aligned} E[b_1 b_2^2] &= E\left[\left(\sum_{i=0}^{2\beta-1} x_i x_{i+1}\right)\left(\sum_{k=0}^{2\beta-1} y_k x_{k+1}\right)^2\right] \\ &= E\left[\left(\sum_{i=0}^{2\beta-1} x_i x_{i+1}\right)\left(\sum_{k=0}^{2\beta-1} y_k^2 x_{k+1}^2 + \sum_{\substack{0 \le k,p \le 2\beta-1 \\ k \ne p}} x_{k+1}x_{p+1}y_k y_p\right)\right] \\ &= \sum_{0 \le i,k \le 2\beta-1} E[x_i x_{i+1} x_{k+1}^2]E[y_k^2] + \sum_{\substack{0 \le i,k,p \le 2\beta-1 \\ k \ne p}} E[x_i x_{i+1} x_{k+1} x_{p+1}]E[y_k y_p] \\ &= \frac{1}{2}\sum_{0 \le i,k \le 2\beta-1} E[x_i x_{i+1} x_{k+1}^2] \qquad \text{[see (3.5) and (3.7)]} \\ &= \frac{1}{16}(2\beta - 2). \end{aligned} \tag{3.51}$$

The identity above is obtained by computing $E[x_i x_{i+1} x_{k+1}^2]$ for the cases outlined in Table 3.1.

For (3.48), we have:

$$
\begin{aligned}
E[b_1^3] &= E\left[\left(\sum_{i=0}^{2\beta-1} x_i x_{i+1}\right)^3\right] \\
&= \sum_{i=0}^{2\beta-1} E[x_i^3 x_{i+1}^3] + 3\sum_{i=0}^{2\beta-2} E[x_i^2 x_{i+1}^2 x_{i+1} x_{i+2}] \\
&\quad + 3\sum_{i=1}^{2\beta-1} E[x_i^2 x_{i+1}^2 x_{i-1} x_i] \\
&\quad + 3\sum_{\substack{0\le i,j\le 2\beta-1 \\ i\ne j-1, i\ne j, i\ne j+1}} E[x_i^2 x_{i+1}^2 x_j x_{j+1}] \\
&\quad + 6\sum_{\substack{0\le i,j,k\le 2\beta-1 \\ i\ne j, j\ne k, k\ne i}} E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] \\
&= \sum_{i=0}^{2\beta-1} E[x_i^3 x_{i+1}^3] + 3\sum_{i=0}^{2\beta-2} E[x_i^2 x_{i+1}^3 x_{i+2}] \\
&\quad + 3\sum_{i=1}^{2\beta-1} E[x_{i-1} x_i^3 x_{i+1}^2] \\
&\quad + 3\sum_{\substack{0\le i,j\le 2\beta-1 \\ i\ne j-1, i\ne j, i\ne j+1}} E[x_i^2 x_{i+1}^2 x_j x_{j+1}] \\
&\quad + 6\sum_{\substack{0\le i,j,k\le 2\beta-1 \\ i\ne j, j\ne k, k\ne i}} E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] \\
&= 3\sum_{i=0}^{2\beta-2} E[x_i^2 x_{i+1}^3 x_{i+2}] \\
&\quad + 3\sum_{\substack{0\le i,j\le 2\beta-1 \\ i\ne j-1, i\ne j, i\ne j+1}} E[x_i^2 x_{i+1}^2 x_j x_{j+1}] && \text{[see (3.28),(3.30) and (3.31)]} \\
&= \frac{3}{8}(2\beta-1) + 3\sum_{0\le i\le 2\beta-3} E[x_i^2 x_{i+1}^2 x_{i+2} x_{i+3}] && \text{[see (3.29) and (3.33)]} \\
&= \frac{3}{8}(2\beta-1) + \frac{3}{16}(2\beta-2) && \text{[see (3.32)]} \\
&= \frac{9}{16}(2\beta) - \frac{3}{4}.
\end{aligned}
$$

Conditions		Identity	Justification
$0 \le i \le 2\beta - 1$	$k > i - 1$	$E[x_i x_{i+1} x_{k+1}^2] = 0$	Eq. (3.26)
$1 \le i \le 2\beta - 1$	$k = i - 1$	$E[x_i^3 x_{i+1}] = 0$	Eq. (3.34)
$i = k + 2$	$0 \le k \le 2\beta - 3$	$E[x_{k+1}^2 x_{k+2} x_{k+3}] = 1/8$	Eq. (3.24)
$i = k + m$ $(2 < m)$	$0 \le k \le 2\beta - m - 1$	$E[x_{k+1}^2 x_{k+m} x_{k+m+1}] = 0$	Eq. (3.27)

Table 3.1: Results needed for the proof of the identity in (3.47).

In the equations above, we employed the identity $E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] = 0$ from (3.31). However, the argument we used for proving (3.31) is not true in the particular case when $i = 0$, $j = 1$, $k = 2\beta - 1$ and $E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] = E[x_1^2 x_2 x_{2\beta-1} x_{2\beta}^2]$. Even in this case, it can be shown that $E[x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}] = 0$ when β is large enough, but we do not outline here the calculations.

Next we calculate $E[b_2^3]$:

$$
\begin{aligned}
E[b_2^3] &= E\left[\left(\sum_{i=0}^{2\beta-1} y_i x_{i+1}\right)^3\right] \\
&= \sum_{i=0}^{2\beta-1} E[y_i^3 x_{i+1}^3] + 3\sum_{i=1}^{2\beta-1} E[y_i^2 x_{i+1}^2 y_{i-1} x_i] + 3\sum_{i=0}^{2\beta-2} E[y_i^2 x_{i+1}^2 y_{i+1} x_{i+2}] \\
&\quad + 3 \sum_{\substack{0\le i,j\le 2\beta-1 \\ j\ne i-1, j\ne i, j\ne i+1}} E[y_i^2 x_{i+1}^2 y_j x_{j+1}] + 6 \sum_{\substack{0\le i,j,k\le 2\beta-1 \\ i\ne j, j\ne k, k\ne i}} E[y_i x_{i+1} y_j x_{j+1} y_k x_{k+1}] \\
&= \sum_{i=0}^{2\beta-1} E[x_{i+1}^3] E[y_i^3] + 3\sum_{i=1}^{2\beta-1} E[x_i x_{i+1}^2] E[y_{i-1} y_i^2] + 3\sum_{i=0}^{2\beta-2} E[x_{i+1}^2 x_{i+2}] E[y_i^2 y_{i+1}] \\
&\quad + 3 \sum_{\substack{0\le i,j\le 2\beta-1 \\ j\ne i-1, j\ne i, j\ne i+1}} E[x_{i+1}^2 x_{j+1}] E[y_i^2 y_j] + 6 \sum_{\substack{0\le i,j,k\le 2\beta-1 \\ i\ne j, j\ne k, k\ne i}} E[x_{i+1} x_{j+1} x_{k+1}] E[y_i y_j y_k] \\
&= 3\sum_{i=0}^{2\beta-2} E[x_{i+1}^2 x_{i+2}] E[y_i^2 y_{i+1}] \qquad (3.52) \\
&= 3(2\beta - 2)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) \qquad (3.53)
\end{aligned}
$$

$$= \frac{3}{16}(2\beta) - \frac{3}{8}.$$

In (3.52), we have used the fact that $E[x_{i+1}^3] = 0$ (see (3.4)) and $E[x_{i+1}^2 x_{j+1}] = 0$ when $\{i\} \cap \{j-1, j, j+1\} = \emptyset$ (see (3.10)). We have also employed the identities $E[x_i x_{i+1}^2] = 0$ (see again (3.10)) and $E[x_{i+1} x_{j+1} x_{k+1}] = 0$ when $i \neq j$, $j \neq k$, $k \neq i$ (see (3.21)). The result in (3.53) follows from (3.9) and the fact that $E[y_0^2 y_1] = E\left[\left(x_0^{(2)}\right)^2 x_1^{(2)}\right] = E\left[x_1^{(2)} \left(x_{2\beta}^{(2)}\right)^2\right] = 0$.

□

The moments of B are given in the next lemma.

Lemma 3.2.4. *For $N > 2$, we have:*

$$E[B] = 0, \tag{3.54}$$

$$E[B^2] = \frac{1}{4}N(2\beta), \tag{3.55}$$

$$E[B^3] = \frac{3}{8}N(2\beta) + \frac{3}{16}(2\beta) - \frac{3}{4}N. \tag{3.56}$$

Proof. The identity in (3.54) follows from (3.44), whilst (3.55) is an immediate consequence of (3.45) and (3.46). For proving (3.56), we apply a formula which is similar to the one employed in Section 3.2.1 for computing $E[A^3]$:

$$\begin{aligned} E[B^3] = & E[b_1^3] + 3(N-1)E[b_1^2 b_2] + 3(N-1)E[b_1 b_2^2] \\ & + (N-1)E[b_2^3] + 3(N-1)(N-2)E[b_2^2 b_3] \\ & + N(N-1)(N-2)E[b_1 b_2 b_3]. \end{aligned}$$

It is easy to verify that $E[b_1^2 b_2] = E[b_2^2 b_3] = E[b_1 b_2 b_3] = 0$. All that remains is to use, in the expression above, the results from (3.47)-(3.49) :

$$\begin{aligned} E[B^3] &= \left[\frac{9}{16}(2\beta) - \frac{3}{4}\right] + 3(N-1)\left[\frac{1}{16}(2\beta - 2)\right] + (N-1)\left[\frac{3}{16}(2\beta) - \frac{3}{8}\right] \\ &= \frac{3}{8}N(2\beta) + \frac{3}{16}(2\beta) - \frac{3}{4}N. \end{aligned}$$

□

3.3 Some More Calculations

In this section, we prove one more lemma concerning moments of random variables A, B and C:

Lemma 3.3.1. *Employing the definitions from Section 1.1, we have:*

$$E[C] = 0, \tag{3.57}$$

$$E[C^2] = \frac{N_0}{2}(2\beta), \tag{3.58}$$

$$E[AB] = \frac{1}{8}(2\beta - 2), \tag{3.59}$$

$$E[AC] = 0, \tag{3.60}$$

$$E[BC] = 0. \tag{3.61}$$

Proof. The identity in (3.57) can be obtained without difficulties. For proving (3.58), we use the notation $\mathbf{x}^{(1)} = [x_1, x_2, \ldots, x_{2\beta}]^\top$, as we have already done previously:

$$\begin{aligned} E[C^2] &= \sum_{i=1}^{2\beta} E[x_i^2]E[\xi_i^2] + \sum_{\substack{1\le i,j\le 2\beta \\ i\neq j}} E[x_i x_j]E[\xi_i]E[\xi_j] \\ &= \frac{N_0}{2}(2\beta) \qquad \text{[see (3.5)]}. \end{aligned}$$

The proof for (3.59) is given below.

$$E[AB] = \sum_{1\le m,n\le N} E[a_m b_n] = E[a_1 b_1]. \tag{3.62}$$

Proving the identity above reduces to:

$$E[a_m b_n] = 0 \quad \text{if } m \neq n, \tag{3.63}$$

$$E[a_m b_m] = 0 \quad \text{if } m \neq 1. \tag{3.64}$$

We show that (3.63) is true when $m = 1$ and $n = 2$ because all other cases can be treated similarly:

$$E[a_1 b_2] = E\left[\left(\mathbf{x}^{(1)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top\right] E\left[\mathbf{x}_d^{(2)}\right] = 0.$$

For (3.64), we take $m = 2$ and introduce the notation: $\mathbf{x}_d^{(1)} = [x_0, x_2, \ldots, x_{2\beta}]^\top$, $\mathbf{x}^{(2)} = [y_1, y_2, \ldots, y_{2\beta}]^\top$ and $\mathbf{x}_d^{(2)} = [y_0, y_1, \ldots, y_{2\beta-1}]^\top$. Note that $x_0 = x_{2\beta}$ and $y_0 = y_{2\beta}$. As

$\mathbf{x}^{(1)} = [x_1, x_2, \ldots, x_{2\beta}]^\top$, we have:

$$E[a_2 b_2] = \sum_{1 \le i,j \le 2\beta} E[x_i x_j] E[y_i y_{j-1}] = 0.$$

For computing $E[a_1 b_1]$, we observe that

$$E[a_1 b_1] = \sum_{1 \le i,j \le 2\beta} E[x_i^2 x_j x_{j-1}] \tag{3.65}$$
$$= \sum_{i=1}^{2\beta-2} E[x_i^2 x_{i+1} x_{i+2}]$$
$$= \frac{1}{8}(2\beta - 2). \tag{3.66}$$

Bearing in mind that $x_0 = x_{2\beta}$, we let k be $\min\{i, j, j-1\}$ for any term $x_i^2 x_j x_{j-1}$ within (3.65). If the power of x_k in $x_i^2 x_j x_{j-1}$ is odd, then $E[x_i^2 x_j x_{j-1}] = 0$ due to the arguments of the same type as those used when proving (3.21), (3.22) and (3.25). It follows that the only terms within (3.65) which can potentially have values different from zero are of the form $E[x_i^2 x_{i+m} x_{i+m+1}]$ $(m > 0)$. However, these terms equal 1/8 when $m = 1$ [see (3.24)], and otherwise are zero. This observation completes the proof for (3.66). From (3.62) and (3.66), we get (3.59).

For the last two identities, it is easy to observe that

$$E[AC] = \sum_{n=1}^{N} E\left[\left(\mathbf{x}^{(n)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top\right] E[\xi] = 0,$$
$$E[BC] = \sum_{n=1}^{N} E\left[\left(\mathbf{x}_d^{(n)}\right)^\top \mathbf{x}^{(1)} \left(\mathbf{x}^{(1)}\right)^\top\right] E[\xi] = 0.$$

3.4 Computation of Skewness for the Terms Within (1.4)

Skewness of A: We apply the definition from [2, Eq. (3.89)]:

$$\eta_3(A) = \frac{\kappa_3(A)}{[\kappa_2(A)]^{3/2}} \tag{3.67}$$
$$= \frac{E[A^3] - 3E[A^2]E[A] + 2(E[A])^3}{\{E[A^2] - (E[A])^2\}^{3/2}} \tag{3.68}$$
$$= \frac{\frac{3}{8}N(2\beta) - \frac{3}{16}N - \frac{9}{32}(2\beta) + \frac{3}{32}}{\left\{\frac{1}{4}N(2\beta) - \frac{1}{8}(2\beta)\right\}^{3/2}}$$

$$= \frac{1}{(2\beta)^{1/2}} \left[\frac{3(N-3/4)}{(N-1/2)^{3/2}} + o(1) \right] \quad (2\beta \gg 1).$$

In (3.67) we denoted $\kappa_3(A)$ and $\kappa_2(A)$ the cumulants of order 3 and 2, respectively. It is known that $\kappa_3(A)$ coincides with the central moment of order 3 of A, and we computed it in (3.68) with formula from [2, Eq. (3.41)]. In the same equation, for evaluating $\kappa_2(A)$ we used the identity $\kappa_2(A) = \mathrm{Var}[A]$. For the calculation of $E[A^q]$ when $q \in \{1, 2, 3\}$, we applied Lemma 3.2.2.

Skewness of B: Using Lemma 3.2.4, we get:

$$\begin{aligned}
\eta_3(B) &= \frac{\kappa_3(B)}{[\kappa_2(B)]^{3/2}} \\
&= \frac{E[B^3]}{\{E[B^2]\}^{3/2}} \\
&= \frac{\frac{3}{8}N(2\beta) + \frac{3}{16}(2\beta) - \frac{3}{4}N}{\left\{\frac{1}{4}N(2\beta)\right\}^{3/2}} \\
&= \frac{1}{(2\beta)^{1/2}} \left[\frac{3(N+1/2)}{N^{3/2}} + o(1) \right] \quad (2\beta \gg 1).
\end{aligned}$$

Skewness of $(\delta_1 B)$: Since δ_1 and B are statistically independent, it is straightforward to write down the following chain of identities:

$$\begin{aligned}
\eta_3(\delta_1 B) &= \frac{\kappa_3(\delta_1 B)}{[\kappa_2(\delta_1 B)]^{3/2}} \\
&= \frac{E[(\delta_1 B)^3]}{\{E[(\delta_1 B)^2]\}^{3/2}} \\
&= \frac{E[\delta_1^3]}{\left\{E[\delta_1^2]\right\}^{3/2}} \frac{E[B^3]}{\{E[B^2]\}^{3/2}} \\
&= \frac{\tau^3 3\tilde{b}^3 (\pi/2)^{1/2}}{\left(\tau^2 2\tilde{b}^2\right)^{3/2}} \kappa_3(B) \\
&= \frac{3\pi^{1/2}}{4} \kappa_3(B).
\end{aligned}$$

In the calculations above, we used (1.6) and (1.7) (after replacing b with $\tilde{b}$).

All results obtained so far in this section are shown more compactly in (1.10).

	$\tau = 0$
$(2\beta)^3$	$((2\pi)^{1/2}s^3(\pi-3)(b^3+\tilde{b}^3))/16$
$(2\beta)^2$	$(3(2\pi)^{1/2}s^3(2N-1)(b+\tilde{b})(3b\tilde{b}+b^2+\tilde{b}^2-\pi b\tilde{b}))/32$
$(2\beta)^1$	$(9(2\pi)^{1/2}s^3(4N-3)(b+\tilde{b})(b^2+b\tilde{b}+\tilde{b}^2))/64$
$(2\beta)^0$	$-(9(2\pi)^{1/2}s^3(2N-1)(b+\tilde{b})(b^2+b\tilde{b}+\tilde{b}^2))/64$
$(2\beta)^2$	$-(s^2(\pi-4)(b^2+\tilde{b}^2))/8$
$(2\beta)^1$	$s^2(N/4-1/8)(2b^2+\pi b\tilde{b}+2\tilde{b}^2)$
	$\tau = s/2$
$(2\beta)^3$	$((2\pi)^{1/2}s^3(8b^3+\tilde{b}^3)(\pi-3))/128$
$(2\beta)^2$	$(3(2\pi)^{1/2}s^3(2N-1)(2b+\tilde{b})(6b\tilde{b}+4b^2+\tilde{b}^2-2\pi b\tilde{b}))/256$
$(2\beta)^1$	$(9(2\pi)^{1/2}s^3(4N-3)(2b+\tilde{b})(4b^2+2b\tilde{b}+\tilde{b}^2))/512$
$(2\beta)^0$	$-(9(2\pi)^{1/2}s^3(2N-1)(2b+\tilde{b})(4b^2+2b\tilde{b}+\tilde{b}^2))/512$
$(2\beta)^2$	$-(s^2(4b^2+\tilde{b}^2)(\pi-4))/32$
$(2\beta)^1$	$(s^2(2N-1)(4b^2+\pi b\tilde{b}+\tilde{b}^2))/16$
	$\tau = s$
$(2\beta)^3$	$((2\pi)^{1/2}b^3s^3(\pi-3))/16$
$(2\beta)^2$	$(3(2\pi)^{1/2}b^3s^3(2N-1))/32$
$(2\beta)^1$	$(9(2\pi)^{1/2}b^3s^3(4N-3))/64$
$(2\beta)^0$	$-(9(2\pi)^{1/2}b^3s^3(2N-1))/64$
$(2\beta)^2$	$-(b^2s^2(\pi-4))/8$
$(2\beta)^1$	$2b^2s^2(N/4-1/8)$

Table 3.2: Expressions of $\kappa_3(\delta_0 A)$ and $\kappa_2(\delta_0 A)$ when $\tau \in \{0, s/2, s\}$. The results are presented as polynomials of variable (2β). For each value of τ, in the second column, we outline the coefficients of the polynomial corresponding to $\kappa_3(\delta_0 A)$ followed by the coefficients of the polynomial corresponding to $\kappa_2(\delta_0 A)$. The expression of each polynomial can be obtained by multiplying each coefficient with the power of (2β) shown on the same row and then summing the resulting terms. Remark that the degree of the polynomial for $\kappa_3(\delta_0 A)$ is three, whilst $\kappa_2(\delta_0 A)$ is a polynomial of degree two. In the expression of $\kappa_2(\delta_0 A)$, the term of degree zero is zero.

Skewness of $(\delta_0 A)$**:** For this term, the calculations are much more difficult than for the previous ones. This is why we restrict our attention to the case when $\tau \in \{0, s/2, s\}$ and use Symbolic Math Toolbox in Matlab. The expressions obtained for $\kappa_3(\delta_0 A)$ and $\kappa_2(\delta_0 A)$ are outlined in Table 3.2. Note that both $\kappa_3(\delta_0 A)$ and $\kappa_2(\delta_0 A)$ are regarded as polynomials of variable (2β). For all three values of τ, the results from Table 3.2 are further used to find $\eta_3(\delta_0 A)$ when N is fixed and $\beta \to \infty$ (see 1.11).

□

3.5 Proof of Lemma 1.2.1

The identity in (1.42) can be easily obtained by using (1.41) and the results presented in Table 3.3. For (1.43), we use the results in Table 3.4 along with those which are outlined below:

Under the hypothesis that $m \neq n$, we have:

$$\begin{aligned} E[a_n x_1^{(1)} x_{2\beta}^{(m)}] &= E[a_n x_1^{(1)}] E[x_{2\beta}^{(m)}] \\ &= 0 \quad \text{[see (3.4)]}. \end{aligned}$$

For $m = n$ ($n \neq 1$), we get:

$$\begin{aligned} E[a_n x_1^{(1)} x_{2\beta}^{(m)}] &= E[a_n x_1^{(1)} x_{2\beta}^{(n)}] \\ &= E\left[\left(\sum_{j=1}^{2\beta} x_j^{(n)} x_j^{(1)}\right) x_{2\beta}^{(n)} x_1^{(1)}\right] \\ &= \sum_{j=1}^{2\beta} E\left[x_j^{(n)} x_{2\beta}^{(n)}\right] E\left[x_j^{(1)} x_1^{(1)}\right] \\ &= 0, \end{aligned}$$

because we cannot have simultaneously $j = 2\beta$ and $j = 1$ [see also (3.7)]. Then we take $m = n = 1$:

$$\begin{aligned} E[a_n x_1^{(1)} x_{2\beta}^{(m)}] &= E[a_1 x_1^{(1)} x_{2\beta}^{(1)}] \\ &= E\left[x_1^{(1)} x_{2\beta}^{(1)} \sum_{j=1}^{2\beta} \left(x_j^{(1)}\right)^2\right] \\ &= E\left[\left(x_1^{(1)}\right)^3 x_{2\beta}^{(1)}\right] + E\left[x_1^{(1)} \left(x_{2\beta}^{(1)}\right)^3\right] + \sum_{j=1}^{2\beta-1} E\left[x_1^{(1)} \left(x_j^{(1)}\right)^2 x_{2\beta}^{(1)}\right] \\ &= 0. \end{aligned}$$

It can be easily shown that each term within equation above is zero by using the same approach as in the proof of (3.21).

We continue our analysis by computing $E[b_m x_1^{(1)} x_{2\beta}^{(n)}]$. It is clear that

$$E[b_m x_1^{(1)} x_{2\beta}^{(n)}] = 0 \quad \text{if} \quad m \neq n.$$

When $m = n$ and $n \neq 1$, we have:

$$\begin{aligned}
E[b_m x_1^{(1)} x_{2\beta}^{(m)}] &= E\left[x_1^{(1)} x_{2\beta}^{(n)} \sum_{j=1}^{2\beta} x_j^{(1)} x_{j-1}^{(n)}\right] \\
&= E\left[\left(x_1^{(1)}\right)^2\right] E\left[\left(x_{2\beta}^{(n)}\right)^2\right] + \sum_{j=2}^{2\beta} E\left[x_{j-1}^{(n)} x_{2\beta}^{(n)}\right] E\left[x_j^{(1)} x_1^{(1)}\right] \\
&= \frac{1}{4} \quad \text{[see (3.5) and (3.7)]}.
\end{aligned}$$

The last case which we should consider is $m = n = 1$:

$$\begin{aligned}
E[b_m x_1^{(1)} x_{2\beta}^{(m)}] &= E\left[x_1^{(1)} x_{2\beta}^{(1)} \sum_{j=1}^{2\beta} x_j^{(1)} x_{j-1}^{(1)}\right] \\
&= E\left[\left(x_1^{(1)}\right)^2 \left(x_{2\beta}^{(n)}\right)^2\right] + E\left[\left(x_1^{(1)}\right)^2 x_2^{(1)} x_{2\beta}^{(n)}\right] + \sum_{j=3}^{2\beta} E\left[x_1^{(1)} x_{j-1}^{(1)} x_j^{(1)} x_{2\beta}^{(1)}\right] \\
&= \frac{1}{4} \quad \text{[see (3.6) and (3.23)]}.
\end{aligned}$$

Using the definition in (1.41), we obtain:

$$\begin{aligned}
&E\left[z_i^2 | \alpha_{00}, \alpha_{01}, \tau, \gamma_i^{(n)}, \gamma_{i-1}^{(n)}\right] \\
&= \quad \delta_0^2 \left\{ \left(\gamma_i^{(1)}\right)^2 \left[\frac{1}{4}(2\beta)^2 + \frac{1}{8}(2\beta)\right] + \frac{1}{4}(2\beta) \sum_{n=2}^{N} \left(\gamma_i^{(n)}\right)^2 \right\} \\
&\quad + \frac{\delta_1^2}{4}(2\beta) \sum_{n=1}^{N} \left(\gamma_i^{(n)}\right)^2 + \frac{\delta_1^2}{4} \sum_{n=1}^{N} \left(\gamma_{i-1}^{(n)} - \gamma_i^{(n)}\right)^2 + s^2 \frac{N_0}{2}(2\beta) \\
&\quad + \quad 2\delta_0\delta_1 \left(\gamma_i^{(1)}\right)^2 \frac{1}{8}(2\beta - 2) + 2\delta_1^2 \sum_{n=1}^{N} \left[\gamma_i^{(n)} \left(\gamma_{i-1}^{(n)} - \gamma_i^{(n)}\right) \frac{1}{4}\right] \\
&= \quad \delta_0^2 \left[\frac{N}{4}(2\beta) + \frac{1}{4}(2\beta)^2 - \frac{1}{8}(2\beta)\right] + \delta_1^2 \frac{N}{4}(2\beta) + s^2 \frac{N_0}{2}(2\beta) + 2\delta_0\delta_1 \frac{1}{8}(2\beta - 2) \\
&\quad + \frac{\delta_1^2}{4} \sum_{n=1}^{N} \left[\left(\gamma_{i-1}^{(n)}\right)^2 - \left(\gamma_i^{(n)}\right)^2\right].
\end{aligned}$$

Note that $\left(\gamma_i^{(n)}\right)^2 = 1$ for all i and n. A simple comparison of the expression above with the one in (1.14) leads to the conclusion that (1.43) is true.

Identity	Justification
$E[a_1] = \beta$	Eq. (3.38)
$E[a_n] = 0,\, n \neq 1$	Lemma 3.2.2
$E[b_n] = 0$	Eq. (3.44)
$E\left[x_1^{(1)} x_{2\beta}^{(1)}\right] = 0$	Eq. (3.7)
$E\left[x_1^{(1)} x_{2\beta}^{(n)}\right] = 0,\, n \neq 1$	Eq. (3.4)
$E[C] = 0$	Eq. (3.57)

Table 3.3: Auxiliary results for proving the identity in (1.42).

Identity	Justification
$E[a_1^2] = \frac{1}{4}(2\beta)^2 + \frac{1}{8}(2\beta)$	Eq. (3.39)
$E[a_n^2] = \frac{1}{4}(2\beta),\, n \neq 1$	Eq. (3.41)
$E[a_m a_n] = 0,\, m \neq n$	Lemma 3.2.2
$E[b_n^2] = \frac{1}{4}(2\beta)$	Eq. (3.45)
$E[b_m b_n] = 0,\, m \neq n$	Lemma 3.2.4
$E\left[\left(x_1^{(1)} x_{2\beta}^{(n)}\right)^2\right] = \frac{1}{4}$	Eqs. (3.5),(3.6)
$E\left[\left(x_1^{(1)}\right)^2 x_{2\beta}^{(m)} x_{2\beta}^{(n)}\right] = 0,\, m \neq n, m \neq 1, n \neq 1$	Eq. (3.4)
$E\left[\left(x_1^{(1)}\right)^2 x_{2\beta}^{(1)} x_{2\beta}^{(n)}\right] = 0,\, n \neq 1$	Eq. (3.4)
$E[C^2] = \frac{N_0}{2}(2\beta)$	Eq. (3.58)
$E[a_m b_n] = 0,\, m \neq n$	Eq. (3.63)
$E[a_m b_m] = 0,\, m \neq 1$	Eq. (3.64)
$E[a_1 b_1] = \frac{1}{8}(2\beta - 2)$	Eq. (3.66)
$E[a_n C] = 0$	Lemma 3.3.1
$E[b_n C] = 0$	Lemma 3.3.1

Table 3.4: Auxiliary results for proving the identity in (1.43).

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