Automatic Structures, Part 2

Bakh Khoussainov
Computer Science Department, The University of Auckland, New Zealand

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Plan: Tutorial 2

Proof methods

- Automatic well-founded partial orders
- Automatic linear orders and trees
- Automatic Boolean algebras
- Automatic Finitely generated groups

Automatic partial orders

Definition

A **partially ordered set** $\mathcal{A} = (A, \leq)$ is automatic if A and $\leq$ are recognized by word automata.

Examples:

1. Small ordinals ω^n, where n is finite.
2. $(\{0, 1\}^{\star}; \preceq)$.
3. Finite or co-finite subset of ω under inclusion.

Well-founded relations

Structures

Definition

A relation R is called **well-founded** if there is no infinite sequence $x_1, x_2, x_3, \ldots$ such that $(x_{i+1}, x_i) \in R$ for $i \in \omega$.

Define the **height** function as follows:

1. For the R-minimal elements x, set $h_{\mathcal{A}}(x) = 0$.
2. Put $h_{\mathcal{A}}(z) = \sup\{h(y) + 1 : (y, z) \in R\}$.

The height of $\mathcal{A} = (A, R)$, is $\sup\{h_{\mathcal{A}}(x) \mid x \in A\}$.

Heights of well-founded partial orders

Goal: Study heights of automatic well founded partial orders.

Lemma

- *For each $\alpha < \omega_1^{CK}$ there is a computable well-founded partial order of height α.*
- *The height of each computable well founded relation is below ω_1^{CK}.* □

Lemma

For a structure $\mathcal{A} = (A; R)$ where R is well-founded, if $h(\mathcal{A}) = \alpha$ and $\beta < \alpha$ then there is an $x \in A$ such that $h_{\mathcal{A}}(x) = \beta$. □

The natural sum of ordinals

Definition

The natural sum of ordinals α, β, $\alpha +' \beta$, is defined recursively by putting $\alpha +' \beta$ as the least ordinal strictly greater than $\gamma +' \beta$ for all $\gamma < \alpha$ and strictly greater than $\alpha +' \gamma$ for all $\gamma < \beta$.

This sum can also be defined as follows:

$$(\omega^{\beta_1} c_1 + \ldots + \omega^{\beta_k} c_k) + (\omega^{\beta_1} b_1 + \ldots + \omega^{\beta_k} b_k) =$$
$$\omega^{\beta_1}(c_1 + b_1) + \ldots + \omega^{\beta_k}(c_k + b_k).$$

- Let $\mathcal{A} = (A, \leq)$ be a well founded partial order.
- Let A_1 and A_2 be disjoint subsets of A such that $A = A_1 \cup A_2$.
- Consider $\mathcal{A}_1 = (A_1, \leq_1)$ and $\mathcal{A}_2 = (A_2, \leq_2)$ obtained by restricting $\leq$ to A_1 and A_2 respectively.
- Let $\alpha_1 = h(\mathcal{A}_1)$ and $\alpha_2 = h(\mathcal{A}_2)$.

Height Lemma

Lemma (Height Lemma)

Under the assumptions above, $h(\mathcal{A}) \leq \alpha_1 +' \alpha_2$.

Proof. For each $x \in A$, define function $f(x)$:

Let $\mathcal{A}_{1,x} = \{z \in A_1 \mid z < x\}$ and $\mathcal{A}_{2,x} = \{z \in A_2 \mid z < x\}$.

Set $f(x) = h(\mathcal{A}_{1,x}) +' h(\mathcal{A}_{2,x})$.

The range of this ranking function is in $\alpha_1 +' \alpha_2$. □

Corollary

If $h(\mathcal{A}) = \omega^n$ then either $h(\mathcal{A}_1) = \omega^n$ or $h(\mathcal{A}_2) = \omega^n$. □

A Characterization Theorem

Theorem (Khoussainov, Minnes, 2007)

An ordinal α is the height of an automatic well-founded partial order if and only if $\alpha < \omega^\omega$.

Proof. One direction is clear because ordinals ω^n do the job.

For the other direction, assume there is an automatic well-founded po $\mathcal{A} = (A, \leq)$ such that $r(\mathcal{A}) = \alpha \geq \omega^\omega$.

- Let $(S_A, \iota_A, \Delta_A, F_A)$ be word automata for A.
- Let $(S_\leq, \iota_\leq, \Delta_\leq, F_\leq)$ be word automata for $\leq$.

Proof: continued (Delhomme's technique)

- For $a \in A$, define $a \downarrow = \{x \in A : x < a\}$.
- For $a, p \in \Sigma^\star$, set

$$X_p^a = \{pw \in A : w \in \Sigma^\star \ \& \ pw < a\}.$$

- Thus, $a \downarrow$ can be partitioned as follows:

$$a \downarrow = \{x \in A : |x| < |a| \ \& \ x < a\} \cup \cup_{p \in \Sigma^\star : |p| = |a|} X_p^a.$$

Proof: continued

- For each n, select $a_n \in A$ such that $h_{\mathcal{A}}(a_n) = \omega^n$.
- By the corollary, select p_n such that
  - $|a_n| = |p_n|$ and
  - $h(X_{p_n}^{a_n}) = h(a_n \downarrow) = \omega^n$.

Proof: continued

Define the following relation on (a, p) such that $|a| = |p|$:
$(a, p) \sim (a', p') \iff$

- $\Delta_A(\iota_A, p) = \Delta_A(\iota_A, p')$, and
- $\Delta_{\leq}(\iota_{\leq}, \binom{p}{a}) = \Delta_{\leq}(\iota_{\leq}, \binom{p'}{a'})$.

There are at most $|S_A| \times |S_{\leq}|$ equivalence classes.

Proof: continued

Therefore, in the sequence $(a_1, p_1), (a_2, p_2), \ldots$ there are m, n such that $m \neq n$ and $(a_m, p_m) \sim (a_n, p_n)$.

Lemma

For any $a, p, a', p' \in \Sigma^\star$, if $(a, p) \sim (a', p')$ then $h(X_p^a) = h(X_{p'}^{a'})$.

Proof. The function $f : X_p^a \to X_{p'}^{a'}$ defined by $f(pw) = p'w$ is well-defined, bijective, and order preserving. □

Thus, $\omega^m = h(X_{p_m}^{a_m}) = h(X_{p_n}^{a_n}) = \omega^n$, and we proved the theorem.

Automatic Linear orders

Fact

Let $f(x)$ be either $a^{b\cdot x+c}$ with $a, b, c \in \omega$ or polynomial with positive integer coefficients. Let L be an automatic linear order. The order $\Sigma_{x\in\omega}(L + f(x) + L)$ is automatic.

- The order of rational numbers is automatic.
- The sum and product of automatic linear orders are automatic.

Definition

Let $(L, \leq)$ be a lo set. Elements $x, y \in L$ are $\equiv_F$**-equivalent** if there are finitely many elements between them.

Factorize $(L, \leq)$ with respect to $\equiv_F$; Continue this process.

Definition

The first ordinal at which the fix point is reached is called the **Cantor-Bendixson rank** of $(L, \leq)$. We denote it by $CB(L, \leq)$.

The fix point is either **1** or the order type of rational numbers.

Cantor Bendixson ranks

Lemma

If L is an automatic linear order then so is its factor $L/\equiv_F$. □

The proof of the heights theorem is adapted to prove this:

Theorem (Khoussainov, Rubin, Stephan, 2003)

An ordinal α *is a CB rank of an automatic linear order if and only if* α *is finite.*

Corollaries

Corollary

Let L be an automatic linearly ordered set.

- *One can compute the Cantor Bendixson rank of L.*
- *It is decidable if L is scattered.*
- *If L is not scattered then one can compute an automatic dense suborder of L.*

Corollaries

Corollary

Let L be an automatic linearly ordered set.

- *It is decidable if L is an ordinal.*
- *If L is an ordinal, one can compute its Cantor normal form.*

Corollary

The isomorphism problem for automatic ordinals is decidable.

Open problem: We do not know whether the isomorphism problem for automatic linear orders is decidable.

Automatic partial order trees

Definition

A tree is $\mathcal{T} = (T, \leq)$ where $\leq$ is partial order such that $\mathcal{T}$ has the least element and the set $x \downarrow$ is linearly ordered and finite for all $x \in T$.

Examples:

1. $(L, \preceq)$, where L is prefix closed regular language.
2. Let L be regular language. Consider $(L \cup \{\lambda\}, \leq)$, where $x \leq y \iff x = y$ or
$$(|x| < |y|) \& \forall z (z \in L \& |x| = |z| \rightarrow x \preceq_{llex} z)$$
3. $(\{0,1\}^\star \cdot 1, \preceq)$ is isomorphic to $\omega^{<\omega}$.

Cantor-Bendixson ranks of trees

Definition

Let $\mathcal{T} = (T, \leq)$ be a tree. $d(\mathcal{T})$ is the subtree of all nodes x such that x belongs to two distinct infinite paths of $\mathcal{T}$. Set

- $d^{\alpha+1} = d(d^{\alpha}(\mathcal{T}))$, and
- for limit ordinal α, set $d^{\alpha} = \cap_{\beta<\alpha} d^{\beta}(\mathcal{T})$.

Definition

The first α for which $d^{\alpha+1}(\mathcal{T}) = d^{\alpha}(\mathcal{T})$ is called the **CB rank of** $\mathcal{T}$ denoted by $CB(\mathcal{T})$.

Cantor-Bendixson ranks of trees

Definition

Let $\mathcal{T} = (T, \leq)$ be an automatic finitely branching tree. Set $x \leq_{KB} y$ if $x = y$ or $y \leq x$ or there are u, v, w such that $v, w \in Successor(u)$ and $v \leq_{llex} w$ and $v \leq x$ and $w \leq y$.

The relation $\leq_{KB}$ is regular. Therefore $KB_{\mathcal{T}} = (T, \leq_{KB})$ is an automatic linearly ordered set. This order can be exploited to prove the following theorem:

Theorem (Khoussainov, Rubin, Stephan, 2003)

If $\mathcal{T}$ is an automatic tree then $CB(\mathcal{T}) < \omega$.

Full Automatic version of König's lemma

Suppose that $\mathcal{T}$ is an automatic tree. An element x is **scattered** if $|[\mathcal{T}_x]| \leq \omega$ and $[\mathcal{T}_x] \neq \emptyset$.

Theorem (Khoussainov, Rubin, Stephan, 2003)

There is a ternary regular relation $R(x, y, z)$ such that:

1. $\exists y \exists z R(x, y, z) = \{x \in \mathcal{T} \mid x \text{ is scattered}\}$.
2. *For each scattered x and $y \in \Sigma^\star$, the set $R_y = \{z \mid R(x, y, z)\}$ is an infinite path through $\mathcal{T}_x$.*
3. *For each scattered x, if η is an infinite path through $\mathcal{T}_x$ there is a y such that $R_y = \eta$.*

The Constant Growth Lemma

Lemma (Khoussainov, Nerode 1994)

Let $f : D^n \to D$ be a function such that the graph of f is a regular relation. There exists a constant C such that for all $x_1, \ldots, x_n \in D$, we have

$$|f(x_1, \ldots, x_n)| \leq max\{|x_1|, \ldots, |x_n|\} + C.$$

Proof. The Pumping lemma does the job. □

Generating sets

Let $\mathcal{A} = (A, F_0, F_1, \ldots, F_n)$ be an automatic structure. Let $X \subset A$ be such that in the $\leq_{llex}$ listing $x_1, x_2, \ldots$ of X we have $|x_n| \leq C' \cdot n$ for some constant C'.

Define $G_n(X)$ as follows:

1. $G_1(X) = \{x_1\}$.
2. $G_{n+1}(X) = G_n(X) \cup \{F_i(\bar{a}) \mid \bar{a} \in G_n(X)\} \cup \{x_{n+1}\}$.

The growth of generation theorem

Theorem (Khoussainov, Nerode, 1994; Blumensath, Gradel, 2000)

There exists a constant C such that

$$|a| \leq C \cdot n$$

for all $a \in G_n(X)$. In particular, $G_n(X) \subseteq \Sigma^{\leq C \cdot n}$ when $|\Sigma| > 1$; and $|G_n(X)| \leq C \cdot n$ when $|\Sigma| = 1$. □

Corollaries

Corollary

The following structures are not word automatic:

- *The free semigroup* $(\Sigma^\star; \cdot)$.
- $(\omega; f)$, *where* $f : \omega^2 \rightarrow \omega$ *is a bijection.*
- *The free group* $F(n)$ *with* $n > 1$ *generators.*
- $(\omega; \times)$.
- $(\omega;\ Div(x, y))$.
- $(\omega; \leq, \{n! \mid n \in \omega\})$.

Examples:

1. The Boolean algebra $\mathcal{B}_\omega$, the collection of all finite or co-finite subsets of ω.
2. The Boolean algebra $\mathcal{B}_\omega^n$, where $n \geq 1$.

The Generation Lemma for monoids

Lemma (Khoussainov, Rubin, Stephan, 2003)

Let $(M, \cdot)$ be an automatic monoid. There is a constant C such that for every $s_1, \ldots, s_n \in M$ we have

$$|s_1 \cdot s_2 \cdot \ldots \cdot s_n| \leq max\{|s_1|, |s_2|, \ldots, |s_n|\} + C \cdot log(n).$$

Proof. Use the constant growth lemma and associativity of the monoid operation. □

The Characterization Theorem for Boolean algebras

Theorem (Khoussainov, Nies, Rubin, Stephan, 2003)

A Boolean algebra is automatic if and only if it is isomorphic to $\mathcal{B}_\omega^n$ *for some* $n \geq 1$*.*

Proof. One direction is clear. We prove the other direction for the atomless Boolean algebra.

Proof: Continued

Construct a sequence embedded trees $\{T_n\}_{n\in\omega}$:

- $T_0 = \{\lambda\}$, $b_\lambda = \mathbf{1}$.
- The induction hypothesis on T_n is that the number of leaves in $\mathcal{T}_n$ is 2^n.
- For each leaf σ, the associated element b_σ is not empty.

Proof: Continued

Define $\mathcal{T}_{n+1}$ as follows:

- For each leaf b_σ in $\mathcal{T}_n$ find the first x such that $b_{\sigma 0} := b_\sigma \cap x$ and $b_{\sigma 1} := b_\sigma \cap \bar{x}$ both not empty.

Proof: Continued

- By the constant growth Lemma we have

$$|b_{\sigma 0}| \leq |b_\sigma| + C_1 \text{ and } |b_{\sigma 1}| \leq |b_\sigma| + C_1.$$

- Hence $X_n \subseteq \Sigma^{C_2 \cdot n}$, where X_n is the set of leaves of $\mathcal{T}_n$.
- Hence, by the generation lemma for monoids $\mathcal{B}(X_n) \subseteq \Sigma^{C_3 \cdot n}$.
- However, $|\mathcal{B}(X_n)| \geq 2^{2^n}$.

We have a contradiction. □

An application

Corollary

The isomorphism problem for word automatic Boolean algebras is decidable. □

Proof Elements $a, b \in B$ are $\equiv_F$**-equivalent** if their symmetric difference $(a \cap \bar{b}) \cup (\bar{a} \cap b)$ is a finite union of atoms.

By the theorem, the factor algebra $\mathcal{B}/F$ is finite if $\mathcal{B}$ is automatic. Also $\equiv_F$ is regular. Thus, $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic iff $\mathcal{B}/F$ and $\mathcal{B}'/F'$ are isomorphic. □

Automatic finitely generated groups

Examples:

1. Finitely generated Abelian groups are automatic.
2. $F(n)$, with $n > 1$, is not automatic.

Definition

A group is **virtually Abelian** if it has an Abelian subgroup of finite index.

Automatic finitely generated groups

Lemma

Virtually Abelian finitely generated groups G are automatic.

Proof. Say, $A =< x_1, x_2 >$ is an Abelian torsion free normal subgroup of finite index of the group G.

Each $g \in G$ is of the form

$$g = t_j x_1^{m_1} x_2^{m_2}, \quad j = 1, \ldots, s.$$

Proof: continued

We have:

$$x_1 t_j = t_j x_1^{a(j)} x_2^{b(j)}, \quad x_2 t_j = t_j x_1^{c(j)} x_2^{d(j)}, \quad \text{and} \quad t_i t_j = t_k x_1^{e(i)} x_2^{e(j)}.$$

Thus,

$$t_i x_1^{m_1} x_2^{m_2} \cdot t_j x_1^{n_1} x_2^{n_2} = t_i t_j x_1^{m_1 a(i) + m_2 c(j) + n_1} x_2^{m_1 b(j) + m_2 d(j) + n_2}.$$

So, the group is automatic. □

Theorem

Our goal is to prove the following

Theorem (Thomas, Oliver, 2003)

A finitely generated group is automatic if and only if the group is virtually Abelian.

Proof. One direction is given by the previous lemma. We prove the other direction.

Proof: Continued

Define:

1. $G^0 = G$, $G^{k+1} = [G_k, G_k]$, and
2. $\gamma_0(G) = G$, $\gamma_{k+1}(G) = [\gamma(G_k), G]$.

Definition

The group G is **solvable** if $G^n = \{e\}$ for some n. The group G is **nilpotent** if $\gamma_n(G) = \{e\}$ for some n.

If G is nilpotent then G is solvable.

Proof: continued

Let $\Delta = \{a_1, \ldots, a_k\}$ be a generating set of G. By the generation lemma for monoids, we have

$$G_n(\Delta) \subseteq \Sigma^{C \cdot log(n)}, \text{ and hence } |G_n(\Delta)| \leq n^C.$$

Theorem (Gromov)

If a finitely generated group has a polynomial growth then it is virtually nilpotent.

Theorem (Ershov)

A nilpotent group has a decidable FO theory if and only if it is virtually Abelian.

Proof: continued

Theorem (Romanovski, Novikov)

A virtually solvable group has a decidable FO theory if and only if it is virtually Abelian.

Thus, if G is automatic and finitely generated then:

1. G has a polynomial growth.
2. By Gromov G is virtually nilpotent. Hence G is virtually solvable.
3. By Romanovski, G is virtually Abelian. □

Open problems

1. Is the isomorphism problem for finitely generated automatic groups decidable?
2. Is the isomorphism problem for torsion free Abelian groups decidable?
3. Is the group $(Q, +)$ automatic?

Automatic groups by Thurston

Let A be a finite set of generators of a group $\mathcal{G}$ and $A = A^{-1}$.

Definition

The **Cayley graph** of $\mathcal{G}$ is the structure $(G, f_a)_{a \in A}$, where $f_a(x) = x \cdot a$ for $x \in G$.

Definition

The group $\mathcal{G}$ is **Thurston automatic** if there is a language $Rep \subseteq A^{\star}$ such that

1. Rep is regular and for each $g \in G$ there is a $v \in Rep$ such that $v = g$.
2. The set $\{(u, v) \mid \mathcal{G} \models u = v \;\&\; u, v \in Rep\}$ is regular.
3. For each $a \in A$, the set $\{(ua, v) \mid \mathcal{G} \models ua = v \;\&\; u, v \in Rep\}$ is regular.

Thurston automatic vs automatic

Here we restrict ourselves to finitely generated groups. We have the following:

- If $\mathcal{G}$ is automatic then $\mathcal{G}$ is Thurston automatic.
- There is a Thurston automatic group which is not automatic. The group $F(n)$ is such an example.
- If $\mathcal{G}$ is Thurston automatic then its Cayley graph is automatic.
- There is a group $\mathcal{G}$ such that its Cayley graph is automatic but $\mathcal{G}$ is not Thurston automatic. The Heisenberg group is such an example.