COMPSCI 773 S1C

Dynamic Programming / Min-Sum Belief Propagation Stereo

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Stereo Reconstruction: Practical Goal

Due to ill-posedness, an original 3D scene simply cannot be reconstructed from its stereo pair or even multiple images...

Reconstruction goal is more limited and practical:

3D surfaces close enough to visual or photogrammetric perception.

Human vision combines binocular stereo with multiple monocular depth cues!

- Human vision fails on uniform/ repetitive texture, large depth gradients, contrast differences, large y-disparities, etc.

Computational stereo relies on regularising constraints to cope with multiple equivalent solutions

- Due to a large variety of observed scenes, only very general prior knowledge can be used to constrain optical surfaces
- E.g. smoothness, depth gradient (curvature), continuity, ...

Stereo by Global Optimisation

Pros: Low sensitivity to local errors via constraints on disparities in neighbouring points along scanlines or over entire images.

Cons: Generally, an NP-hard problem for 2D constraints.

- 2D constraints on corresponding signals make it even harder!
- Feasible solutions only in rare cases when direct exhaustion of surface variants with exponential complexity can be avoided.
  - Profile-wise MAP/ML reconstruction by 1D dynamic programming (**DP**) – no constraints across the profiles.
  - Similar MAP/ML or MPM/MWM reconstruction by 1D belief propagation (**BP**) – the same drawback.
- Most of the global optimisation algorithms are of cubic or larger complexity and cannot be applied to large-size images.

Global Optimisation: Popular Tools

Exact 1D MAP/ML:

- DP; max- or min-sum BP.

Approximate 2D MAP/ML: Iterative Graph Cut; loopy BP (LBP).

- Exact minimum cut / maximum flow solution on networks with non-negative edge capacities and two special source and sink nodes.
- Selecting a minimum subset of edges that separate the source from the sink and carry the maximum flow through the network.
- Binary optimisation on the lattice: reduces to the min-cut/max-flow for the cuts associated with 0–1 transitions.

Exact 1D MPM/MWM: Sum-product BP

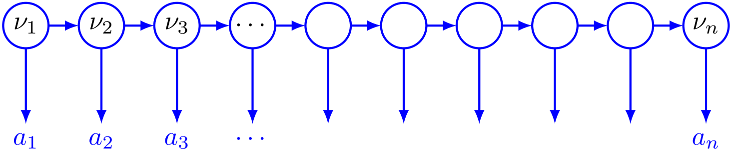
Approximate 2D MPM/MWM: Loopy sum-product BP

Convergence for the LBP is not guaranteed, but often exists in practice.

1D Optimisation on a Homogeneous Markov Chain

Hidden Markov Model (HMM): n nodes, or stages ν_i; $i = 1, \ldots, n$ with unmeasurable inner states and measurable outputs:

- m possible hidden states $u_i \in \mathbb{U}$ at each node ν_i.
- n measured (known) output signals a_i; $i = 1, \ldots, n$.



Constrained interstate transitions: $u_{i-1} \in \mathbb{C}_i(u_i)$; $i = 2, \ldots, n$.

- $\mathbb{C}_i(u_i) \subseteq \mathbb{U}$ – a subset of admissible interstate transitions, i.e. states of the preceding node ν_{i-1}, given the state u_i of the node ν_i.
- Equivalently, a subset $\mathbb{H}_i(u_i) \in \mathbb{U}$ of admissible states for the next node ν_{i+1}, given the state u_i of the node ν_i.

1D Optimisation on a Homogeneous Markov Chain

Generally, the subsets of admissible interstate transitions may also depend on the measurements (a_i, a_{i-1}) or (a_i, a_{i+1}), respectively.

- Admissible sequences $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{U} \bigcup\limits_{i=1}^{n-1} \mathbb{H}_i(u_i) \subseteq \mathbb{U}^n$.
- Each sequence supports an additive n-variate real-valued energy:

$$E(\mathbf{u}, \mathbf{a}) = \sum_{i=1}^{n} \varphi_i(u_i, a_i) + \sum_{i=1}^{n-1} f_i(u_i, a_i, u_{i+1}, a_{i+1})$$

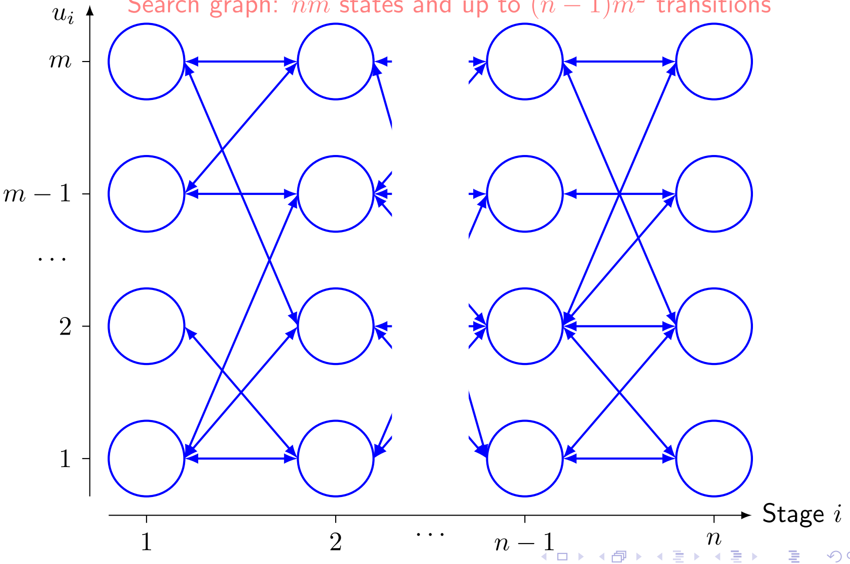
The energy terms are non-negative and depend on the hidden states $\mathbf{u}$ and known measurements $\mathbf{a} = (a_1, \ldots, a_n)$.

Energy minimisation to estimate, or restore the hidden states:

$$\mathbf{u}^*_{\mathbf{a}} \in \arg\min_{\mathbf{u} \in \mathbb{U}^n} E(\mathbf{u}, \mathbf{a});\ E^*(\mathbf{a}) = E(\mathbf{u}^*_{\mathbf{a}}, \mathbf{a})$$

State Space (i, u_i) to Search for the Optimal Solutions

Search graph: nm states and up to $(n-1)m^2$ transitions



Dynamic Programming (DP)

Forward DP pass (the measurements $\mathbf{a}$ are omitted for brevity):
For each $i = 2, \ldots, n$ and each $u_i \in \mathbb{U}$, compute the potentially optimal (candidate) partial energy, $F_i(u_i)$, and one backward pointer, $B_i(u_i)$:

$$\begin{cases} F_i(u_i) = \varphi_i(u_i) + \min\limits_{s \in \mathbb{C}_i(u_i)} \{f_{i-1}(s, u_i) + F_{i-1}(s)\}\,; \\ B_i(u_i) \in \mathbb{B}_i(u_i) = \arg \min\limits_{s \in \mathbb{C}_i(u_i)} \{f_{i-1}(s, u_i) + F_{i-1}(s)\} \end{cases}$$

Here,

$\mathbb{B}_i(u_i)$ – the set of pointers to the candidate states of the node ν_{i-1};

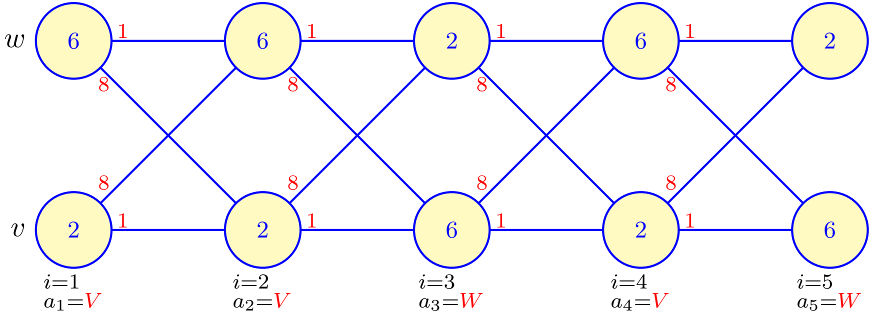
$F_i(u_i)$ – the candidate partial energy for the candidate sequence of states ending at ν_i in the state u_i; $F_1(u_1) = \varphi_1(u_1)$; $u_1 \in \mathbb{U}$.

Backward DP pass to restore the single optimal solution by using the computed candidate backward pointers, or interstate transitions:

$$\begin{cases} u_n^* \in \arg\min\limits_{s \in \mathbb{U}} F_n(s); \quad E^* = F_n(u_n^*) \\ u_{i-1}^* = B_i(u_i^*);\ i = n, n-1, \ldots, 2 \end{cases}$$

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

$f(u, u') = 1$ if $u = u'$ and 8 otherwise for $u, u' \in \mathbb{U}$
$\varphi(u, a) = 2$ if $(u, a) \in \{(v, V), (w, W)\}$ and 6 otherwise for $u \in \mathbb{U}$



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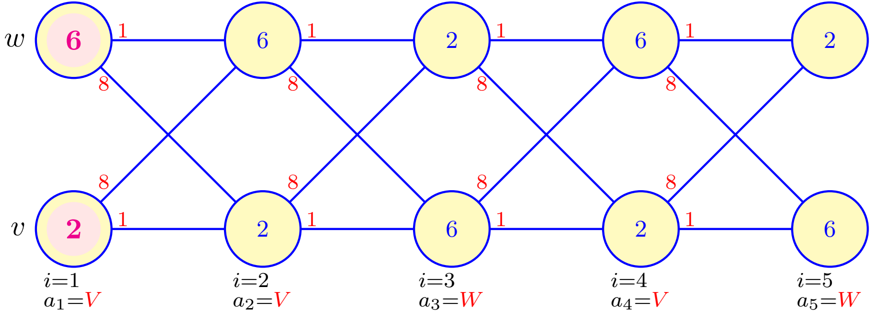

Step $i = 1$: Initialisation $F_1(u_1) = \varphi(u_1, a_1)$; $u_1 \in \mathbb{U}$

$F_1(v) = \varphi(v, V) = \mathbf{2}$; $F_1(w) = \varphi(w, V) = \mathbf{6}$

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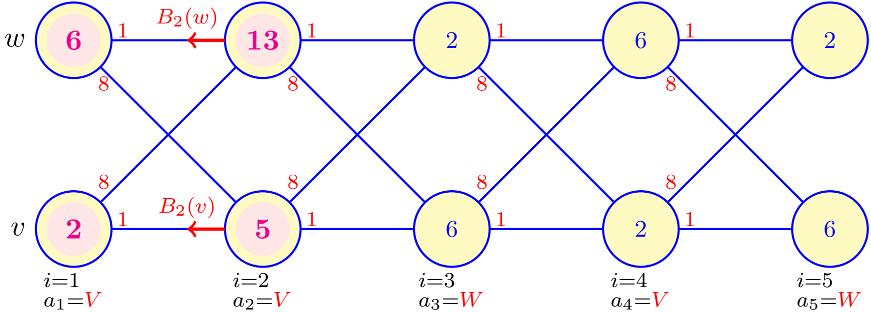

Step $i = 2$: $F_2(u_2) = \varphi(u_2, a_2) + \min_{s \in \mathbb{U}} \{f(s, u_2) + F_1(s)\}$

$$F_2(v) = 2 + \min \left\{ \begin{array}{c} 1 + \mathbf{2}, \\ 8 + \mathbf{6} \end{array} \right\} = \mathbf{5};\; F_2(w) = 6 + \min \left\{ \begin{array}{c} 8 + \mathbf{2}, \\ 1 + \mathbf{6} \end{array} \right\} = \mathbf{13}$$

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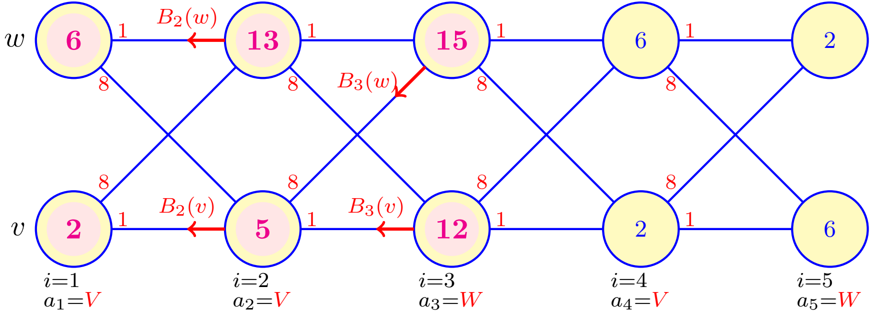

Step $i = 3$: $F_3(u_3) = \varphi(u_3, a_3) + \min_{s \in \mathbb{U}} \{f(s, u_3) + F_2(s)\}$

$$F_3(v) = 6 + \min \left\{ \begin{array}{c} 1 + \mathbf{5}, \\ 8 + \mathbf{13} \end{array} \right\} = \mathbf{12}; \; F_3(w) = 2 + \min \left\{ \begin{array}{c} 8 + \mathbf{5}, \\ 1 + \mathbf{13} \end{array} \right\} = \mathbf{15}$$

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

$f(u, u') = 1$ if $u = u'$ and 8 otherwise for $u, u' \in \mathbb{U}$

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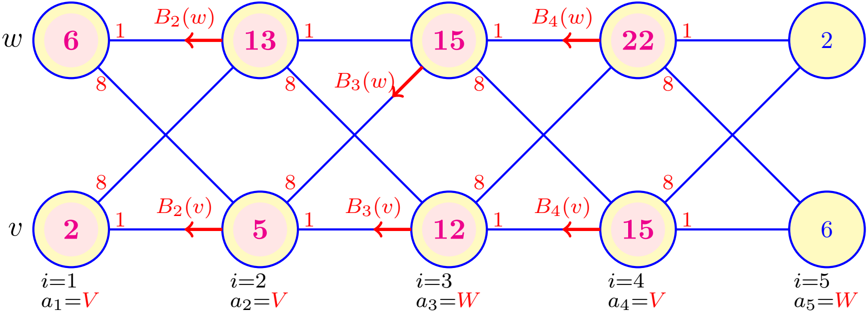

Step $i = 4$: $F_4(u_4) = \varphi(u_4, a_4) + \min_{s \in \mathbb{U}} \{f(s, u_4) + F_3(s)\}$

$F_4(v) = 2 + \min \left\{ \begin{matrix} 1 + \mathbf{12}, \\ 8 + \mathbf{15} \end{matrix} \right\} = \mathbf{15}$; $F_4(w) = 6 + \min \left\{ \begin{matrix} 8 + \mathbf{12}, \\ 1 + \mathbf{15} \end{matrix} \right\} = \mathbf{22}$

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

$f(u, u') = 1$ if $u = u'$ and 8 otherwise for $u, u' \in \mathbb{U}$
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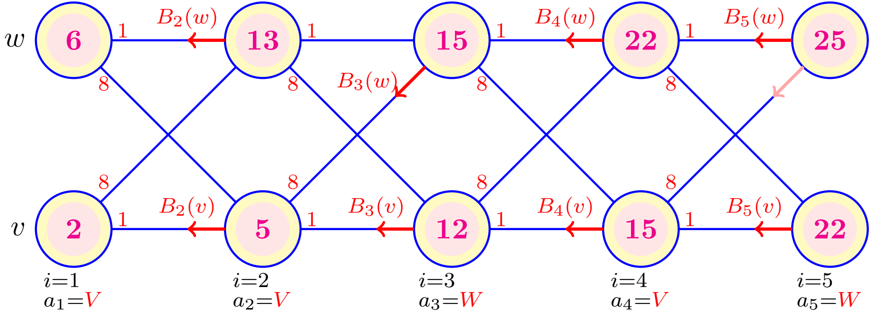

Step $i = 5$: $F_5(u_5) = \varphi(u_5, a_5) + \min_{s \in \mathbb{U}} \{f(s, u_5) + F_4(s)\}$

$$F_5(v) = 6 + \min \left\{ \begin{array}{c} 1 + \mathbf{15}, \\ 8 + \mathbf{22} \end{array} \right\} = \mathbf{22}; \;\; F_5(w) = 2 + \min \left\{ \begin{array}{c} 8 + \mathbf{15}, \\ 1 + \mathbf{22} \end{array} \right\} = \mathbf{25}$$

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

$f(u, u') = 1$ if $u = u'$ and 8 otherwise for $u, u' \in \mathbb{U}$
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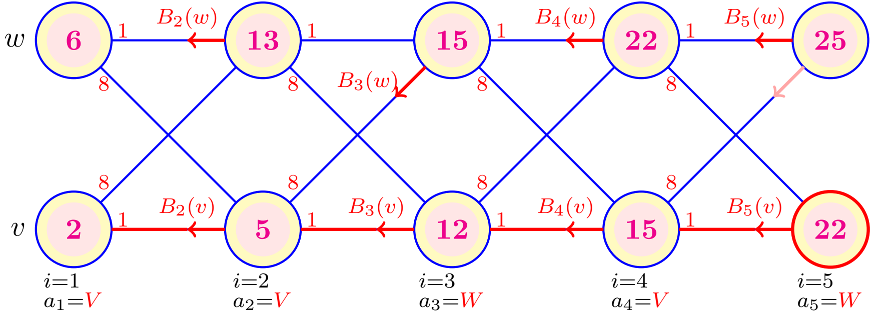


Backtracking: $\mathbf{u}^* = [v, v, v, v, v]$

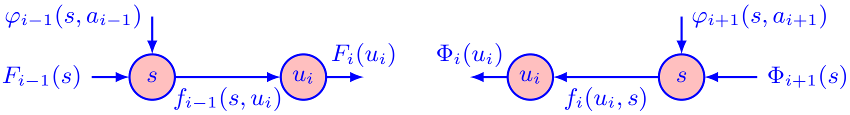
$u_5^* = v \equiv \arg\min\{F_5(v) = 22, F_5(w) = 25\}$; $u_4^* = B_5(u_5^*) = v$; ...

Belief Propagation (BP): Forward / Backward Passes

Forward BP pass– message propagation from $F_1(u_1) = 0;\ u_1 \in \mathbb{U}$:

For each $i = 2, \ldots, n$ and each state $u_i \in \mathbb{U}$, compute the potentially optimal forward partial energy, $F_i(x_i)$, called the forward message:

$$F_i(u_i) = \min_{s \in \mathbb{C}_i(u_i)} \{f_{i-1}(s, u_i) + F_{i-1}(s) + \varphi_{i-1}(s, a_{i-1})\}$$

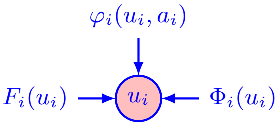


Backward BP pass – message propagation from $\Phi_n(u_n) = 0;\ u_n \in \mathbb{U}$:

For each $i = n-1, \ldots, 1$ and each $u_i \in \mathbb{U}$, compute the potentially optimal backward partial energy, $\Phi_i(x_i)$, called the backward message:

$$\Phi_i(u_i) = \min_{s \in \mathbb{H}_i(u_{i+1})} \{f_i(u_i, s) + \Phi_{i+1}(s) + \varphi_{i+1}(s, a_{i+1})\}$$

Belief Propagation (BP)



For the well-posed energy minimisation problem the BP gives the unique optimiser $\mathbf{u}^*$.

Total conditionally minimal energies with the one fixed nodal state:

$$E_i(u_i) = \min_{\{\mathbf{u} \setminus u_i\} \in \mathbb{U}^{n-1}} \{E(\mathbf{u}) = F_i(u_i) + \varphi_i(u_i, a_i) + \Phi_i(u_i)\}$$

The globally minimum energy $E^* = \min\limits_{\mathbf{u} \in \mathbb{U}^n} E(\mathbf{u})$ is found from these conditionals for each $i = 1, \ldots, n$: $E^* = \min\limits_{u_i \in \mathbb{U}} E_i(u_i)$

Nodal states u_i^* for the global optimiser(s):

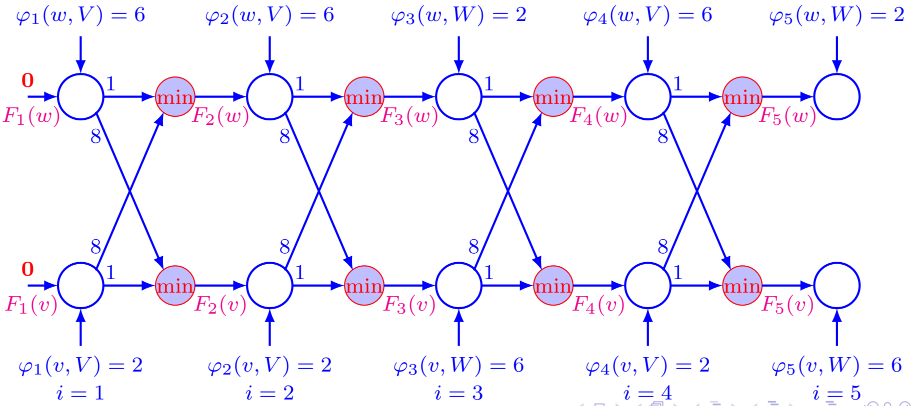
Marking states u_i^*, which belong to the optimiser(s) $\mathbf{u}^* = \arg\min_{\mathbf{u}} E(\mathbf{u})$:

$$\text{for each } i = 1, \ldots, n \quad u_i^* \in \mathbb{U}_i^* = \{u_i : u_i \in \mathbb{U};\ E_i(u_i) = E^*\}$$

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

$f_i(u_i, u_{i+1}) = 1$ if $u_i = u_{i+1}$ and 8 otherwise for $u_i, u_{i+1} \in \mathbb{U}$
$\varphi_i(u_i, a_i) = 2$ if $(u_i, a_i) \in \{(v, V), (w, W)\}$ and 6 otherwise for $u_i \in \mathbb{U}$

Forward pass: $F_i(u) = \min\limits_{s \in \{v,w\}} \{f_{i-1}(s, u) + F_{i-1}(s) + \varphi_{i-1}(s, a_{i-1})\}$:



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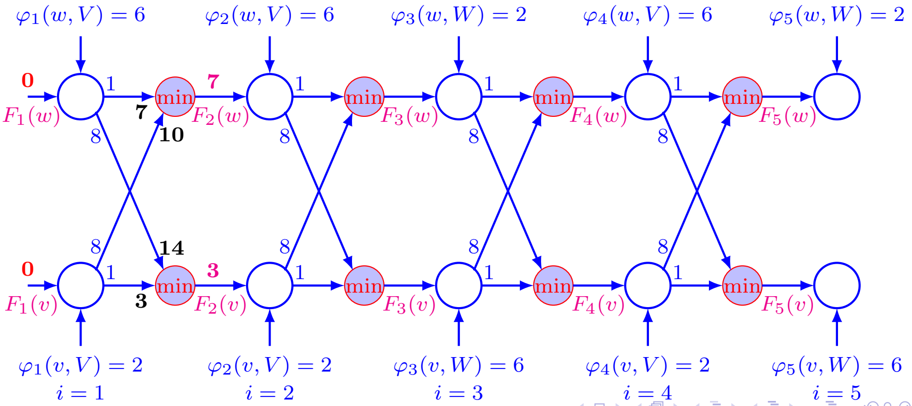

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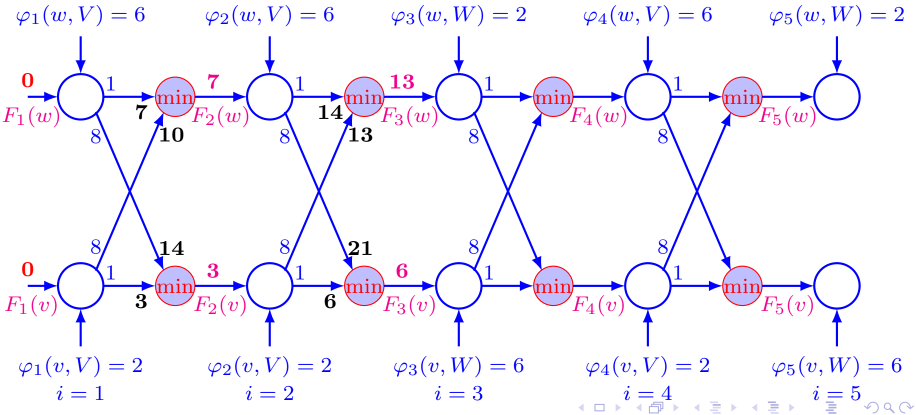

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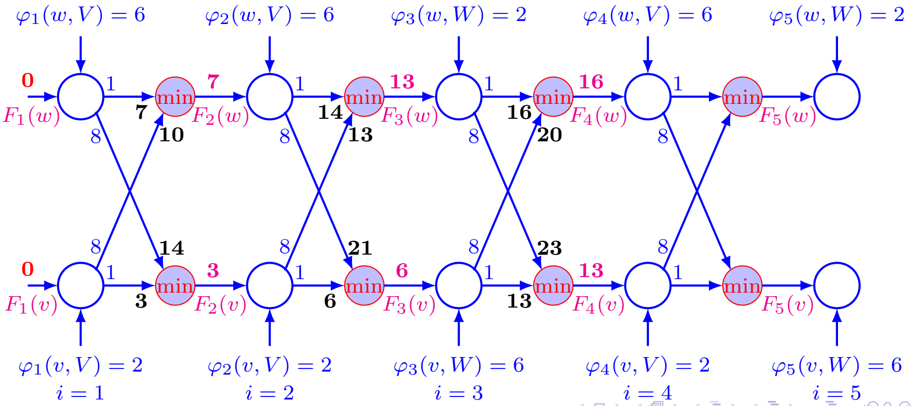

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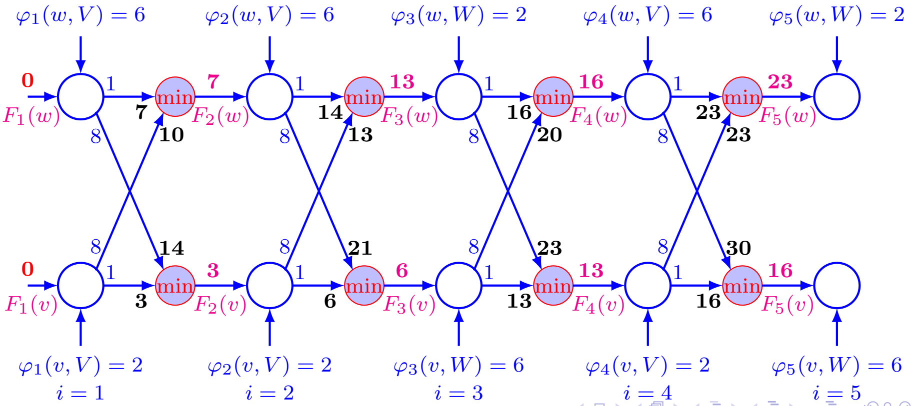

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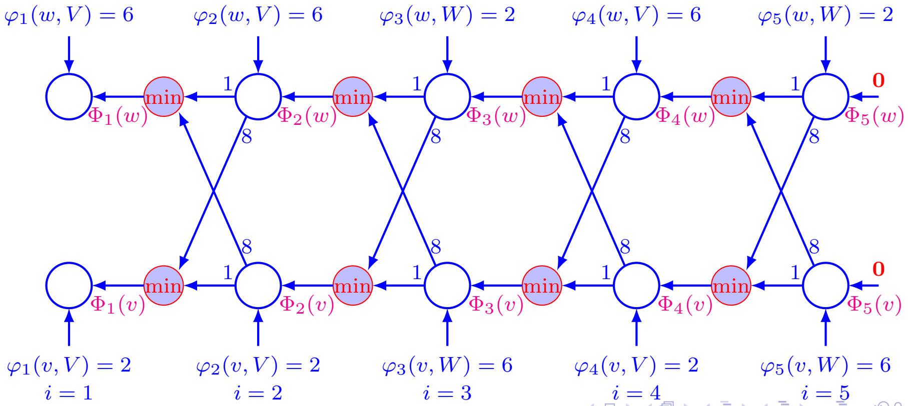
Backward pass: $\Phi_i(u) = \min\limits_{s \in \{v,w\}} \{f_i(u, s) + \Phi_{i+1}(s) + \varphi_{i+1}(s, a_{i+1})\}$:



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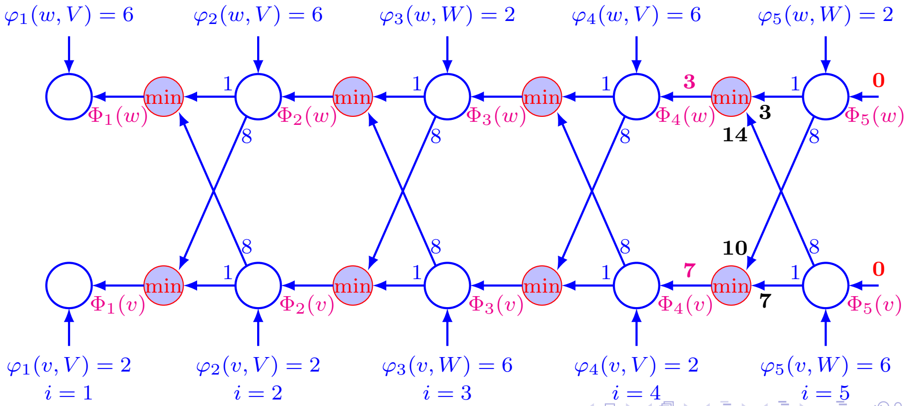

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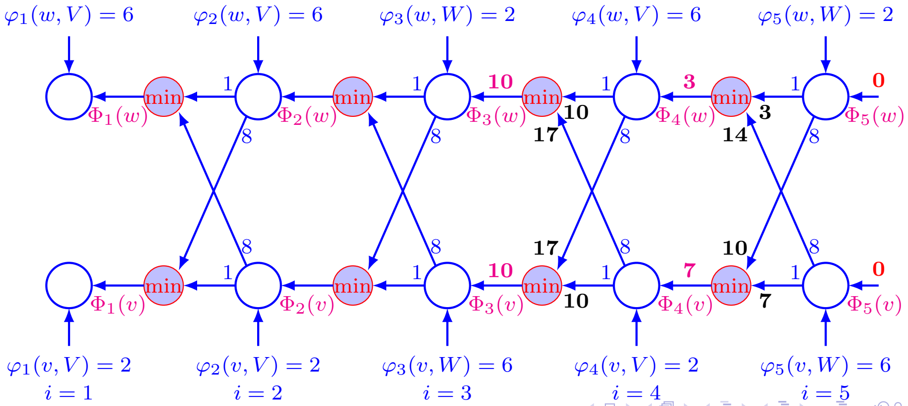

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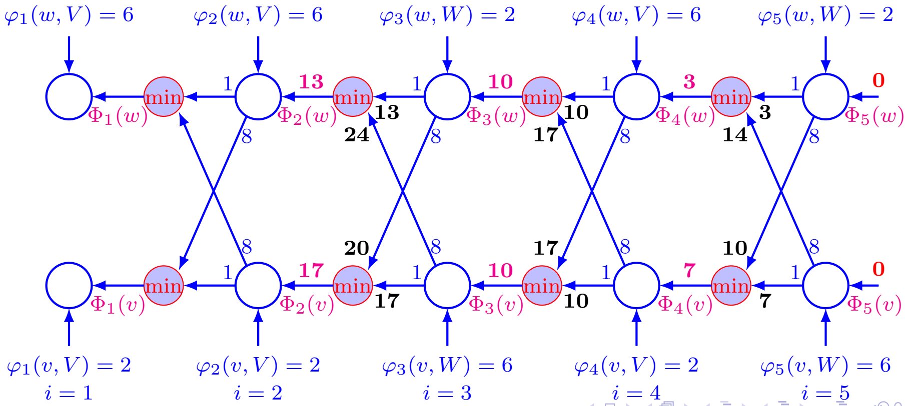

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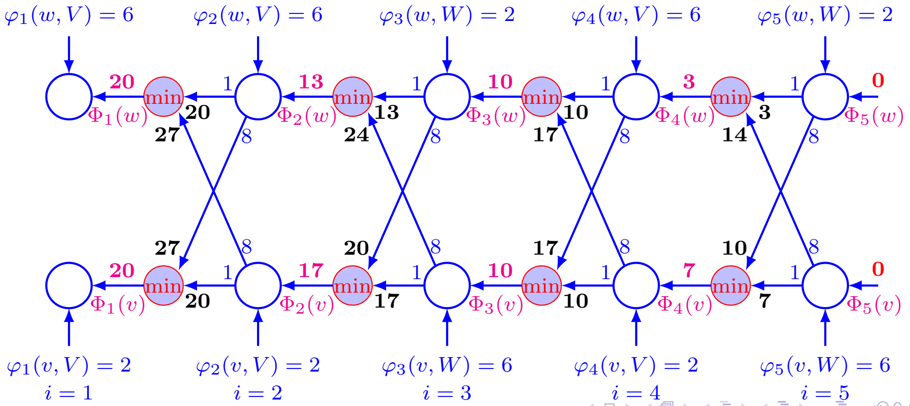
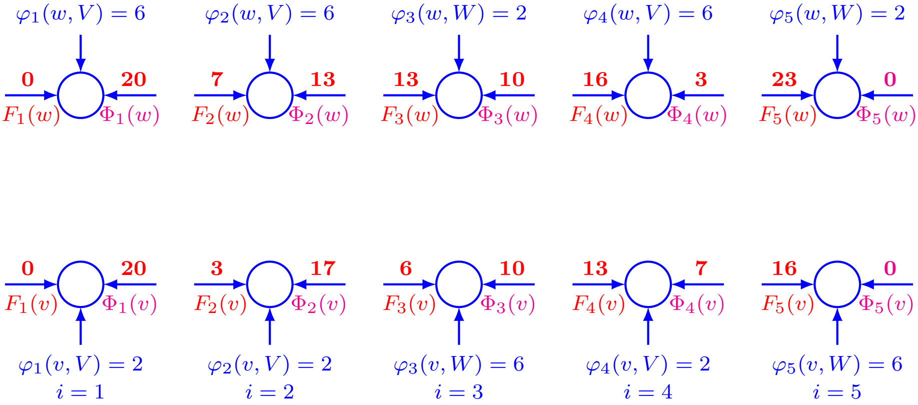


Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

Conditional global minima: $E_i(u_i) = F_i(u_i) + \varphi_i(u_i, a_i) + \Phi_i(u_i)$



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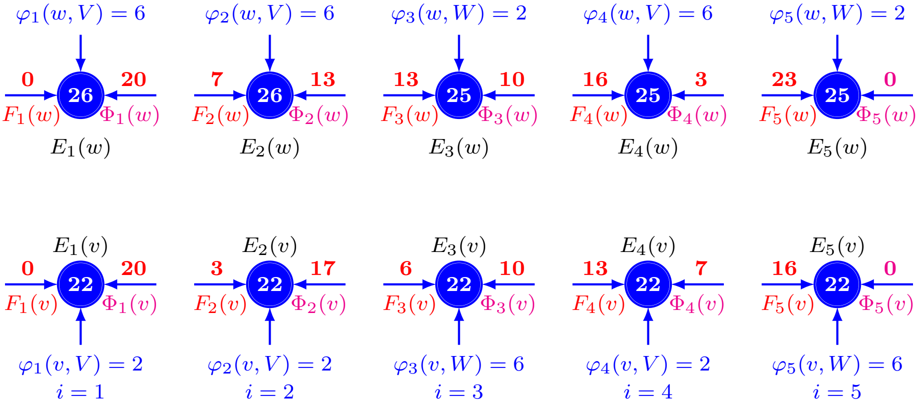

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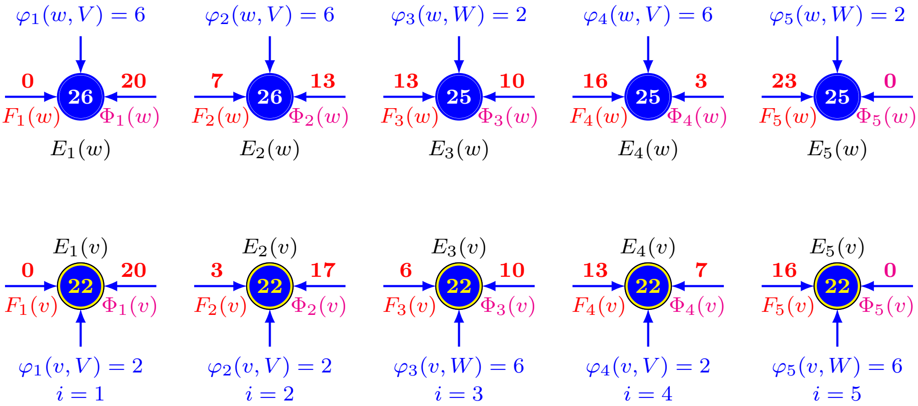
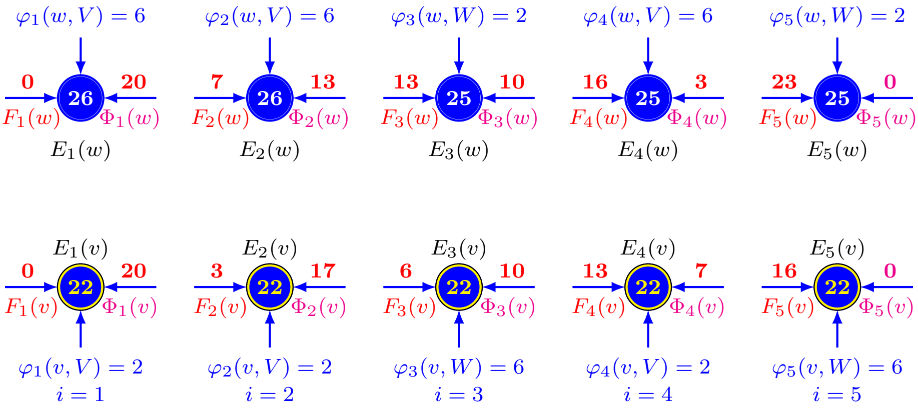


The global energy minimum $E^* = \min_{u_i \in \{v,w\}} E_i(u_i) = 22$ for $i = 1, \ldots, 5$.

Example: $\mathbb{U} = \{v, w\}$; $\mathbb{A} = \{V, W\}$; $\mathbf{a} = VVWVW$

Conditional global minima: $E_i(u_i) = F_i(u_i) + \varphi_i(u_i, a_i) + \Phi_i(u_i)$

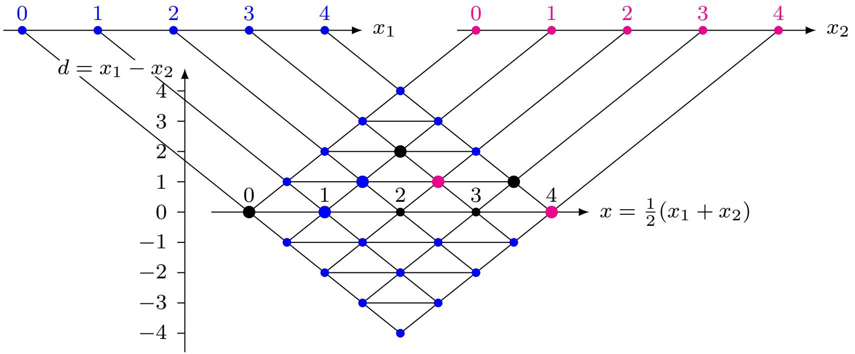


The marked optimum chain of the hidden states: $\mathbf{u}^* = [v, v, v, v, v]$.

Markov Chain Profile Model: Symmetric (x, d) Coordinates

- Accounts for symmetry of stereo channels, visibility of 3D points and discontinuities due to occlusions
- Simplifying assumption: **only a single continuous surface**

Graph of profile variants (GPV)



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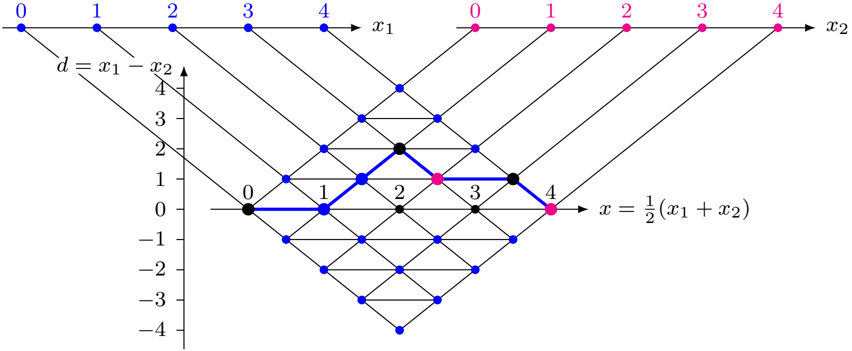

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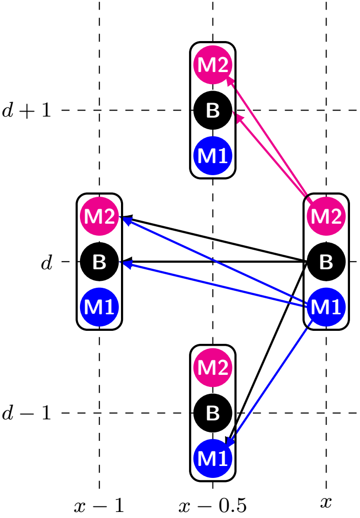


- Nodes along a continuous profile are subject to two constraints:
  1. Ordering constraint
  2. Visibility constraint
- $\{(x-1, d); (x-0.5, d \pm 1)\} \leftarrow (x, d)$
- Each node (x, d) has 3 states s
  - B – BVP (binocularly visible point): Point-wise matching score
  - Mi – MVP (monocularly visible point): Regularizing weight for partially occluded points with no correspondence
  - 8 allowable transitions
  - Popular constant MVP weight
    - More adequate MVP weights depend on related BVP scores

Allowable Transitions Along a Profile

- BVP (x, d, B): the pixel-wise energy $e_x(d, s, \mathbf{g}_{1,2})$ depends on the signals:
  - $\left(g_{1:x+\frac{d}{2}}, g_{2:x-\frac{d}{2}}\right)$ for the current BVP ($s = \mathrm{B}$ or (possibly)
  - also on the signals $\left(g_{1:x_{\mathrm{pr}}+\frac{d_{\mathrm{pr}}}{2}};\; g_{2:x_{\mathrm{pr}}-\frac{d_{\mathrm{pr}}}{2}}\right)$ for the immediate preceding BVP ($s_{\mathrm{pr}} = \mathrm{B}$).

$$e_x(d, s = \mathrm{B}, \mathbf{g}_{1,2}) = \begin{cases} \varphi_x\left(g_{1:x+\frac{d}{2}}, g_{2:x-\frac{d}{2}}\right) & \text{or} \\ f_x\left(g_{1:x+\frac{d}{2}}, g_{2:x-\frac{d}{2}}; g_{1:x_{\mathrm{pr}}+\frac{d_{\mathrm{pr}}}{2}};\; g_{2:x_{\mathrm{pr}}-\frac{d_{\mathrm{pr}}}{2}}\right) & \end{cases}$$

- MVP $(x, d, \mathrm{M1})$; MVP $(x, d, \mathrm{M2})$: the fixed "occlusion" weight e_{occl} (generally, it may depend on the related BVPs, too):

$$e_x(d, s = \mathrm{M1}, \mathbf{g}_{1,2}) = e_x(d, s = \mathrm{M2}, \mathbf{g}_{1,2}) = e_{\mathrm{occl}}$$

Simple Markov Chain x-Profile Model

Probability of a profile $\mathbf{d} = [(x_i, d_i, s_i) : \ i = 1, \ldots, n]$:

$$\Pr(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2) = p\left(x_1, d_1, s_1 | \mathbf{g}_1, \mathbf{g}_2\right) \prod_{i=2}^{n} p\left(x_i, d_i, s_i | x_{i-1}, d_{i-1}, s_{i-1}; \mathbf{g}_1, \mathbf{g}_2\right)$$

- Each term depends on a transition from the GVP-node (x_{i-1}, d_{i-1}) in state s_{i-1} to the node (x_i, d_i) in state s_i along the profile
- Transitions are limited by the visibility states along a GVP
- The probability $p\left(x_i, d_i, \mathrm{B} | x_{i-1}, d_{i-1}, s_{i-1}; \mathbf{g}_1, \mathbf{g}_2\right)$ of a transition to state B depends on dissimilarity between the corresponding image signals for the current BVP on a profiles variant
  - Generally, it can also depend on the signals for the immediate preceding BVP along this variant
- Transition probabilities to the MVP can relate to those to the BVP
  - Typical simplification: a constant MVP probability

Simple Probability Models of Corresponding Signals

Symmetric model:

- Signal deviations in $\mathbf{g}_1$ and $\mathbf{g}_2$ w.r.t. an unobserved noiseless Cyclopean image (or ortho-image) $\mathbf{g}$ of a 3D scene

$$g_{1:x_1,y_1} = g_{x,y} + \nu_{1:x,y}; \quad g_{2:x_2,y_2} = g_{x,y} + \nu_{2:x,y}$$

- Independent central-symmetric random noise $\nu_{\dots}$: monotone decrease of the probability densities $p(\nu_{1:x,y}) \propto \exp(-\gamma\nu^2_{1:x,y})$ and $p(\nu_{2:x,y}) \propto \exp(-\gamma\nu^2_{2:x,y})$

Asymmetric models:

$$g_{1:x,y} = g_{2:x,y} + \nu_{x,y} \qquad \text{or} \qquad g_{2:x,y} = g_{1:x,y} + \nu_{x,y}$$

More Realistic Models of Corresponding Signals

Contrast deviation model – positive transfer factors, a, varying over the each camera field of view and pixel-wise independent random noise, n, of image sensors:

$$g_{1:x_1,y_1} = a_{1:x,y}g_{x,y} + \nu_{1:x,y}; \quad g_{2:x_2,y_2} = a_{2:x,y}g_{x,y} + \nu_{2:x,y}$$

- Transfer factors: strong interdependence for adjacent BVPs to account for visual resemblance of corresponding areas
- **Symmetric difference model** of the interdependence:
  - Limited direct proportion of the noiseless signal increments between the neighbouring BVPs (x, y) and (x', y) along the same epipolar profile: for $k = 1, 2$

$$\begin{aligned} \min_{e_k \in \varepsilon} \{e_k (g_{x,y} - g_{x',y})\} &\leq a_{k:x,y}g_{x,y} - a_{k:x',y}g_{x',y} \\ &\leq \max_{e_k \in \varepsilon} \{e_k (g_{x,y} - g_{x',y})\} \end{aligned}$$

  - $\varepsilon = [e_{\min}, e_{\max}]$ – a fixed range $0 < e_{\min} \leq e_{\max}$ of the difference factors e

Dynamic Programming (DP) Stereo

Simplified notation with the omitted y-coordinate:

- For brevity:

$$g_i \equiv g_{x_i,y_i}; \quad g_{1:i} \equiv g_{1:x_i+\frac{d_i}{2},y}; \quad g_{2:i} \equiv g_{2:x_i-\frac{d_i}{2},y}$$

- $\mathbf{d} = ((x_i, d_i, s_i) : i = 1, 2, \ldots, N)$ – a digital profile (with allowable transitions between the adjacent GVP nodes)
- $\mathbf{g} = (g_i : i = 1, 2, \ldots, N)$ – the sequence of Cyclopean image signals along the profile $\mathbf{d}$
- $\mathbf{g}_1 = (g_{1:i} : i = 1, 2, \ldots, N_1)$ – the sequence of the corresponding left image signals for the profile $\mathbf{d}$
- $\mathbf{g}_2 = (g_{2:i} : i = 1, 2, \ldots, N_2)$ - the sequence of the corresponding right image signals for the profile $\mathbf{d}$

Pixel-wise Signal Dissimilarity

The simplest symmetric signal model for BVPs:

- Noisy grayscale signals: $g_{1:i} = g_i + n_{1:i}$; $g_{2:i} = g_i + n_{2:i}$

$$\min_{g_i} \left\{ \max \left\{ (g_{1:i} - g_i)^2, (g_{2_i} - g_i)^2 \right\} \right\}$$
$$\Rightarrow \quad g_i = \frac{1}{2}(g_{1:i} + g_{2:i}) \Rightarrow D_i = (g_{1:i} - g_{2:i})^2$$

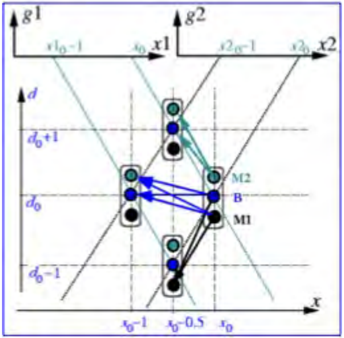
- Noisy colour (RGB) signals: $D_i = \sum_{c \in \{R,G,B\}} (g_{c:1:i} - g_{c:2:i})^2$

The simplest dissimilarity for MVPs: $D_{\text{occ}} = \text{const}$

- The constant weight: an expected signal mismatch for partially occluded points observed only in one image
- A varying MVP weight depending on mismatches for the relevant BVPs might be more adequate

Constraints on Total Signal Dissimilarity

$$D_y(\mathbf{d}|\mathbf{g}_1,\mathbf{g}_2) = \sum_{i=1}^{N} D_{s_i:i} \equiv \sum_{i=1}^{N} D_y(x_i, d_i, s_i; x_{i-1}, d_{i-1}, s_{i-1}|\mathbf{g}_1,\mathbf{g}_2)$$



Visibility and ordering constraints on transitions between successive GPV-nodes along a continuous profile

⇒ Subsets Ω_{x_i,d_i,s_i} of GPV-nodes, which can precede any node $\mathbf{v}_i = (x_i, d_i, s_i)$ along a profile:

$$\Omega_{\mathbf{v}_i} \equiv \Omega_{x_i,d_i,s_i} =$$

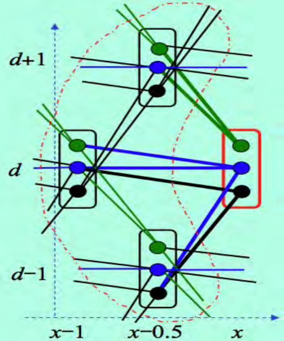
$$\begin{cases} (x_i - \frac{1}{2}, d_i + 1, \mathrm{B}); (x_i - \frac{1}{2}, d_i + 1, \mathrm{M}_2) & \text{if} \quad s_i = \mathrm{M}_2 \\ (x_i - \frac{1}{2}, d_i - 1, \mathrm{M}_1); (x_i - 1, d_i, \mathrm{B}); (x_i - 1, d_i, \mathrm{M}_2 & \text{if} \quad s_i = \mathrm{B} \\ (x_i - \frac{1}{2}, d_i - 1, \mathrm{M}_1); (x_i - 1, d_i, \mathrm{B}); (x_i - 1, d_i, \mathrm{M}_2 & \text{if} \quad s_i = \mathrm{M}_1 \end{cases}$$

DP to Minimise Signal Dissimilarity

Finds the profile for the globally minimal dissimilarity $D_y(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2)$

- DP exhausts all the profiles $\mathbf{d}$, which are possible in a GPV under the constrained transitions $\mathbf{\Omega}$
  - Forward pass along the x-axis of a GPV to find the minimum dissimilarity $D^* = \min\limits_{\mathbf{d}} D_y(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2)$
  - Backward pass to get the profile $\mathbf{d}^* = \arg\min\limits_{\mathbf{d}} D_y(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2)$
- At any current location, x_i, all GPV-nodes $\mathbf{v}_i = (x_i, d_i, s_i)$ are examined in order to calculate and store current **potentially optimal** total dissimilarities $D_{\mathrm{po}:i}(\mathbf{v}_i)$
  - Potential optimality: the stored dissimilarity is optimal if the node belongs to the globally optimal solution
  - $D_{\mathrm{po}:i}(\mathbf{v}_i)$ – the **minimal total signal dissimilarity** for the potentially optimal backward path from $\mathbf{v}_i = (x_i, d_i, s_i)$
- For each node, a potentially optimal backward transition $B_i(\mathbf{v}_i)$ to one of the preceding nodes $\mathbf{v}_{i-1}$ in $\mathbf{\Omega}$ is stored

Dynamic Programming (DP)

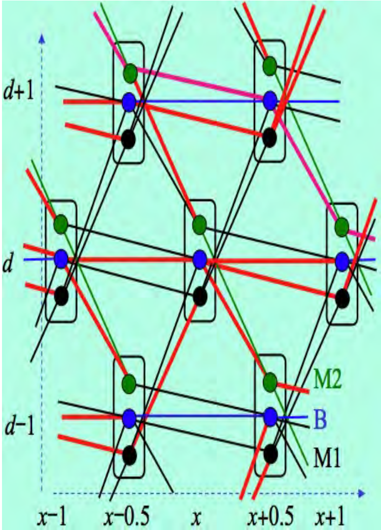


Recurrent DP computation:

$$\begin{cases} D_{\mathrm{po}:i}(\mathbf{v}_i) \\ \quad = \min\limits_{\mathbf{v}\in\Omega_{\mathbf{v}_i}} \{D_y(\mathbf{v}_i, \mathbf{v}|\mathbf{g}_1, \mathbf{g}_2) + D_{\mathrm{po}:i-1}(\mathbf{v})\} \\ \quad = D_y(\mathbf{v}_i, \mathbf{v}^*_{i-1}|\mathbf{g}_1, \mathbf{g}_2) + D_{\mathrm{po}:i-1}(\mathbf{v}^*_{i-1}) \\ \\ B_i(\mathbf{v}_i) \equiv \mathbf{v}^*_{i-1} \\ \quad = \arg\min\limits_{\mathbf{v}\in\Omega_{\mathbf{v}_i}} \{D_y(\mathbf{v}_i, \mathbf{v}|\mathbf{g}_1, \mathbf{g}_2) + D_{\mathrm{po}:i-1}(\mathbf{v})\} \end{cases}$$

$$\begin{aligned} D_{\mathrm{po}:i}(x,d,\mathrm{M}_2) &= D_{\mathrm{occ}} + \min\{D_{\mathrm{po}:i-1}(x-0.5,d+1,\mathrm{M}_2); D_{\mathrm{po}}(x-0.5,d+1,\mathrm{B})\} \\ D_{\mathrm{po}:i}(x,d,\mathrm{B}) &= \min\{D_{\mathrm{po}:i-1}(x-1,d,\mathrm{M}_2) + D_y(x,d;x_{\mathrm{prB:M}_2},d_{\mathrm{prB:M}_2}); \\ &\qquad D_{\mathrm{po}:i-1}(x-1,d+1,\mathrm{B}) + D_y(x,d;x-1,d); \\ &\qquad D_{\mathrm{po}:i-1}(x-0.5,d-1,\mathrm{M}_1) + D_y(x,d;x_{\mathrm{prB:M}_1},d_{\mathrm{prB:M}_1})\} \\ D_{\mathrm{po}:i}(x,d,\mathrm{M}_1) &= D_{\mathrm{occ}} + \min\{D_{\mathrm{po}:i-1}(x-0.5,d-1,\mathrm{M}_1); D_{\mathrm{po}:i-1}(x-1,d,\mathrm{B})\} \end{aligned}$$

Basic Recurrent DP Computation



- $B_i(x_i, d_i, s_i)$ – a potentially optimal backward transition from a GPV-node
- Optimal profile: a sequence of potentially optimal backward transitions such that
  1. Begins (i.e. the profile ends) at point $x_N = x_{\max}$ or $x_{\max} - 0.5$ and
  2. Minimises the total dissimilarity $D_{\text{po}}(x_N, d_N, s_N)$ for all the GPV nodes x_N, d_N, s_N
- $D_{\text{po}}(x^*_N, d^*_N, s^*_N)$ – the minimal total signal dissimilarity

Basic Recurrent DP Computation

After the forward pass through a given x-coordinate range $[x_{\min}, x_{\max}]$, the optimal profile is recovered by the backward pass through the stored potentially optimal backward transitions:

$$D_y(\mathbf{d}^*|\mathbf{g}_1, \mathbf{g}_2) = D_{\text{po}:N}(x_N^*, d_N^*, s_N^*)$$

$$(x_N^*, d_N^*, s_N^*) = \arg\min_{\substack{x_N \in \{x_{\max}-0.5, x_{\max}\} \\ d_N \in [d_{\min}, d_{\max}] \\ s_N \in \{\mathrm{M}_1, \mathrm{B}, \mathrm{M}_2\}}} \{D_{\text{po}:N}(x_N, y_N, s_N)\}$$

$$(x_{i-1}^*, d_{i-1}^*, s_{i-1}^*) = B_i\left(x_i^*, d_i^*, s_i^*\right); \quad i = N, N-1, \ldots, 2$$

DP stereo matching by energy minimization

- Markov models of surfaces / signals $\Longrightarrow$ Energy function $E(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2)$ as a matching score:

$$E(\mathbf{d}|\mathbf{g}_1, \mathbf{g}_2) = \sum_{i=1}^{n} \phi(x_i, d_i, s_i; x_{i-1}, d_{i-1}, s_{i-1}|\mathbf{g}_1, \mathbf{g}_2)$$

$$(x_{i-1}, d_{i-1}, s_{i-1}) \in \Omega(x_i, d_i, s_i) =$$

$$\begin{cases} \{(x_i - 0.5, d_i + 1, B); (x_i - 0.5, d_i + 1, M2)\} & \text{if} \quad s_i = M2 \\ \{(x_i - 0.5, d_i - 1, M1); (x_i - 1, d_i, B), (x_i - 1, d_i, M2)\} & \text{if} \quad s_i = B \\ \{(x_i - 0.5, d_i - 1, M1); (x_i - 1, d_i, B), (x_i - 1, d_i, M2)\} & \text{if} \quad s_i = M1 \end{cases}$$

  - Energy $=$ signal dissimilarity $+$ surface continuity $+$ surface smoothness $+$ occlusions $+ \ldots$
  - SDPS $\rightarrow$ energy accounts for different contrast and offset deviations along scanlines (approximate minimization by DP)
- DP: profile-wise global energy minima

Recurrent DP framework: Summary

Disparity "tube": $d_{\min} \leq d_i \leq d_{\max}$; $s_i \in \mathbb{S}$, for $i = 1, \ldots, n$

1. **Forward pass** along the x-axis of GPV
   - At each x_i, compute and store for each node $\mathbf{v}_i = (x_i, d_i, s_i)$
     - Potentially optimum partial total energy $E_i(\mathbf{v}_i)$
     - Potentially optimum backward transition $B_i(\mathbf{v}_i)$

$$\begin{cases} E_i(\mathbf{v}_i) & = \phi(\mathbf{v}_i; \mathbf{v}^\circ_{i-1} | \mathbf{g}_1, \mathbf{g}_2) + E_{i-1}(\mathbf{v}^\circ_{i-1}) \\ B_i(\mathbf{v}_i) \equiv \mathbf{v}^\circ_{i-1} & = \arg\min\limits_{\mathbf{v}_{i-1}} \{\phi(\mathbf{v}_i; \mathbf{v}_{i-1} | \mathbf{g}_1, \mathbf{g}_2) + E_{i-1}(\mathbf{v}_{i-1})\} \end{cases}$$

2. **Backward pass** along the x-axis of GPV
   - Optimal profile: a sequence of potentially optimum transitions:

$$\begin{aligned} \mathbf{v}^*_n &= \arg\min_{\mathbf{v}_n} E_n(\mathbf{v}_n) \\ \mathbf{v}^*_{i-1} &= B_n(\mathbf{v}^*_i);\ i = n, n-1, \ldots, 2 \end{aligned}$$

Computational complexity $O(N\Delta)$ where $\Delta = d_{\max} - d_{\min} + 1$

Symmetric Dynamic Programming Stereo: An Example