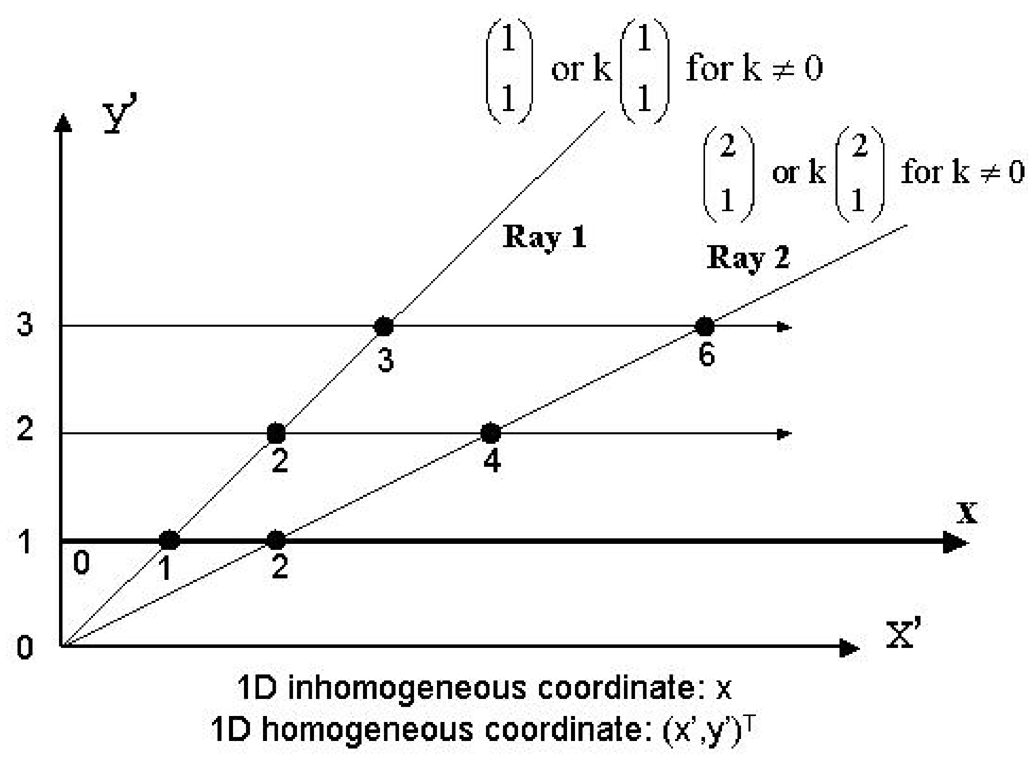
COMPSCI 773S1C – 2012

2D / 3D Geometry and 2D Edge Representation by Lines

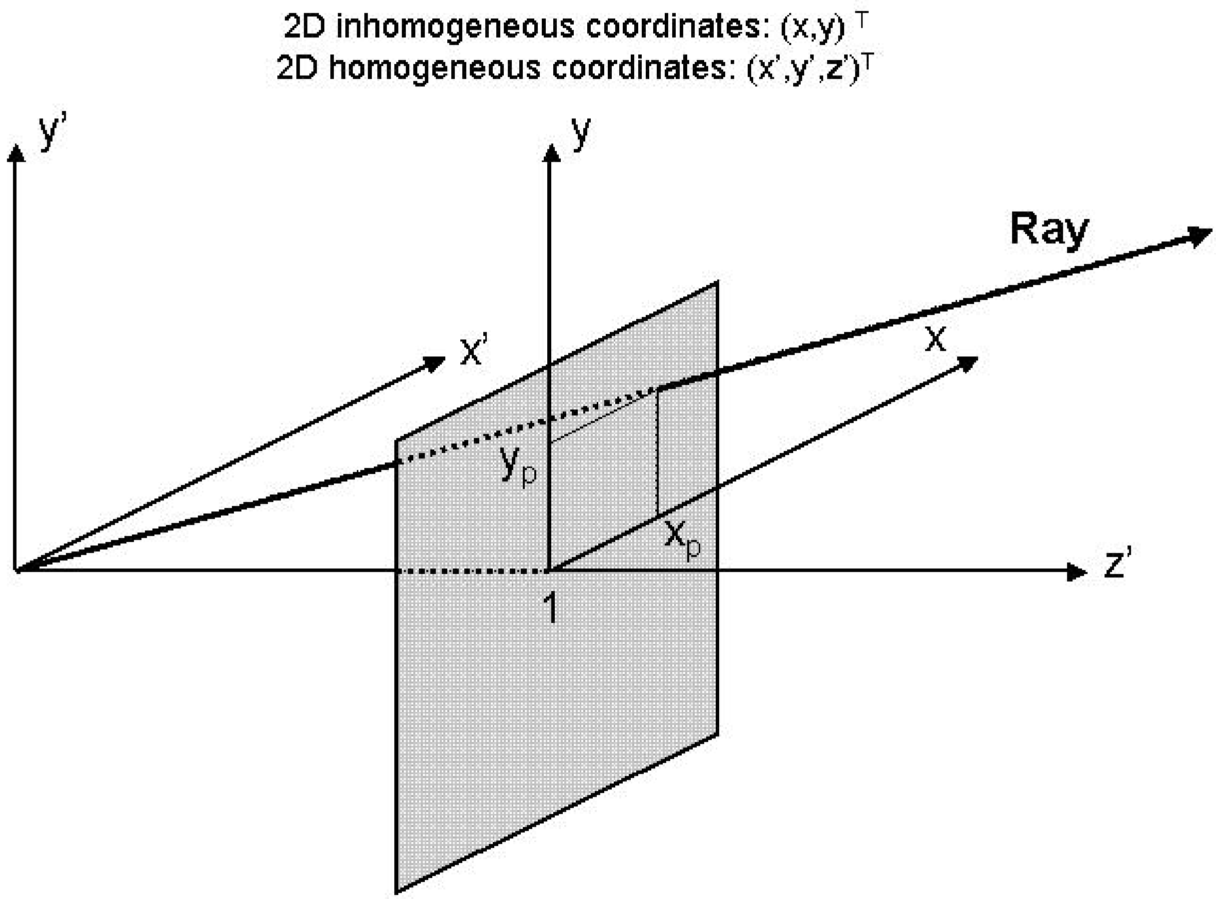
Homogeneous coordinates of a 1D point

- In inhomogeneous 1D (cartesian) coordinates, a point is represented by a single value (for example, x=1).
- In homogeneous coordinates, a 1D point is represented by a 2D vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ or $\begin{pmatrix} \frac{x'}{y'} \\ 1 \end{pmatrix}$, which defines a ray:

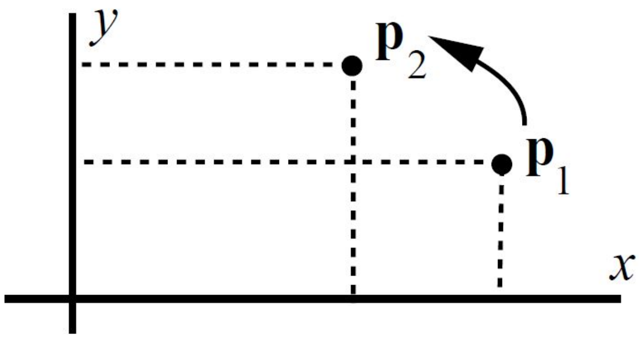


2D Homogeneous coordinates

In inhomogeneous 2D (cartesian) coordinates, a point is represented by a 2D vector $(x, y)^T$. In homogeneous coordinates, it is viewed as a 3D vector $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ or multiple of the vector $\begin{pmatrix} \frac{x'}{z'} \\ \frac{y'}{z'} \\ 1 \end{pmatrix}$.



Translation of a 2D point



$\mathbf{p}_2 = \mathbf{p}_1 + \Delta$ **or**

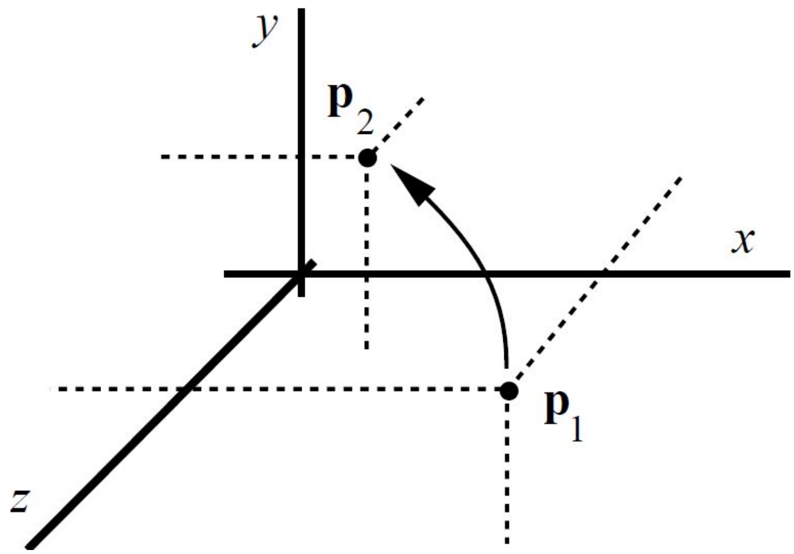
$x_2 = x_1 + \delta_x$

$y_2 = y_1 + \delta_y$

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta_x \\ 0 & 1 & \delta_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

Translation of a 3D point



$\mathbf{p}_2 = \mathbf{p}_1 + \Delta$ or

$x_2 = x_1 + \delta_x$

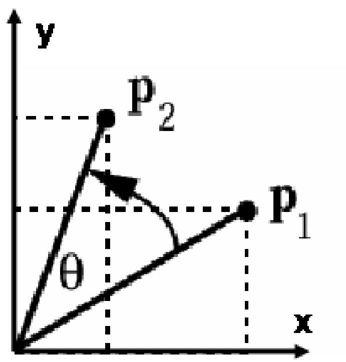
$y_2 = y_1 + \delta_y$

$z_2 = z_1 + \delta_z$

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta_x \\ 0 & 1 & 0 & \delta_y \\ 0 & 0 & 1 & \delta_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix}$$

Rotation of a 2D point around the origin



$$p_2 = R_\theta p_1$$

$$R_\theta = \begin{Vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{Vmatrix}$$

θ – **a rotation angle;** R_θ – **a rotation matrix**

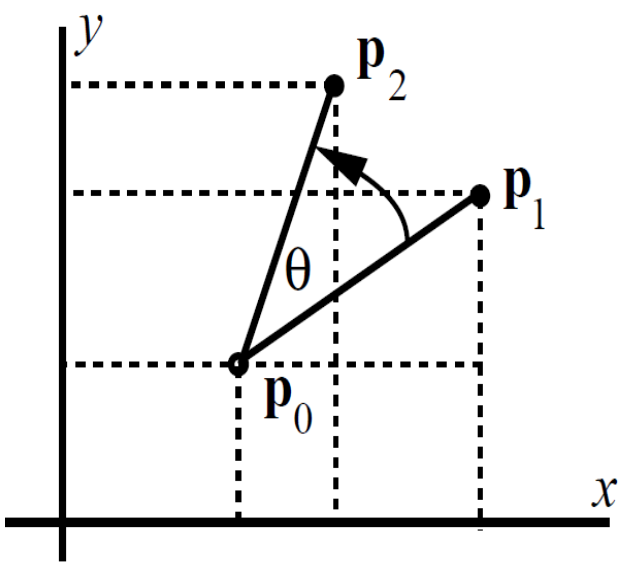
$$x_2 = x_1 \cos\theta - y_1 \sin\theta$$
$$y_2 = x_2 \sin\theta + y_2 \cos\theta$$

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ \\ y_2 \\ \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0. \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ y_1 \\ \\ 1 \end{pmatrix}$$

Rotation of a 2D point around a given center

from point.pdf from point.pdf from point.pdf from point.pdf from point.pdf from point.pdf



$$\mathbf{p}_2 = \mathbf{p}_0 + \mathbf{R}_\theta(\mathbf{p}_1 - \mathbf{p}_0)$$

$$\mathbf{R}_\theta = \begin{Vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{Vmatrix}$$

θ – **a rotation angle;** $\mathbf{R}_\theta$ – **a rotation matrix**

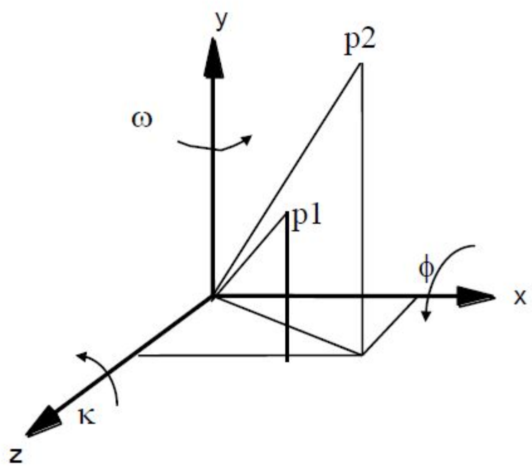
$$x_2 = x_0 + (x_1 - x_0)\cos\theta - (y_1 - y_0)\sin\theta$$

$$y_2 = y_0 + (x_1 - x_0)\sin\theta + (y_1 - y_0)\cos\theta$$

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & \begin{array}{c}[x_0(1-\cos\theta) \\ +y_0\sin\theta]\end{array} \\ \sin\theta & \cos\theta & \begin{array}{c}[-x_0\sin\theta \\ +y_0(1-\cos\theta)]\end{array} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

Rotation of a 3D point around the origin



$$\mathbf{R}_z = \begin{Vmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$ **Rotation (*swing*) matrix : rotation around z–axis**

$$\mathbf{R}_y = \begin{Vmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{Vmatrix}$$ **Rotation (*pan*) matrix : rotation around y–axis**

$$\mathbf{R}_x = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{Vmatrix}$$ **Rotation (*tilt*) matrix : rotation around x–axis**

- Three axes (x, y, z) to rotate about, so three different matrices
- Let C = cos θ and S = sin θ, then the matrices for positive (right handed) rotation are:

Rotation about x-axis: $$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C & -S & 0 \\ 0 & S & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about y-axis: $$\mathbf{R}_y = \begin{pmatrix} C & 0 & S & 0 \\ 0 & 1 & 0 & 0 \\ -S & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about z-axis: $$\mathbf{R}_z = \begin{pmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note on 3 × 3 rotation matrices:

Row and column corresponding to axis of rotation are as for identity **I**

Other elements are C on diagonal, ± S off diagonal, so that $\mathbf{R} = \mathbf{I}$ if $\theta = 0$.

Sign of S can be inferred from the fact that rotation around x,y,z by θ=90° transforms y→z , z→x, x→y, respectively.

3D rotation in homogeneous coordinates

In the general case, the 3D rotation is decomposed to three rotations around the coordinate axes x, y, z. The rotation matrix

$$\mathbf{R}_{\kappa,\phi,\omega} = \mathbf{R}_z(\kappa)\mathbf{R}_y(\phi)\mathbf{R}_x(\omega)$$

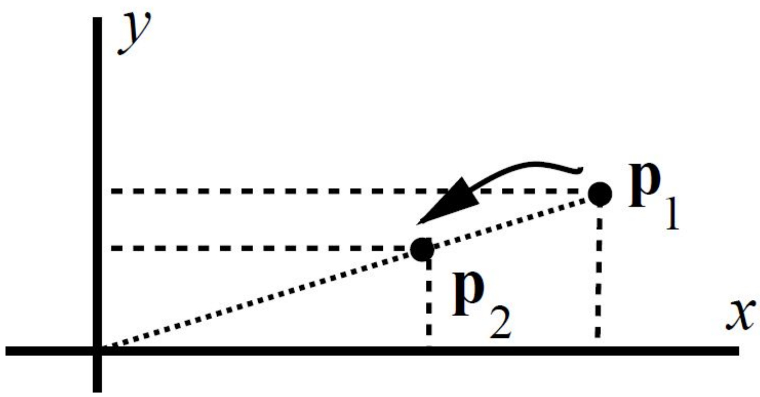
and in homogeneous coordinates the rotation is as follows:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} = \left\Vert \begin{matrix} & & & 0 \\ & \mathbf{R}_{\kappa,\phi,\omega} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\Vert \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix}$$

The resulting matrix $\mathbf{R}_{\kappa,\phi,\omega}$:

$\cos\phi\cos\kappa$	$\sin\omega\sin\phi\cos\kappa - \cos\omega\sin\kappa$	$\cos\omega\sin\phi\cos\kappa + \sin\omega\sin\kappa$
$\cos\phi\sin\kappa$	$\sin\omega\sin\phi\sin\kappa + \cos\omega\cos\kappa$	$\cos\omega\sin\phi\sin\kappa - \sin\omega\cos\kappa$
$-\sin\phi$	$\sin\omega\cos\phi$	$\cos\omega\cos\phi$

Scaling of a 2D point



$$\mathbf{p}_2 = \mathbf{S}\,\mathbf{p}_1 \quad \texttt{or}$$

$$x_2 = s_x\, x_1$$

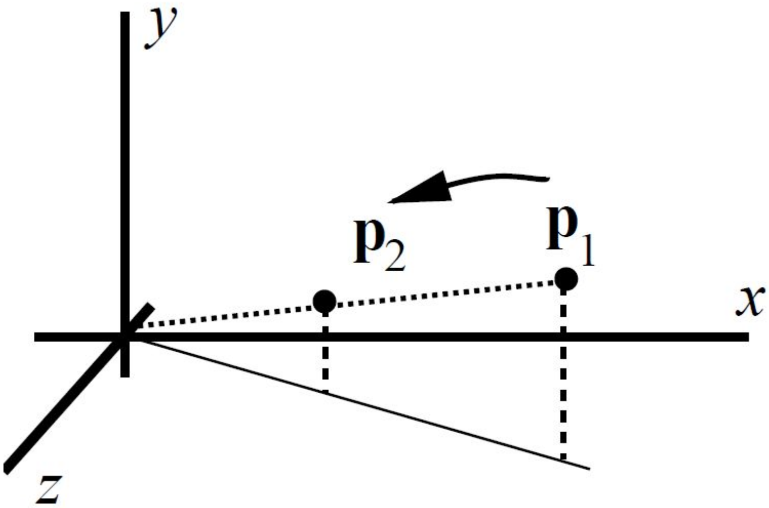
$$y_2 = s_y\, y_1$$

$$\mathbf{S} = \begin{Vmatrix} s_x & 0 \\ 0 & s_y \end{Vmatrix} \; \textbf{– a scaling matrix}$$

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

Scaling of a 3D point



$\mathbf{p}_2 = \mathbf{S}\,\mathbf{p}_1$ or

$x_2 = s_x\, x_1$

$y_2 = s_y\, y_1$

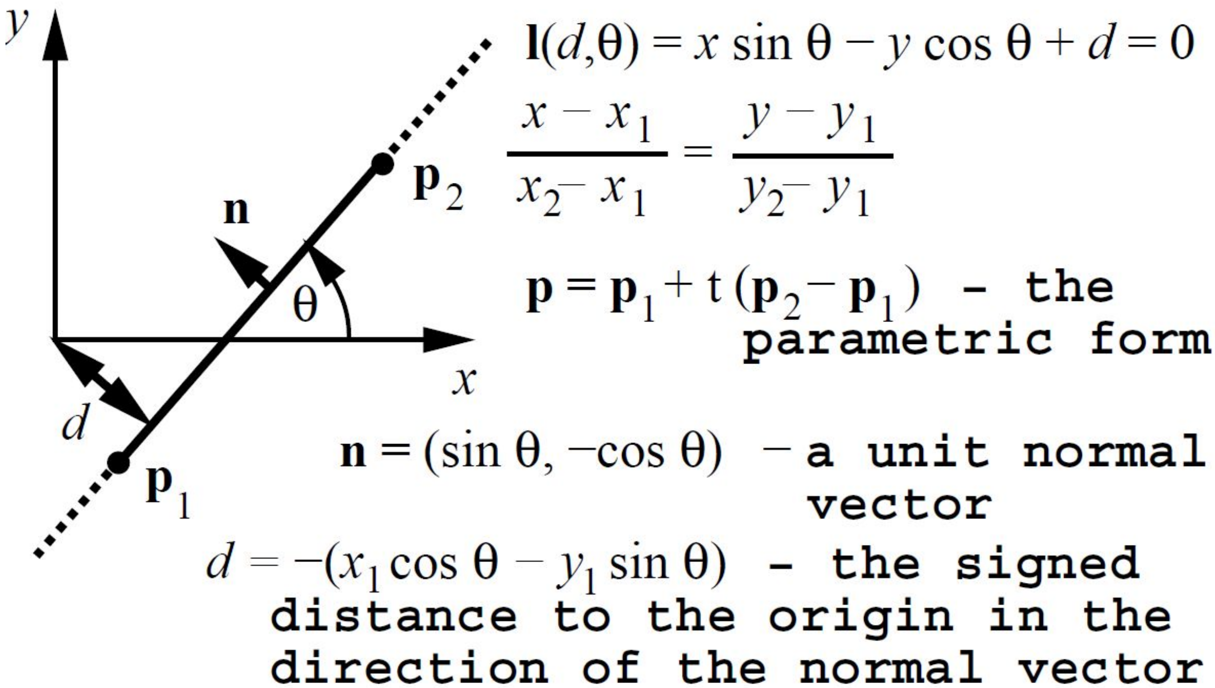
$z_2 = s_z\, z_1$

$$\mathbf{S} = \begin{Vmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{Vmatrix}$$ **– a scaling matrix**

- In homogeneous coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix}$$

Representation of 2D lines and straight-line segments

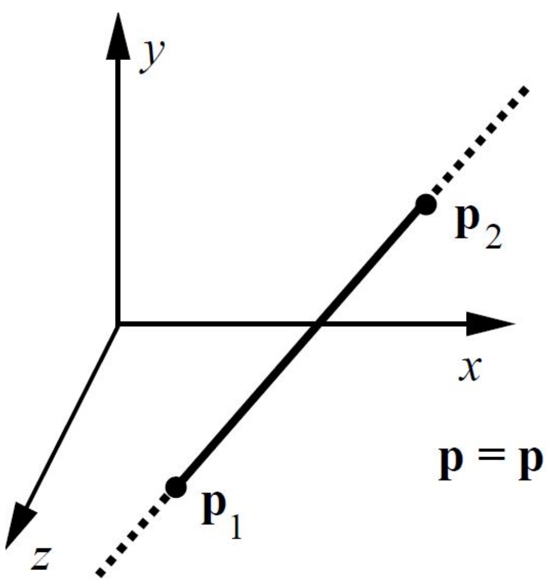


- **Interior** line points: $0 < t < 1$

- **End** points: $t = 0$ and $t = 1$

- **Exterior** line points: $t < 0$ and $t > 1$

Representation of 3D lines and straight-line segments



$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

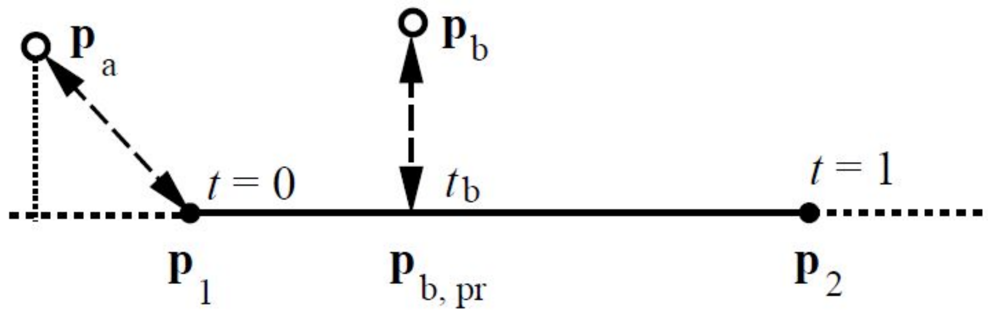
$\mathbf{p} = \mathbf{p}_1 + t\,(\mathbf{p}_2 - \mathbf{p}_1)$ - the parametric form

- **Interior** line points: $0 < t < 1$

- **End** points: $t = 0$ and $t = 1$

- **Exterior** line points: $t < 0$ and $t > 1$

Distance to a 2D or 3D segment



$\mathbf{p}_{b,\,pr}$ – the orthogonal projection of $\mathbf{p}_b$ onto the line $\mathbf{p} = \mathbf{p}_1 + t\,(\mathbf{p}_2 - \mathbf{p}_1)$:

$$t_b = \arg\min_t \left\| \mathbf{p}_b - (\mathbf{p}_1 + t\,(\mathbf{p}_2 - \mathbf{p}_1)) \right\|^2 = \frac{(\mathbf{p}_b - \mathbf{p}_1)^{\mathsf{T}}(\mathbf{p}_2 - \mathbf{p}_1)}{(\mathbf{p}_2 - \mathbf{p}_1)^{\mathsf{T}}(\mathbf{p}_2 - \mathbf{p}_1)}$$

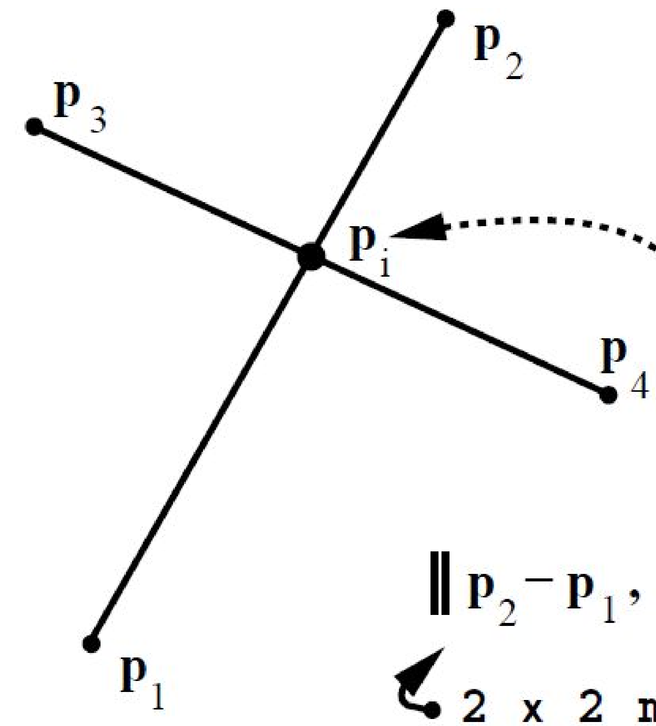
$$= \frac{(x_b - x_1)(x_2 - x_1) + (y_b - y_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ **in 2D case or**

$$= \frac{(x_b - x_1)(x_2 - x_1) + (y_b - y_1)(y_2 - y_1) + (z_b - z_1)(z_2 - z_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

in 3D case. **The projection onto the segment $\overline{\mathbf{p}_1\mathbf{p}_2}$:**

$$\mathbf{p}_{b,\,pr} = \begin{cases} \mathbf{p}_1 & \texttt{if } t_b < 0 \\ \mathbf{p}_1 + t_b\,(\mathbf{p}_2 - \mathbf{p}_1) & \texttt{if } 0 < t_b < 1 \\ \mathbf{p}_2 & \texttt{if } t_b > 1 \end{cases}$$

Intersection of two 2D lines



$$\mathbf{p}_i = \mathbf{p}_1 + t_{i,12}(\mathbf{p}_2 - \mathbf{p}_1)$$

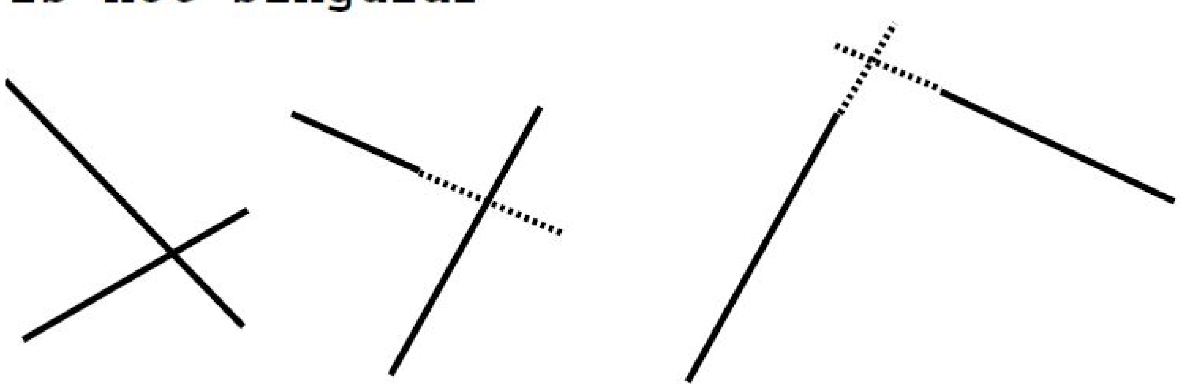
$$\mathbf{p}_i = \mathbf{p}_3 + t_{i,34}(\mathbf{p}_4 - \mathbf{p}_3)$$

$$\mathbf{t}_i = \begin{pmatrix} t_{i,12} \\ t_{i,34} \end{pmatrix}$$

$$\| \mathbf{p}_2 - \mathbf{p}_1 ,\ \mathbf{p}_4 - \mathbf{p}_3 \| \, \mathbf{t}_i = \mathbf{p}_3 - \mathbf{p}_1$$

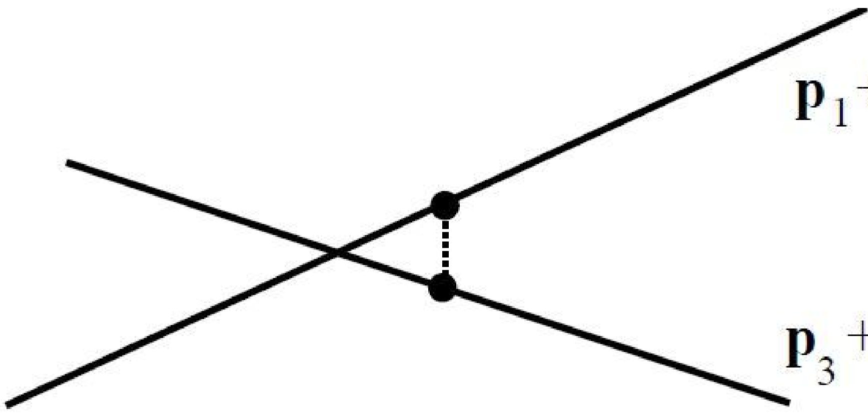
2 x 2 matrix
1 x 2 vector

The intersection exists if the matrix is not singular



Interior and exterior intersections of straight-line segments

Intersection of two 3D lines



$$\min_{t_{12},t_{34}} \left\| \mathbf{p}_1 + t_{12}(\mathbf{p}_2 - \mathbf{p}_1) - \mathbf{p}_3 - t_{34}(\mathbf{p}_4 - \mathbf{p}_3) \right\|^2$$

$$\left\| \begin{matrix} \Delta_{21}^{\mathsf{T}}\Delta_{21} & -\Delta_{21}^{\mathsf{T}}\Delta_{43} \\ -\Delta_{21}^{\mathsf{T}}\Delta_{43} & \Delta_{43}^{\mathsf{T}}\Delta_{43} \end{matrix} \right\| \begin{pmatrix} t_{21} \\ t_{43} \end{pmatrix} = \begin{pmatrix} -\Delta_{13}^{\mathsf{T}}\Delta_{21} \\ \Delta_{13}^{\mathsf{T}}\Delta_{43} \end{pmatrix}$$

where $\Delta_{ij} = \mathbf{p}_i - \mathbf{p}_j$

$$\begin{pmatrix} t_{21} \\ t_{43} \end{pmatrix} = - \left\| \begin{matrix} \Delta_{21}^{\mathsf{T}}\Delta_{21} & -\Delta_{21}^{\mathsf{T}}\Delta_{43} \\ -\Delta_{21}^{\mathsf{T}}\Delta_{43} & \Delta_{43}^{\mathsf{T}}\Delta_{43} \end{matrix} \right\|^{-1} \begin{pmatrix} \Delta_{13}^{\mathsf{T}}\Delta_{21} \\ -\Delta_{13}^{\mathsf{T}}\Delta_{43} \end{pmatrix}$$

Exercise: derive the explicit formulae for t_{12} and t_{34} for the closest intersection point.