DUE: Mon, Oct 18, 4:30pm, Andre's office.

1. Recall that a prefix-free machine W is called *universal* if for every prefix-free machine S there effectively exists a constant c such that for each input string x:

$$K_W(x) \leq K_S(x) + c.$$

   (a) Prove that for every universal prefix-free machines U and V there is a constant d such that for all strings x:
   $$\mid K_U(x) - K_V(x) \mid \leq d.$$
   [3 marks]

   (b) Prove that every universal prefix-free machine machine W is onto, i.e., for every y there exists x such that $W(x) = y$. [3 marks]

   (c) Is every universal prefix-free machine W *infinitely* onto? I.e., do there exist for every y infinitely many x such that $W(x) = y$? [4 marks]

   (d) Let U be a universal prefix-free machine such that $U(x) = y$. Is the restriction of U to $\text{dom}(U) \setminus \{x\}$ is still universal? [2 bonus marks]

2. Let α be a real in $(0, 1]$. Show that the following conditions are equivalent (such reals are called c.e.): [10 marks]

   (a) There is a computable, nondecreasing sequence of rationals which converges to α.

   (b) The set of rationals less than or equal to α is c.e.

   (c) There is an infinite prefix-free c.e. set $A \subseteq B^*$ with $\alpha = \Omega_A$.

3. A real is called computable if its binary representation is computable. Give an example of a c.e. real that is not random but also is not computable. [5 marks]

4. Recall that $C(x)$ is the plain Kolmogorov complexity of string x, where any machine is allowed (warning: this is denoted $K(x)$ in Sipser).

   (a) Explain why there are constants $d, d' \in \mathbb{N}$ such that $C(x) \leq |x| + d$ and $C(x) \leq K(x) + d'$ for each x. [5 marks]

   (b) Show that for each $b \in \mathbb{N}$, for each n, there are at least $2^n - 2^{n-b} + 1$ strings x of length n such that $C(x) \geq n - b$. [5 marks]

   (c) Show that there is a constant d such that $C(xy) \leq K(x) + C(y) + d$ for each strings x, y. [3 bonus marks]