

News

- Office Hours change:
 - From Thu 12 pm – 2 pm
 - To Thu 1 pm – 3 pm
- 15 mins

Parametric Curves

- Using matrices to write parametric curves

$$\mathbf{p}(t) = p_1 + t(p_2 - p_1) = (1 - t)p_1 + t p_2$$

$$\mathbf{p}(t) = \underbrace{\begin{pmatrix} t & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}_M \underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}_G$$

Check:

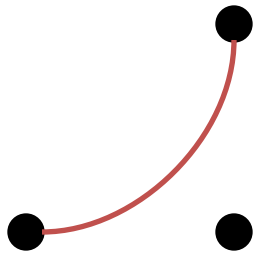
$$\mathbf{p}(t) = \overbrace{\left((-t + 1) \quad (t + 0) \right)}^{T * M} \underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}_G$$

Parametric Curves

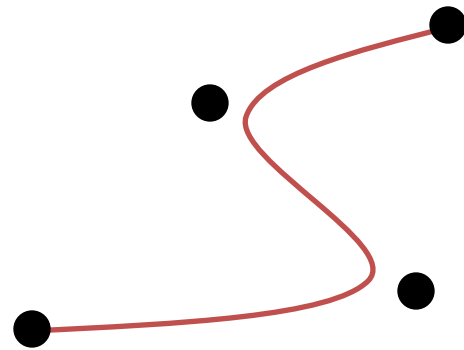
- **T = power basis**, contains the powers of the parameter t
- **M = basis matrix**, contains the coefficients that relate the basis to each control point
- **G = geometric constraint vector**, contains the control points and other constraints
 - Note, G would have 3 points if the curve have 3 control points, regardless of the points being 2D or 3D

General Form

$$\mathbf{p}(t) = (t^2 \quad t \quad 1)(\mathbf{M}: 3 * 3) \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$



$$\mathbf{p}(t) = (t^3 \quad t^2 \quad t \quad 1)(\mathbf{M}: 4 * 4) \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$



Continuity

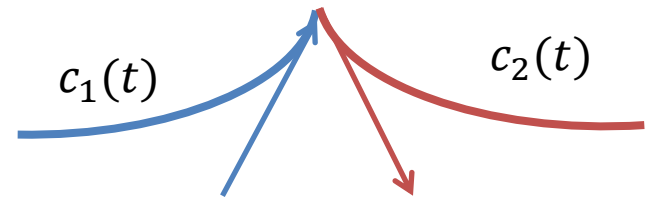
- No continuity



Change curve symbol to c to avoid confusion with point p_1 .

Continuity

- G^0 : 0th order geometric
 - End points match
 - $c_1(1) = c_2(0)$



Continuity

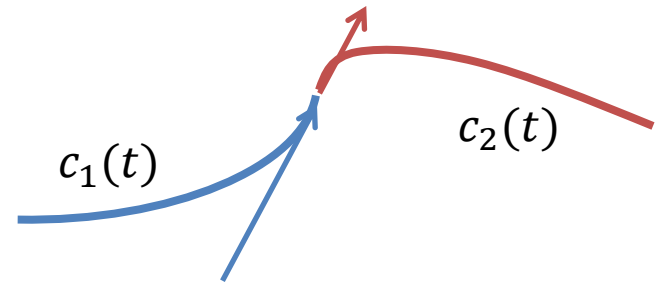
- G^1 : 1st order geometric

- End points match

- $c_1(1) = c_2(0)$

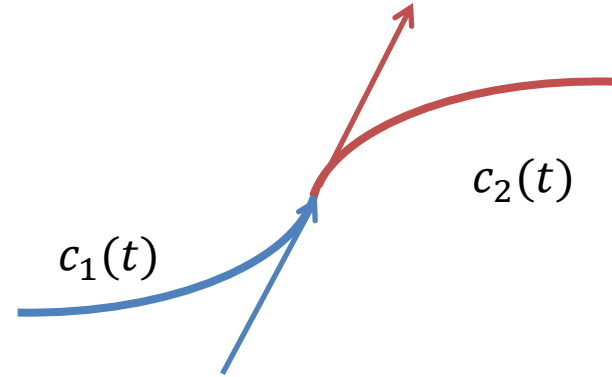
- Gradient in same direction (up to scale k)

- $\frac{\partial c_1}{\partial t}(1) = k * \frac{\partial c_2}{\partial t}(0)$



Continuity

- C^1 : 1st order parametric
 - End points match
 - $c_1(1) = c_2(0)$
 - Gradient exactly same
 - $\frac{\partial c_1}{\partial t}(1) = \frac{\partial c_2}{\partial t}(0)$



Continuity

- C^2 : 2nd order parametric
 - End points match
 - $c_1(1) = c_2(0)$
 - Gradient exactly same
 - $\frac{\partial c_1}{\partial t}(1) = \frac{\partial c_2}{\partial t}(0)$
 - Second derivative same
 - $\frac{\partial^2 c_1}{\partial t^2}(1) = \frac{\partial^2 c_2}{\partial t^2}(0)$

Example

We have two 2D curves defined by ($t \in [0, 1]$):

$$c_1(t) = \begin{pmatrix} -1 + t \\ (1 - t)^2 - 1 \end{pmatrix}$$

$$c_2(t) = \begin{pmatrix} t \\ t^2 - 1 \end{pmatrix}$$

Doing the maths:

$$c_1(1) = \begin{pmatrix} -1 + 1 \\ 0^2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$c_2(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

They are G^0
continuous.

Example

$$\frac{\partial c_1}{\partial t} = \begin{pmatrix} 1 \\ 2t - 2 \end{pmatrix} \Rightarrow \frac{\partial c_1}{\partial t}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{\partial c_2}{\partial t} = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \Rightarrow \frac{\partial c_2}{\partial t}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

They are G^1
and C^1
continuous.

Example

$$\frac{\partial^2 c_1}{\partial t^2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

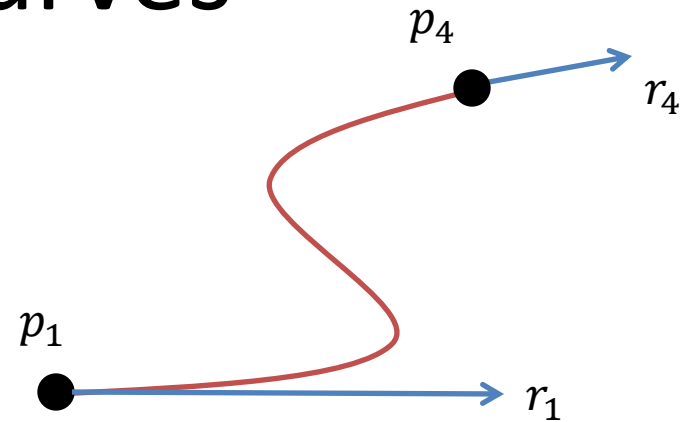
$$\frac{\partial^2 c_2}{\partial t^2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

They are C^2
continuous.

The curves are G^0 , G^1 , C^1 and C^2 continuous.
(Pssst! They are the same curve here...)

Hermite Curves

- Start/End
- 2 Tangents

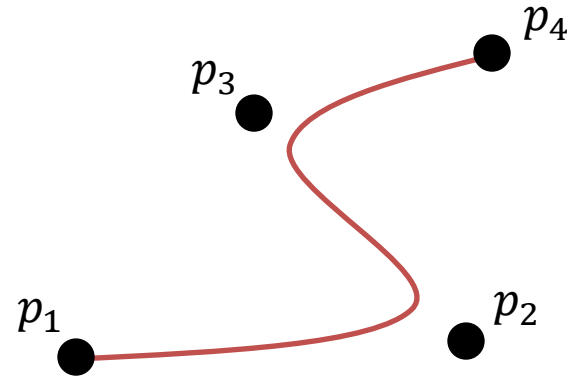


$\mathbf{p}(t)$

$$\begin{aligned}
 &= (t^3 \quad t^2 \quad t \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix} \\
 &= (2t^3 - 3t^2 + 1) p_1 + (-2t^3 + 3t^2) p_4 \\
 &\quad + (t^3 - 2t^2 + t) r_1 + (t^3 - t^2) r_4
 \end{aligned}$$

Bezier Curves

- Start/End
- 2 Control Points



$p(t)$

$$= (t^3 \quad t^2 \quad t \quad 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$
$$= (1-t)^3 p_1 + 3t(1-t)^2 p_2 + 3t^2(1-t) p_3 + t^3 p_4$$

Matrix M from Constraints

- Given constraints, we can find the matrix M.
- Example: Hermite curve constraints
 - Start point is p_1
 - $p_1 = \mathbf{p}(0)$
 - End point is p_4
 - $p_4 = \mathbf{p}(1)$
 - Start tangent is r_1
 - $r_1 = \mathbf{p}'(0)$
 - End tangent is r_4
 - $r_4 = \mathbf{p}'(1)$

Matrix M from Constraints

- Tangent of curve

$$\begin{aligned}\mathbf{p}'(t) &= \frac{d(\mathbf{T} \mathbf{M} \mathbf{G})}{dt} \\ &= \frac{d}{dt} (t^3 \quad t^2 \quad t \quad 1) \mathbf{M} \mathbf{G} \\ &= (3t^2 \quad 2t \quad 1 \quad 0) \mathbf{M} \mathbf{G}\end{aligned}$$

Matrix M from Constraints

- Constraint 1: $p_1 = \mathbf{p}(0)$

$$p_1 = \mathbf{TMG} \quad \text{with } t = 0$$

$$p_1 = (t^3 \quad t^2 \quad t \quad 1) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

$$= (0 \quad 0 \quad 0 \quad 1) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

Matrix M from Constraints

- Constraint 2: $p_4 = \mathbf{p}(1)$

$$p_4 = \mathbf{TMG} \quad \text{with } t = 1$$

$$p_4 = (t^3 \quad t^2 \quad t \quad 1) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

$$= (1 \quad 1 \quad 1 \quad 1) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

Matrix M from Constraints

- Constraint 3: $r_1 = \mathbf{p}'(0)$

$$r_1 = \mathbf{T}'\mathbf{M}\mathbf{G} \quad \text{with } t = 0$$

$$r_1 = (3t^2 \quad 2t \quad 1 \quad 0) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

$$= (0 \quad 0 \quad 1 \quad 0) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

Matrix M from Constraints

- Constraint 4: $r_4 = \mathbf{p}'(1)$

$$r_4 = \mathbf{T}'\mathbf{M}\mathbf{G} \quad \text{with } t = 1$$

$$r_4 = (3t^2 \quad 2t \quad 1 \quad 0) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

$$= (3 \quad 2 \quad 1 \quad 0) \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

Matrix M from Constraints

- Write all 4 constraints as rows of matrix:

$$\begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \mathbf{M} \begin{pmatrix} p_1 \\ p_4 \\ r_1 \\ r_4 \end{pmatrix}$$

- We know from the rules of matrices that $\mathbf{a} = \mathbf{I} \mathbf{a}$ and $\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$.
- Therefore, our matrix here must be \mathbf{M}^{-1} .

Matrix M from Constraints

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hermite to Bezier

- Transform Hermite curve to Bezier curve

$$\begin{pmatrix} p_{H1} \\ p_{H4} \\ r_{H1} \\ r_{H4} \end{pmatrix} \rightarrow \begin{pmatrix} p_{B1} \\ p_{B2} \\ p_{B3} \\ p_{B4} \end{pmatrix}$$

- Constraints:

- $p_{H1} = p_{B1}$
- $p_{H4} = p_{B4}$
- $r_{H1} = 3(p_{B2} - p_{B1})$ or $p_{B2} = p_{B1} + r_{H1}/3$
- $r_{H4} = 3(p_{B4} - p_{B3})$ or $p_{B3} = p_{B4} - r_{H4}/3$

Hermite to Bezier

$$\underbrace{\begin{pmatrix} p_{H1} \\ p_{H4} \\ r_{H1} \\ r_{H4} \end{pmatrix}}_{\mathbf{G}_H} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix}}_{\mathbf{M}_{HB}} \underbrace{\begin{pmatrix} p_{B1} \\ p_{B2} \\ p_{B3} \\ p_{B4} \end{pmatrix}}_{\mathbf{G}_B}$$

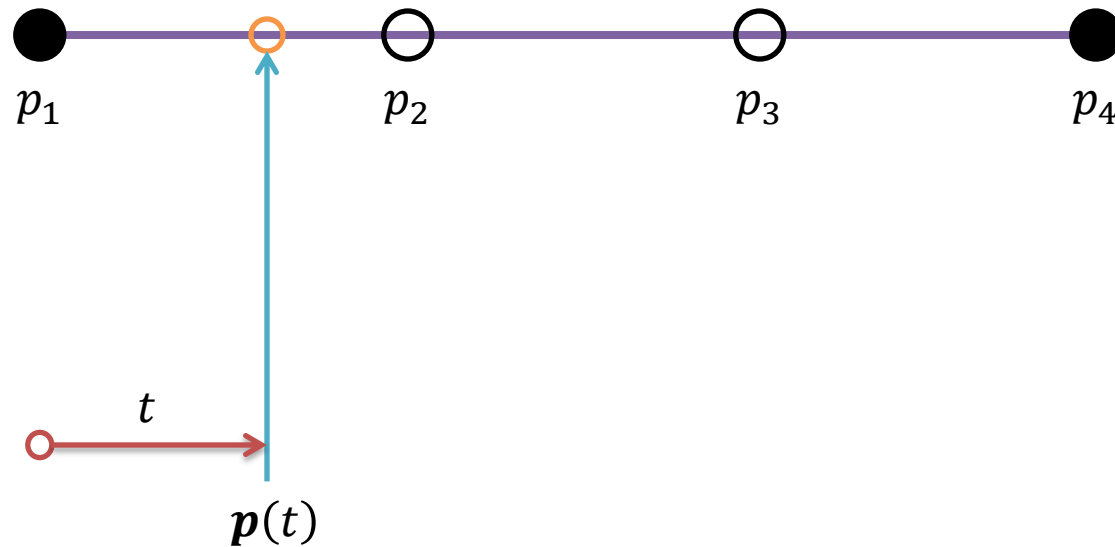
$$\begin{aligned} \mathbf{p} &= \mathbf{T} \mathbf{M}_H \mathbf{G}_H \\ &= \mathbf{T} \mathbf{M}_H \mathbf{M}_{HB} \mathbf{G}_B \\ &= \mathbf{T} \mathbf{M}_B \mathbf{G}_B \end{aligned}$$

Bezier Surface Patches

- Use one variable as the control points of the other.
- Review 1 variable:

$$\mathbf{p}(t) = (t^3 \quad t^2 \quad t \quad 1) \mathbf{M}_B \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

Bezier Surface Patches



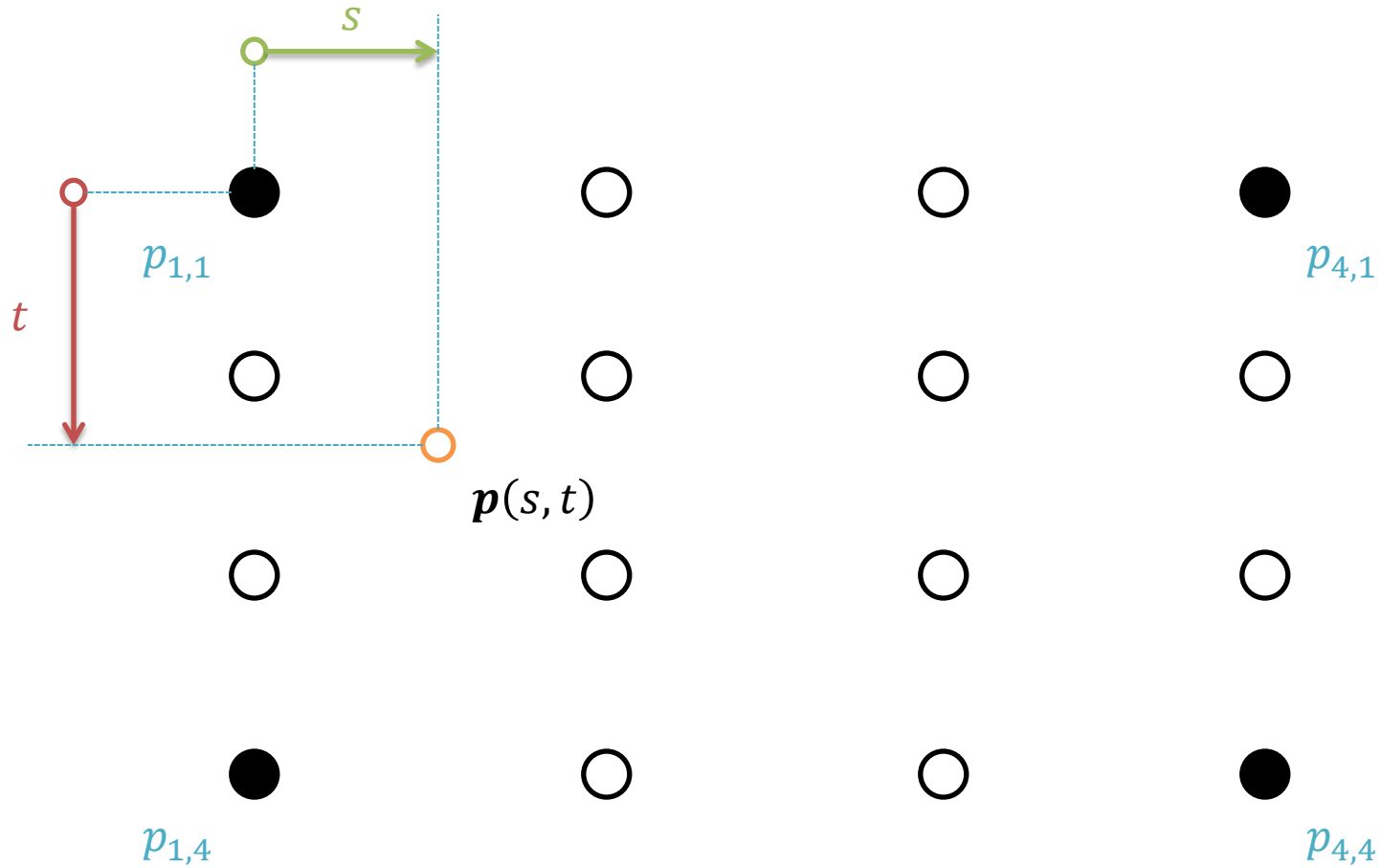
Bezier Surface Patches

- 2 variables:

$$\mathbf{p}(s, t) = (s^3 \quad s^2 \quad s \quad 1) \mathbf{M}_B \begin{pmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \\ \mathbf{p}_3(t) \\ \mathbf{p}_4(t) \end{pmatrix}$$

$$\mathbf{p}_{1..4}(t) = (t^3 \quad t^2 \quad t \quad 1) \mathbf{M}_B \begin{pmatrix} p_{1..4,1} \\ p_{1..4,2} \\ p_{1..4,3} \\ p_{1..4,4} \end{pmatrix}$$

Bezier Surface Patches



Bezier Surface Patches

- 1 variable expanded form:

$$\begin{aligned} \mathbf{p}(x) &= \underbrace{(1-x)^3}_{B_1(x)} p_1 + \underbrace{3x(1-x)^2}_{B_2(x)} p_2 + \underbrace{3x^2(1-x)}_{B_3(x)} p_3 + \underbrace{x^3}_{B_4(x)} p_4 \\ &= \sum_{i=1}^4 B_i(x) p_i \end{aligned}$$

$B_i(x)$ are the (cubic)
Bernstein polynomials

Bezier Surface Patches

- Writing 2 variables in expanded form:

$$\mathbf{p}(s, t) = (1-s)^3 \mathbf{p}_1(t) + 3s(1-s)^2 \mathbf{p}_2(t) + 3s^2(1-s) \mathbf{p}_3(t) + s^3 \mathbf{p}_4(t)$$

$$= \sum_{i=1}^4 B_i(s) \mathbf{p}_i(t)$$

$$\mathbf{p}_i(t) = \sum_{j=1}^4 B_j(t) \mathbf{p}_{i,j}$$

$$\Rightarrow \mathbf{p}(s, t) = \sum_{i=1}^4 B_i(s) \sum_{j=1}^4 B_j(t) \mathbf{p}_{i,j}$$

$$= \sum_{i=1}^4 \sum_{j=1}^4 B_i(s) B_j(t) \mathbf{p}_{i,j}$$

Bezier Surface Patches

- Normals
 - Same idea as parametric surfaces
 - Do the partial derivatives per variable
 - Take cross product

$$\frac{\partial \mathbf{p}(s, t)}{\partial t} = \sum_{i=1}^4 \sum_{j=1}^4 B_i(s) \frac{\partial B_j(t)}{\partial t} p_{i,j}$$

$$\frac{\partial \mathbf{p}(s, t)}{\partial s} = \sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial B_i(s)}{\partial s} B_j(t) p_{i,j}$$

Bezier Surface Patches

- E.g.

$$\begin{aligned} B_2(x) &= 3x(1-x)^2 \\ &= (3x)(x^2 - 2x + 1) \\ &= 3x^3 - 6x^2 + 3x \end{aligned}$$

$$\frac{\partial B_2(x)}{\partial x} = 9x^2 - 12x + 3$$

Bezier Surface Patches

- Code suggestion
- Define the following functions:
 - float Bernstein(int i, float x) // i is the order: 1 -> 4
 - float dBernstein(int i, float x) // for derivatives