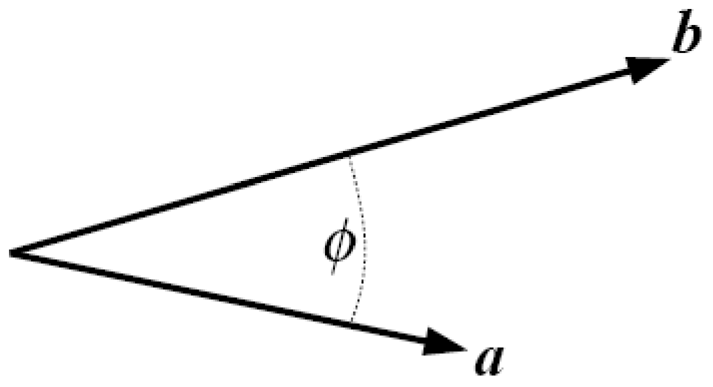
CompSci 373 Tutorial

Geometry Part 2

Vector: Dot Product

$$a \bullet b = a^T b$$
$$= a_1 b_1 + a_2 b_2 + (a_3 b_3)$$
$$= |a||b| \cos\phi$$



- Dot product of **A** and **B** is 0, if **A** and **B** are perpendicular (at 90 degree to each other)
- Dot product of **A** and **B** is positive, if the angle are less than 90
- Dot product of **A** and **B** is negative, if the angle are greater than 90

Dot Product Example

- Given are the two vectors

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

What is the angle between the two vectors?

$$a \bullet b = |a||b| \cos\phi$$

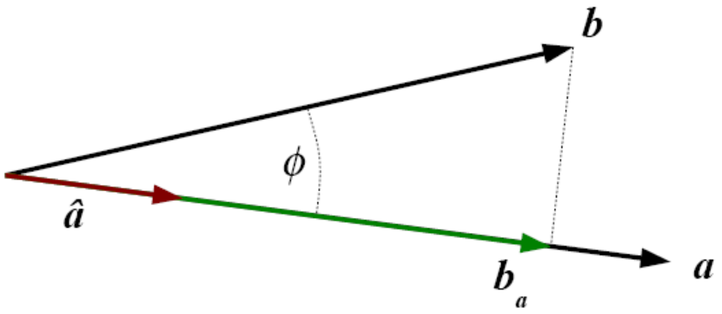
$$\cos\phi = \frac{a \bullet b}{|a||b|}$$

$$\phi = \arccos\frac{a \bullet b}{|a||b|} = \arccos\frac{1\times 2 + 2\times 1 + 3\times 2}{3\times\sqrt{14}} = \arccos\frac{10}{11.225}$$

$$= \arccos 0.891 = 27.02^{\circ}$$

Projection

$$\cos\phi = \frac{|b_a|}{|b|} \Rightarrow |b_a| = |b| \cdot \cos\phi$$



$$b_a = \hat{a}|b_a| = \frac{a}{|a|}|b_a|$$

$$= \frac{a}{|a|}|b|\cos\phi = \frac{a}{|a|}|b|\frac{a \bullet b}{|a||b|} = \frac{a \bullet b}{|a||a|}a$$

$$= \frac{a \bullet b}{a \bullet a}a$$

Projection

- Given are two vectors ***a*** and ***b***.

$$a = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Calculate the projection b_a of b onto a.

$$\boldsymbol{b}_a = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} \boldsymbol{a} = \frac{(1 \cdot 3 + 2 \cdot 4 + 2 \cdot 5)}{(1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2)} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{21}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{7}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

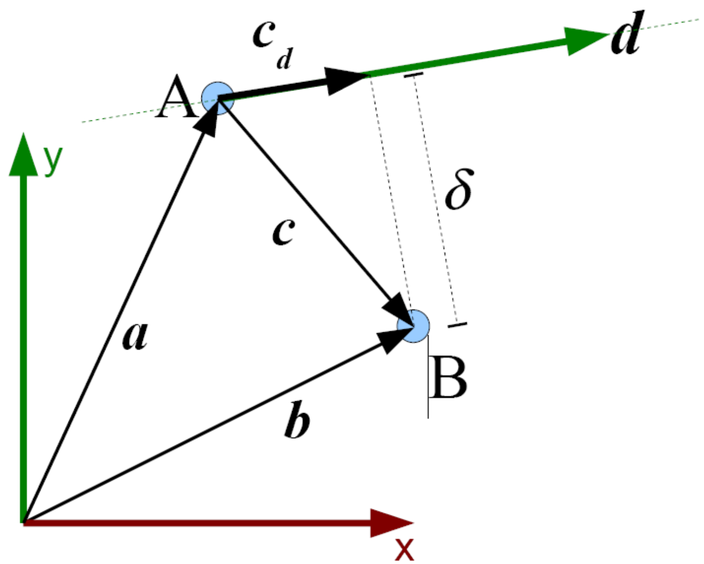
Distance of a Point to a Line

- Given is a line through point A in the direction d. What is the distance from Point B to that line?

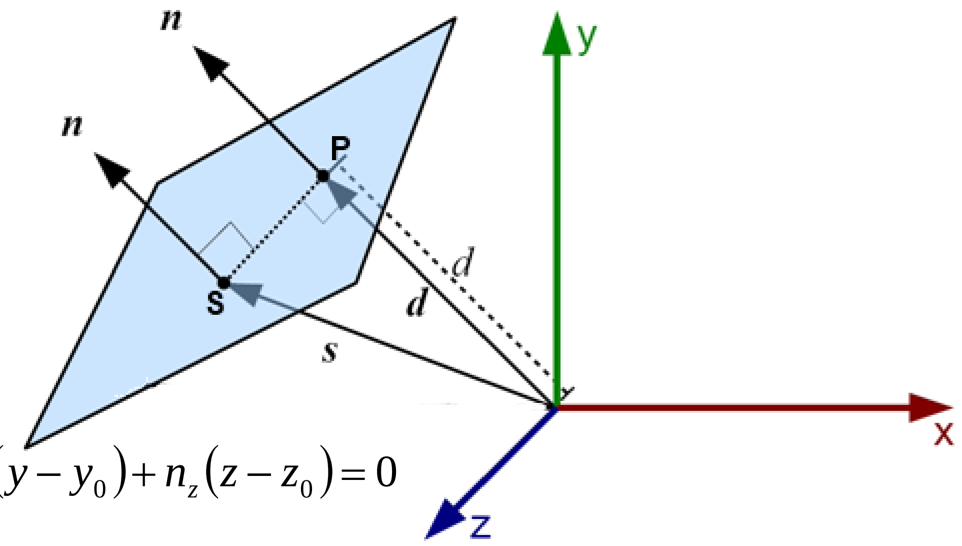
$$c_d = \frac{d \bullet c}{|d||d|} d$$

$$\delta = |c - c_d|$$

$$= \left| c - \frac{d \bullet c}{|d||d|} d \right| = \left| b - a - \frac{d \bullet (b - a)}{|d||d|} d \right|$$



Plane



$$n \bullet (P - S) = 0$$

$$n \bullet \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right) = 0$$

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0 \Rightarrow n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

$$n_x x + n_y y + n_z z = n_x x_0 + n_y y_0 + n_z z_0$$

- This is called scalar equation of a plane, and is often written as

$$n_x x + n_y y + n_z z = d$$

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d \Leftrightarrow n \bullet P = d$$

- Where d is the distance of plane to origin

Plane: Example

- How far is the plane 3x + y - 2z = 5 from the origin?
  - Convert to vector form

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$$

Distance from the origin?

  - The normal vector **must** be in normalised form

$$\frac{1}{\sqrt{3^2+1^2+(-2)^2}} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{5}{\sqrt{3^2+1^2+(-2)^2}}$$

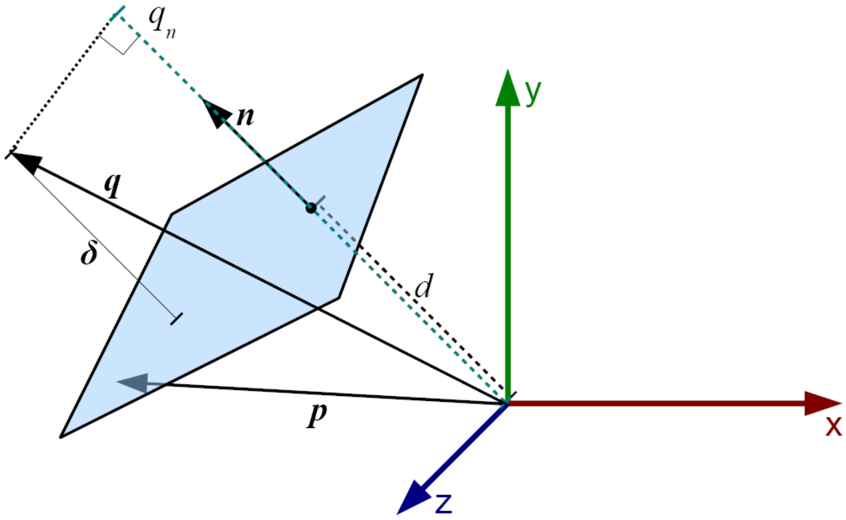
$$\frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{5}{\sqrt{14}}$$

Distance of Point to Plane

$$n \bullet P = d$$

$$n \bullet q = q_n$$

$$\Rightarrow \delta = n \bullet q - d$$



- Meaning δ

$\delta = 0:$ q is on plane $\delta > 0:$ "outside" $\delta < 0:$ "inside"

Example

- How far is point Q=(3,4,2) from the plane 3x + y - 2z = 5?

$$\delta = n \bullet Q - d$$

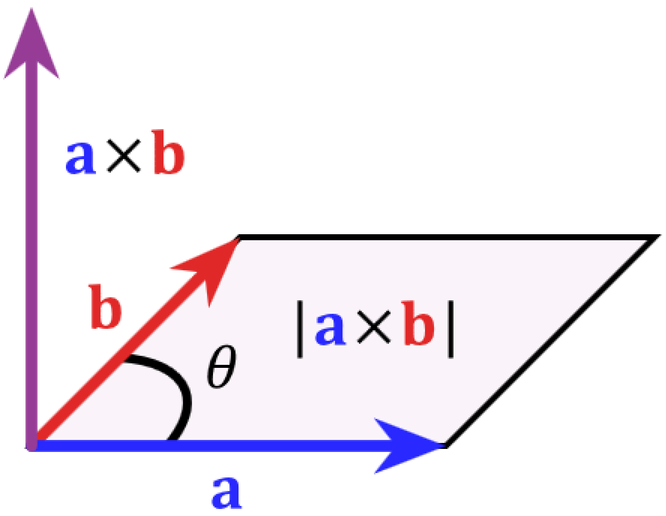
$$\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \frac{5}{\sqrt{14}} = \frac{1}{\sqrt{14}}\left(\left(3\times 3 + 1\times 4 + \left(-2\right)\times 2\right) - 5\right) = \frac{4}{\sqrt{14}}$$

- Is the point outside? **Yes**

- Cross/Vector Product

$$c = a \times b$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ a_1 b_3 - b_1 a_3 \\ a_1 b_2 - b_1 a_2 \end{pmatrix}$$



- c is perpendicular to **a** and **b**

Cross Product: Example

- Given are the two vectors
  - What are the cross product of these 2 vectors?

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

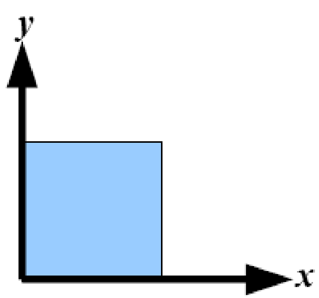
$$a \times b = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ a_1 b_3 - b_1 a_3 \\ a_1 b_2 - b_1 a_2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 - 1 \cdot 3 \\ 3 \cdot 2 - 1 \cdot 2 \\ 1 \cdot 1 - 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

What is the area A of the triangle define by these 2 vectors?

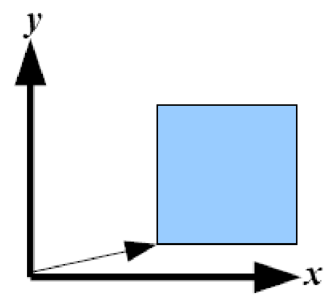
$$A = \frac{1}{2}\left|a \times b\right| = \frac{1}{2}\left|\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}\right| = \frac{1}{2}\sqrt{1^2 + 4^2 + (-3)^2} = 2.5495$$

Affine Transformations

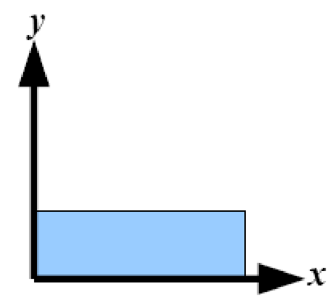
- Preserve parallel lines
- Preserve ratios



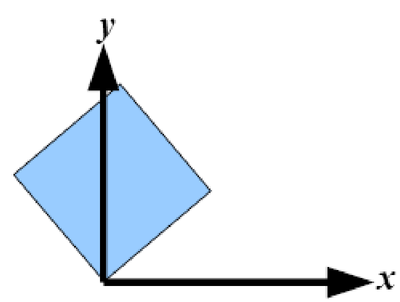
- Translation



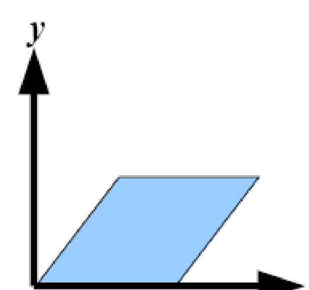
- Scaling



- Rotation



- Shearing



Homogeneous Coordinates

- Convert a vector to homogeneous coordinate system

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} wa_x \\ wa_y \\ wa_z \\ w \end{pmatrix}$$

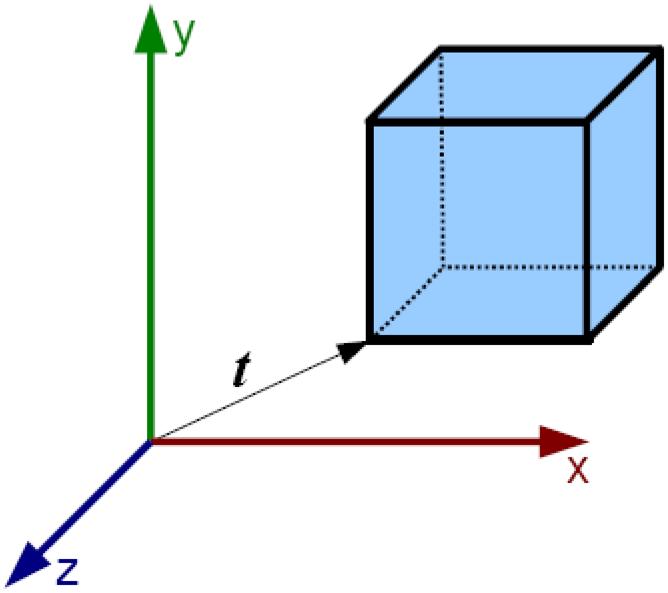
- Combine **M** and ***t***

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad t = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} \rightarrow \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$M_{Translation} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

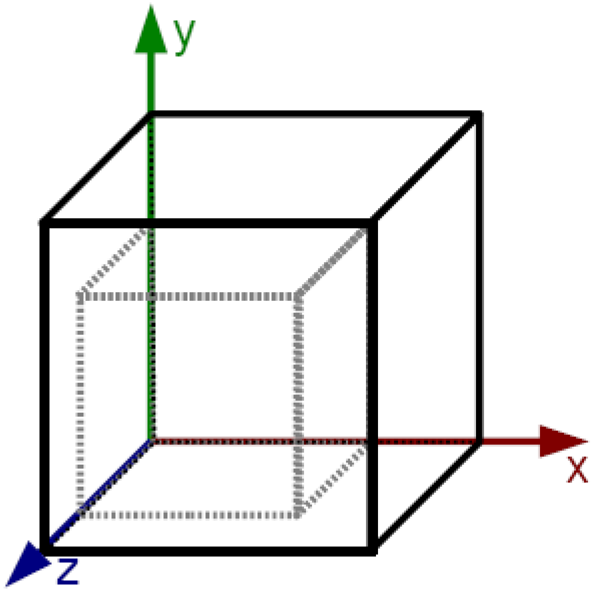
$$M^{-1}_{Translation} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Scaling

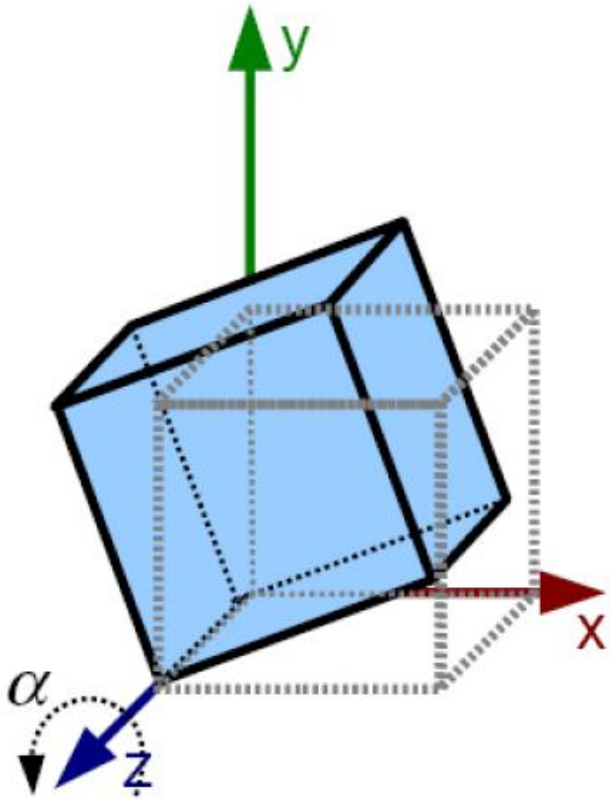
$$M_{Scaling} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^{-1}_{Scaling} = \begin{bmatrix} \frac{1}{S_x} & 0 & 0 & 0 \\ 0 & \frac{1}{S_y} & 0 & 0 \\ 0 & 0 & \frac{1}{S_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Rotation around Z

$$M_{RotateZ} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

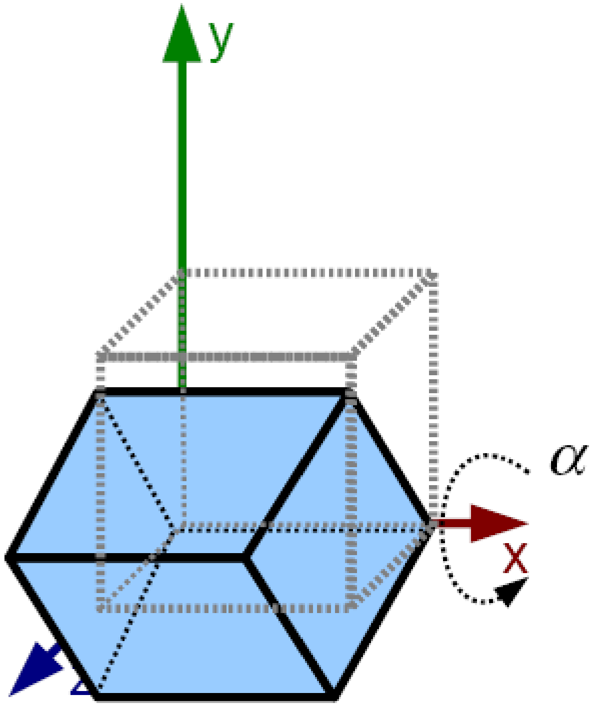


$$M^{-1}_{RotateZ} = M^{T}_{RotateZ} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations <u>DO NOT COMMUTE</u>

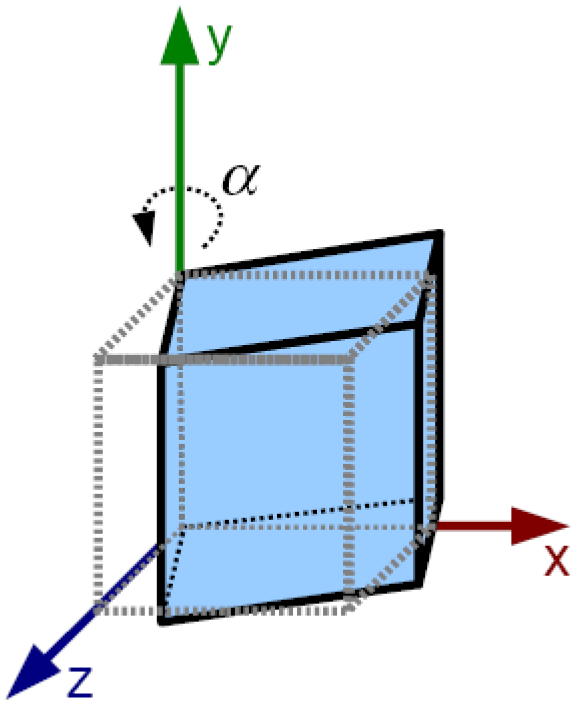
Rotation around X

$$M_{RotateX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



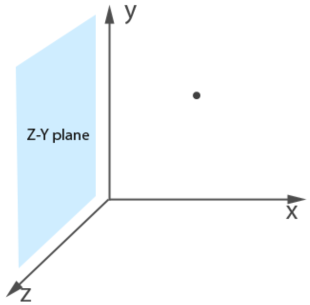
Rotation around Y

$$M_{RotateY} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Shearing

In (y, z) with respect to x value $Shear_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_y & 1 & 0 & 0 \\ sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

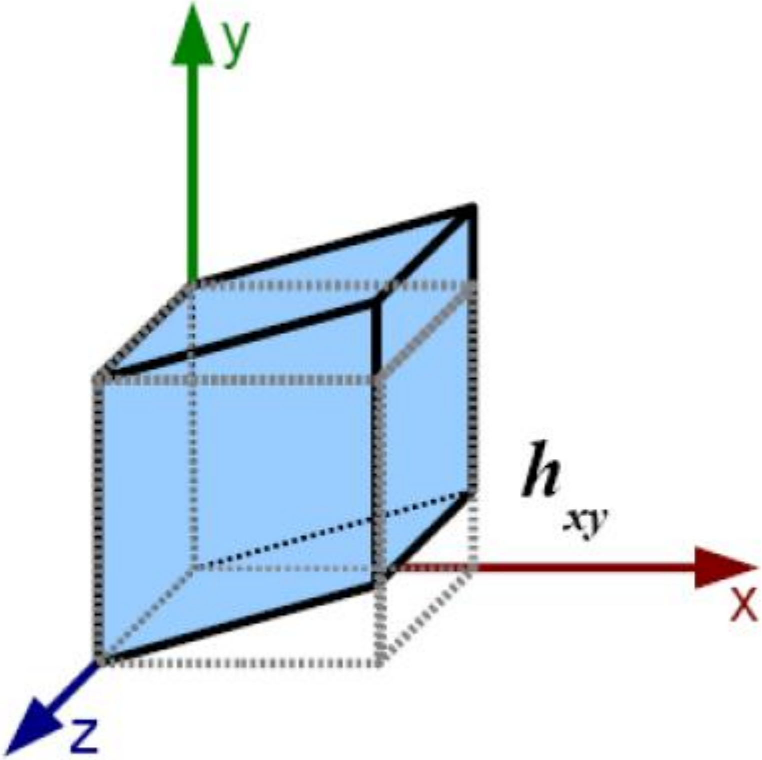


In (z, x) with respect to y value $Shear_{xz} = \begin{bmatrix} 1 & sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In (x, y) with respect to z value $Shear_{xy} = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Shearing

$$M_{Shearing} = \begin{bmatrix} 1 & h_{yx} & h_{zx} & 0 \\ h_{xy} & 1 & h_{zy} & 0 \\ h_{xz} & h_{yz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transformation

- What type are the following matrices?

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translate

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale

$$M_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate around Y

$$M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear

Transformation

- What type are the following matrices?

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translate & Scale

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale (Reflection Z)

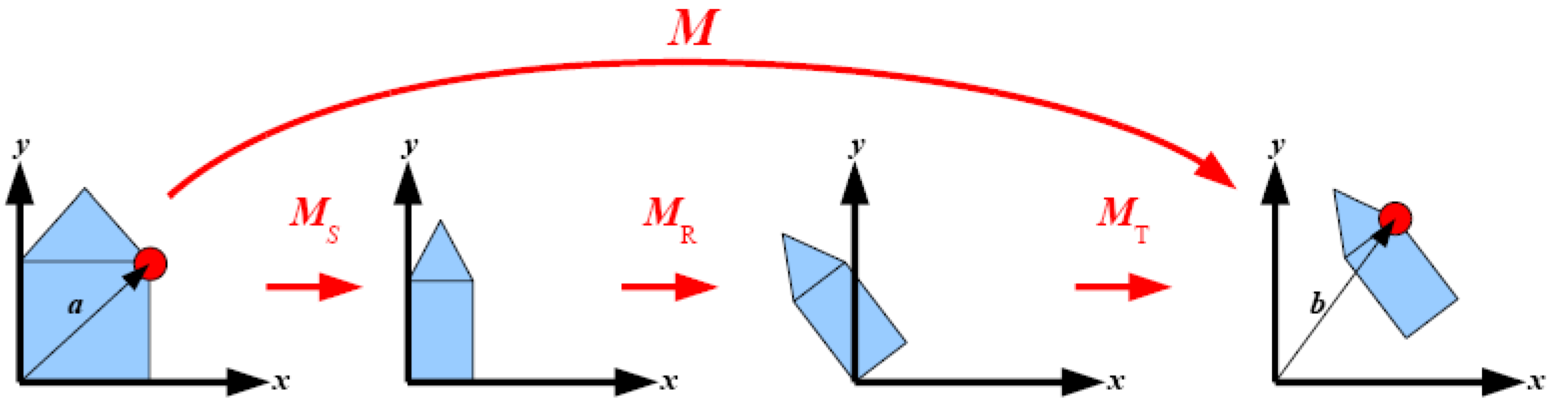
$$M_3 = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation & Rotate Z

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$$

???

Concatenation

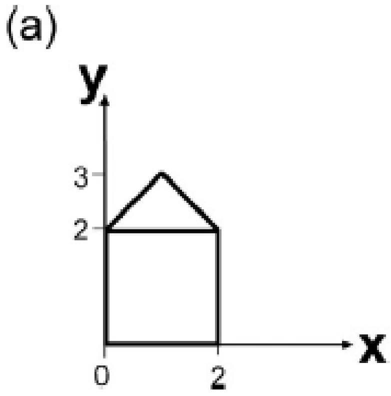


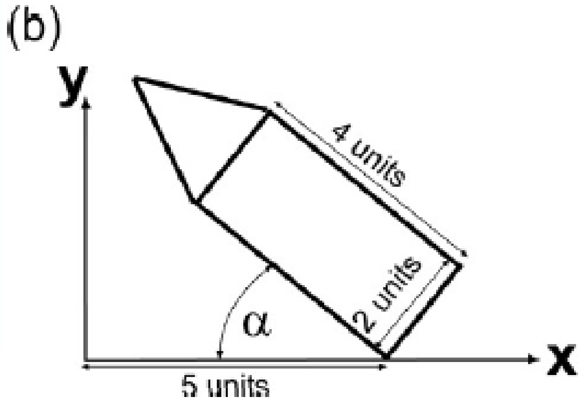
$$M = M_T M_R M_S$$

$$b = Ma = M_T \left(M_R \left(M_S a \right) \right)$$

Concatenation

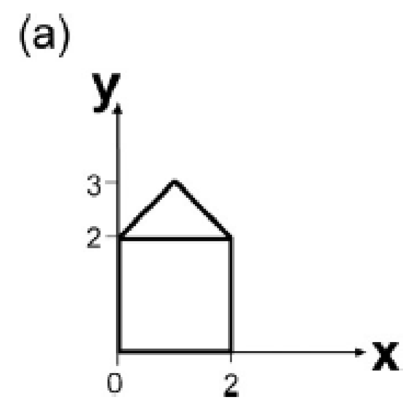
- Which matrix **M** transforms (a) to (b)?

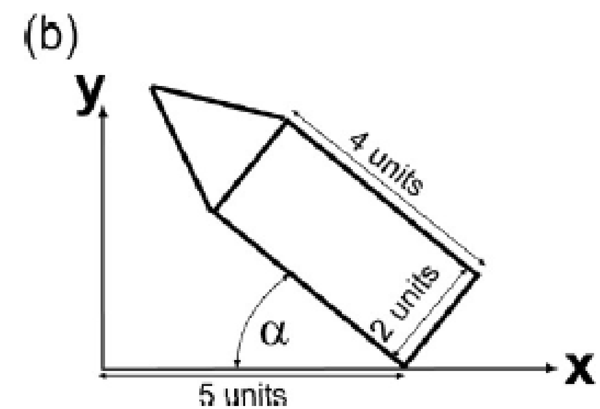




Concatenation

- Which matrix **M** transforms (a) to (b)?
  - What operations are involved?
    - Scaling 2 Y-axis
    - Translation +5 X-axis
    - Rotation $90^{\circ} - \alpha$
  - In which order?
    1. Scaling
    2. Rotation
    3. Translation





$$M = M_T M_R M_S$$

Concatenation

- Set up the matrices:
  - Scaling 2 Y-axis

$$M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Rotation $90^\circ - \alpha$

$$M_R = \begin{bmatrix} \cos(90-\alpha) & -\sin(90-\alpha) & 0 \\ \sin(90-\alpha) & \cos(90-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Translation +5 X-axis

$$M_T = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

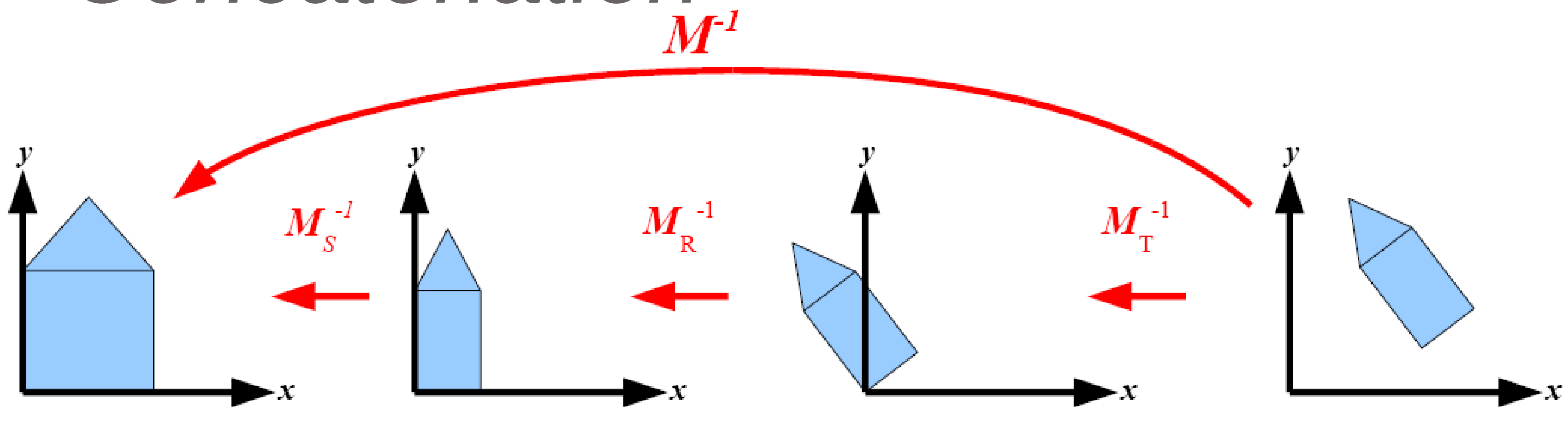
Concatenation

- Put everything together

$$M = M_T M_R M_S$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90-\alpha) & -\sin(90-\alpha) & 0 \\ \sin(90-\alpha) & \cos(90-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenation



$$M = M_T M_R M_S$$

$$I = M_S^{-1} M_R^{-1} M_T^{-1} M_T M_R M_S$$

$$M^{-1} \qquad\qquad M$$

$$M^{-1} = M_s^{-1} M_R^{-1} M_T^{-1}$$

Concatenation

- Let a 3x3 matrix P transform a 2D point M with Cartesian coordinates (x, y) into the 2D point M' by scaling S with scaling factor (s_x, s_y) along the coordinate axes, translation T with Cartesian coordinate (t_x, t_y), and rotation R by angle θ
  - What is the matrix P?

Concatenation

Let a 3x3 matrix P transform a 2D point M with Cartesian coordinates (x, y) into the 2D point M' by scaling S with scaling factor (s_x, s_y) along the coordinate axes, translation T with Cartesian coordinate (t_x, t_y), and rotation R by angle θ

- What is the matrix P?

$$P = RTS$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If $\theta = 45^\circ$, $t_x = 1$, $t_y = 2$, $s_x = 1$ and $s_y = 2$, what is the, matrix P?

Concatenation

Let a 3x3 matrix P transform a 2D point M with Cartesian coordinates (x, y) into the 2D point M' by scaling S with scaling factor (s_x, s_y) along the coordinate axes, translation T with Cartesian coordinate (t_x, t_y), and rotation R by angle θ

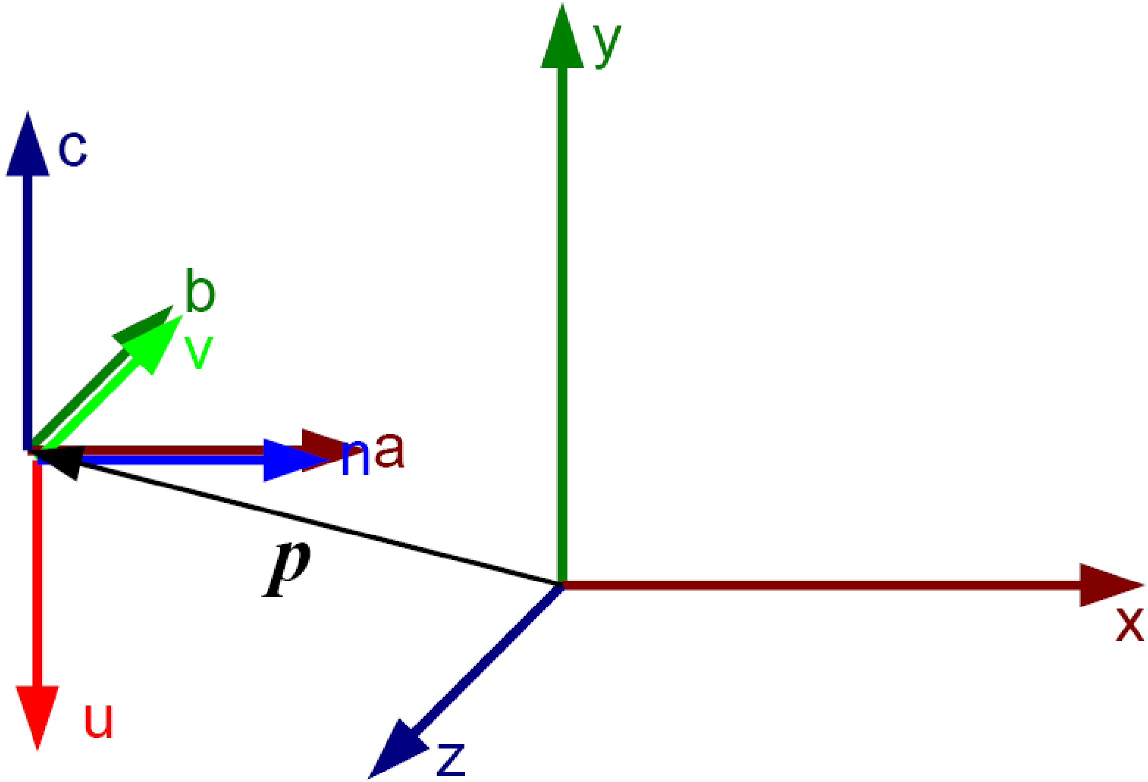
- What is the inverse of matrix P

$$P^{-1} = S^{-1}T^{-1}R^{-1}\cancel{RTS}$$

$$= \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenation

- An object has a local coordinate system a=(1,0,0), b=(0,0,-1), c=(0,1,0) at position p=(-10,2,5). Which homogeneous matrix rotates the object into the new coordinate system u=(0,-1,0), v=(0,0,-1) n=(1,0,0)?



Reminder

- To rotate the coordinate system to align with a new coordinate system (a, b, c) **with the same origin**

$$R_{abc} = \begin{bmatrix} a_x & b_x & c_x & 0 \\ a_y & b_y & c_y & 0 \\ a_z & b_z & c_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If we are actually in a new coordinate system (a, b, c), and we want to go back the original origin, use R_{abc}^{-1} to rotate back

Concatenation

- Steps
  - Translate abc to the origin
    - = Translation matrix with –p
  - Align abc with world coordinate axes
    - Inverse rotation matrix
  - Rotate to align with uvn
    - Use uvn to construct rotation matrix
  - Translate back
    - = Translation matrix with p

$$T_{-p} = \begin{bmatrix} 1 & 0 & 0 & -p_x \\ 0 & 1 & 0 & -p_y \\ 0 & 0 & 1 & -p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{abc}^{-1} = \begin{bmatrix} a_x & a_y & a_z & 0 \\ b_x & b_y & b_z & 0 \\ c_x & c_y & c_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{uvn} = \begin{bmatrix} u_x & v_x & n_x & 0 \\ u_y & v_y & n_y & 0 \\ u_z & v_z & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_p = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation

- Put it all together

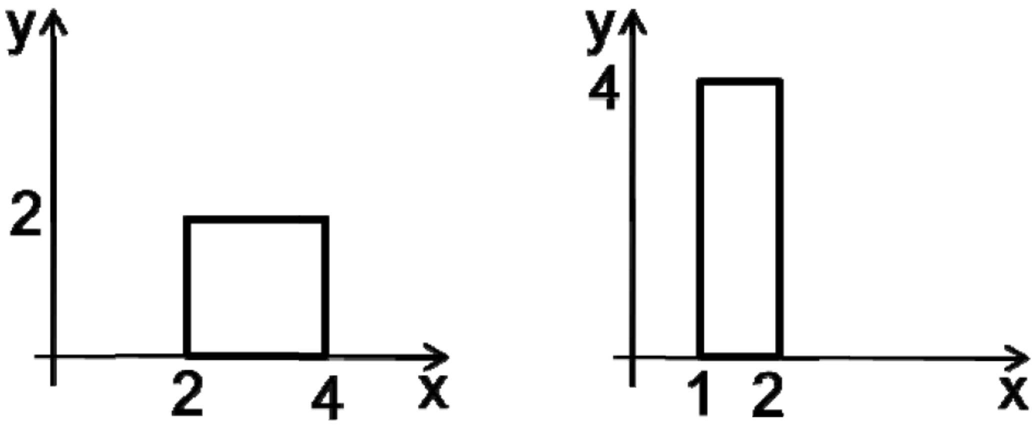
$$M = M_p M_{uvn} M_{abc}^{-1} M_p^{-1}$$

Check Point

- Projection of a vector onto another

$$a = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

- How far is the point P = (4, -4, 3) from the plane 2x –2y + 5z + 8 = 0
- Which homogenous 2D matrix transformation from figure in the left to the one on the right?



Check Point

- Projection of a vector onto another

$$a = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$b_a = \frac{b \cdot a}{a \cdot a} a = \frac{8}{16} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Check Point

- How far is the point P = (4, -4, 3) from the plane 2x –2y + 5z + 8 = 0
  - Rewrite the plane equation into vector form

$$\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8$$

$$\Rightarrow \frac{1}{\sqrt{4+4+25}} \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-8}{\sqrt{4+4+25}}$$

$$\Rightarrow \frac{1}{\sqrt{33}} \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-8}{\sqrt{33}}$$

Check Point

- How far is the point P = (4, -4, 3) from the plane 2x –2y + 5z + 8 = 0
  - Compute the distance

$$\frac{1}{\sqrt{33}}\begin{pmatrix}2\\-2\\5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{-8}{\sqrt{33}}$$

$$\Leftrightarrow\frac{1}{\sqrt{33}}\begin{pmatrix}2\\-2\\5\end{pmatrix}\begin{pmatrix}4\\-4\\3\end{pmatrix}=\frac{-8}{\sqrt{33}}$$

$$\Leftrightarrow\frac{1}{\sqrt{33}}(8+8+15)+\frac{8}{\sqrt{33}}=\frac{31}{\sqrt{33}}+\frac{8}{\sqrt{33}}\approx 6.8$$

Check Point

- Which homogenous 2D matrix transformation from figure in the left to the one on the right?

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$