CompSci 373 Tutorial

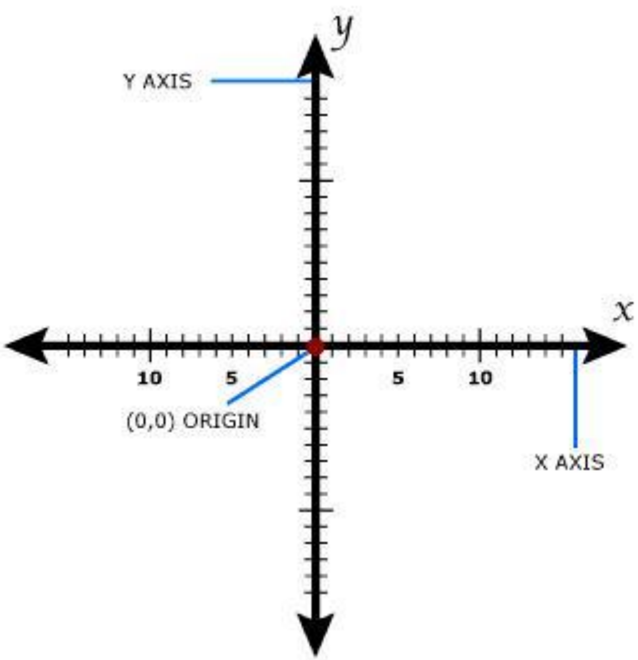
Geometry Part 1

- **Time and Location**
  - Monday, 1:00–2:00pm. Location: Engineering 3.402
  - Monday, 2:00– 3:00pm. Location: Engineering 3.403

- **Website:** http://www.cs.auckland.ac.nz/courses/compsci373s1c/

- Input:
  - Justin: hngu039@aucklanduni.ac.nz
    - Office hours: Tuesday 09:00-11:00pm, Thursday 11:00-01:00pm
    - City campus, Building 303S, Room 496
  - Minh: mngu012@aucklanduni.ac.nz
    - Office hours: Monday 9am - 12pm
    - City Campus, Building 303S, Room 396

Cartesian Coordinate Systems



- The **x axis** runs left to right with larger values to the right.
- The y **axis** runs up and down with larger values above.
- Each point in the plane can be refered to by its x and y coordinate pair (x, y)
- Note that: the coordinates on the computer screen have the origin at the top left corner and the positive y value point downward.

Points and Vectors

- A point defines a position in space.
- A point is defined using a pair (x, y) of coordinate values and (x, y, z) in 3D.
- Example:

2D point: P(1, 2)

3D point: P (1, 2, 3)

- A vector describes both a magnitude and a direction

in 2D

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

in 3D

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Vector Operations

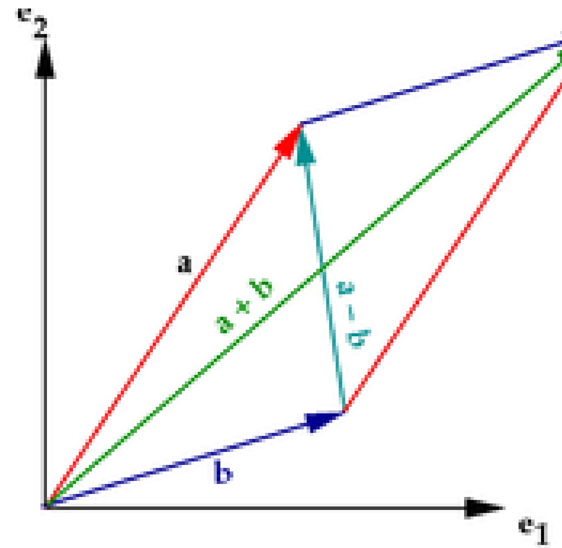
- **Addition/Subtraction**

$$a \pm b = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$$

- Example:

$$u = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$u + v = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$



Vector Operations

- **Scalar Production**

$$sa = \begin{pmatrix} sa_1 \\ sa_2 \end{pmatrix}$$

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- **Magnitude**

$$|a| = \left| \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right| = \sqrt{a_1^2 + a_2^2}$$

Vector Operations

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- **Normal Vector**

$$\bar{a} = \frac{a}{|a|}$$

Matrices

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Matrix Dimension:** $m \times n$
  - $m \Rightarrow$ number of **ROWs**
  - $n \Rightarrow$ number of **COLUMNs**

Matrix Operations

- **Addition/Subtraction**
  - Can only obtained if their size are the same

$$M \pm N = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \pm \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

$$M \pm N = \begin{bmatrix} m_{11} \pm n_{11} & m_{12} \pm n_{12} \\ m_{21} \pm n_{21} & m_{22} \pm n_{22} \end{bmatrix}$$

$$M + N = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & -1 & -1 \end{bmatrix}$$

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$$M + N = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 3+1 & 5+2 \\ 2+4 & (-3)+(-1) & (-1)+(-1) \end{bmatrix}$$

Matrix Operations

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  - Can only obtained if their size are the same

$$M \pm N = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \pm \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

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$$M + N = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 3+1 & 5+2 \\ 2+4 & (-3)+(-1) & (-1)+(-1) \end{bmatrix} = \begin{bmatrix} 7 & 4 & 7 \\ 6 & -4 & -2 \end{bmatrix}$$

Matrix Operations

- **Scalar Production**
  - Can only obtained if their size are the same

$$sM = s\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} sm_{11} & sm_{12} \\ sm_{21} & sm_{22} \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} \Rightarrow 2\begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix}$$

Matrix Operations

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  - Can only obtained if their size are the same

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$$M = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} \Rightarrow 2\begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 4 & 2\times 3 & 2\times 5 \\ 2\times 2 & 2\times(-3) & 2\times(-1) \end{bmatrix}$$

Matrix Operations

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  - Can only obtained if their size are the same

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$$M = \begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix} \Rightarrow 2\begin{bmatrix} 4 & 3 & 5 \\ 2 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times4 & 2\times3 & 2\times5 \\ 2\times2 & 2\times(-3) & 2\times(-1) \end{bmatrix} = \begin{bmatrix} 8 & 6 & 10 \\ 4 & -6 & -2 \end{bmatrix}$$

Matrix Multiplication

- A is a matrix size $m_A \times n_A$ and B with size $m_B \times n_B$

We can computer the product of A and B only if $n_A = m_B$

The resulting matrix C= AB then has the size of $m_A \times n_B$

$$\boldsymbol{N} = \begin{pmatrix} g & h \\ j & k \\ l & m \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad \begin{pmatrix} \oplus & \oplus \\ \oplus & \oplus \end{pmatrix} = \boldsymbol{MN}$$

$$\boldsymbol{MN} = \begin{pmatrix} ag + bj + cl & ah + bk + cm \\ dg + ej + fl & dh + ek + fm \end{pmatrix}$$

Matrix Multiplication

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} =$$

Matrix Multiplication

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Matrix Multiplication

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \times 3 + 0 \times (-1) + (-2) \times 4 \\ ? & ? \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ ? & ? \end{bmatrix}$$

Matrix Multiplication

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ -6 & ? \end{bmatrix}$$

Matrix Multiplication

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1 & 0 & -2 \\ \boxed{0 \quad 3 \quad -1} \end{bmatrix} \begin{bmatrix} 0 & \boxed{3} \\ -2 & \boxed{-1} \\ 0 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ -6 & -7 \end{bmatrix}$$

Transposition

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- $m \times n$ matrix turns into an $n \times m$ matrix
- Some rules

$$(M^T)^T = M$$

$$(sM)^T = s(M^T)$$

$$(M + N)^T = M^T + N^T$$

$$(MN)^T = N^T M^T$$

Determinant and Inverse

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad det(\boldsymbol{M}) = ad - bc$$

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boldsymbol{M}^{-1} = \frac{1}{det(\boldsymbol{M})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$MM^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad MI = M$$

Determinant and Inverse

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad \det(A) = 4 \times 2 - 3 \times 3 = -1$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

- To verify we multiply AA^{-1}

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 4 \times (-2) + 3 \times 3 & 4 \times 3 + 3 \times (-4) \\ 3 \times (-2) + 2 \times 3 & 3 \times 3 + 2 \times (-4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

- Rotation, Scaling, Shearing: **M**
- Translation: **b**
- General form: $x' = Mx + b$

- **Problem:** two parameters – how can we handle Rotation/Scaling/Shearing and Translation with just one matrix?

Homogeneous Coordinates

- Add a homogeneous coordinate w

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} \rightarrow \begin{pmatrix} wa_x \\ wa_y \\ w \end{pmatrix}$$

$$Ex: \begin{pmatrix} 3 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$$Ex: \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} \\ \end{pmatrix}$$

- **Combine M and b**

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{pmatrix} b_x \\ b_y \\ 1 \end{pmatrix} \quad \Rightarrow \begin{bmatrix} m_{11} & m_{12} & b_x \\ m_{21} & m_{22} & b_y \\ 0 & 0 & 1 \end{bmatrix}$$

Check Point

- Find the magnitude of the following vectors

$$u=(4,3) \qquad u=(5,3) \qquad u=(1,0)$$

- Find the product of A and B

a) $A=(1,2)\;\; B=(4,3)^T$ b) $A=(1,1,1)\;\; B=(2,3,4)^T$

- Find the product of 2 matrices: A and B

$$A=\begin{pmatrix}5 & 3 & 2\\ 4 & -1 & -1\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}-2 & 4\\ 5 & 3\\ 4 & -1\end{pmatrix}$$

$$A=\begin{pmatrix}-2 & 4\\ 5 & 3\\ 4 & -1\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}-2 & 4\\ 5 & 3\end{pmatrix}$$

$$A=\begin{pmatrix}4 & 4 & -2 & -3\\ 3 & -1 & -1 & 2\end{pmatrix} \quad \text{and} \quad B=(5\ -4\ \ 3\ \ 2)^T$$

Check Point

- Convert these vectors to homogeneous coordinates

$u = (0,0)$

$u = (\infty, -\infty)$

Answers

- Find the magnitude of the following vectors

$u = (4,3)$ $\quad$ $u = (5,3)$ $\quad$ $u = (1,0)$

$|u| = 5$ $\quad$ $|u| = \sqrt{34}$ $\quad$ $|u| = 1$

- Find the product of A and B
  - a) 10 b) 9
- Find the product of 2 matrices: A and B

$$AB = \begin{pmatrix} 5 & 3 & 2 \\ 4 & -1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \times (-2) + 3 \times 5 + 2 \times 4 & 5 \times 4 + 3 \times 3 + 2 \times (-1) \\ 4 \times (-2) + (-1) \times 5 + (-1) \times 4 & 4 \times 4 + (-1) \times 3 + (-1) \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 27 \\ -17 & 14 \end{pmatrix}.$$

Answers

$$AB = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} (-2)\times(-2)+4\times 5 & (-2)\times 4+4\times 3 \\ 5\times(-2)+3\times 5 & 5\times 4+3\times 3 \\ 4\times(-2)+(-1)\times 5 & 4\times 4+(-1)\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 4 \\ 5 & 29 \\ -13 & 13 \end{pmatrix} .$$

$$AB = \begin{pmatrix} 4 & 4 & -2 & -3 \\ 3 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4\times 5+4\times(-4)+(-2)\times 3+(-3)\times 2 \\ 3\times 5+(-1)\times(-4)+(-1)\times 3+2\times 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 20 \end{pmatrix}$$