

# Computer Graphics and Image Processing Geometry II

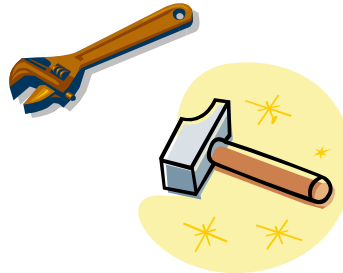
Part 1 – Lecture 3

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## Today's Outline

- Applications of Dot and Cross Products
- The Geometry of Planes
- 2D Affine Transformations

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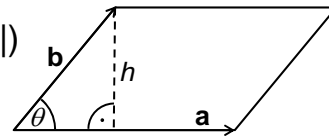


## APPLICATIONS OF DOT AND CROSS PRODUCTS

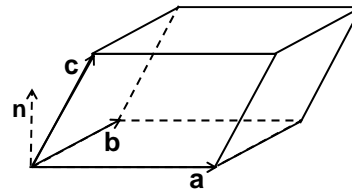
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## Areas and Volumes

- $|\mathbf{a} \times \mathbf{b}|$  is area of parallelogram:  
 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta) |\mathbf{n}| \quad (|\mathbf{n}|=1)$   
 $= |\mathbf{a}| |\mathbf{b}| \sin(\theta) \quad (h = \sin(\theta) |\mathbf{b}|)$   
 $= |\mathbf{a}| h$



- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$  is the volume of a parallelepiped:  
 $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (|\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}) \cdot \mathbf{c}$   
 $= (\text{area of bottom}) \mathbf{n} \cdot \mathbf{c}$   
 $= (\text{area of bottom}) \text{height}$



Reminder:  $\mathbf{n} \cdot \mathbf{c} = |\mathbf{n}| |\mathbf{c}| \cos(\theta)$

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# Coordinate Transformations

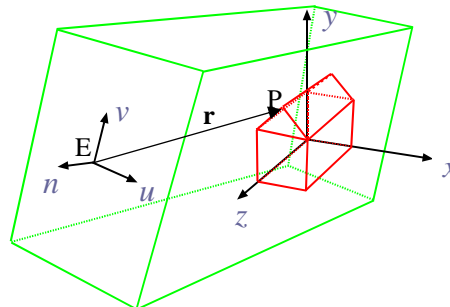
## Given:

New coordinate system with location  $E$  and axis unit vectors  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{n}$

**Wanted:** Coordinates  $P'$  of a point  $P$  in the new coordinate system

## Idea:

1. Find position vector  $\mathbf{r}$  expressing  $P$  relative to  $E$ :  
 $\mathbf{r} = P - E$
2. Project  $\mathbf{r}$  onto each of the axis unit vectors to get the new coordinates:  
 $P' = (\mathbf{r} \cdot \mathbf{u}, \mathbf{r} \cdot \mathbf{v}, \mathbf{r} \cdot \mathbf{n})$



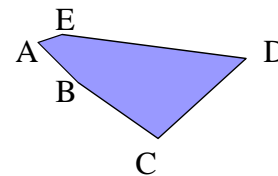
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# The Normal of a Polygon

■ In principle, get normal  $\mathbf{n}$  from the cross product of any two adjacent edge vectors, e.g.  $\mathbf{n} = (D-C) \times (B-C)$

■ But this is *non-robust* - gives erroneous or unrepresentative value when:

1. 3 vertices co-linear (on a straight line)
2. 2 adjacent vertices very close together
3. Polygon not coplanar (i.e. not all points on a plane)



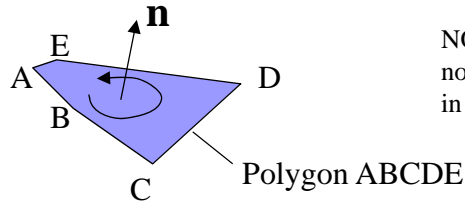
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$$

→ Magnitude of cross product tends to zero and direction is sensitive to slight movement in either point

**Warning:** In computer graphics exceptional conditions (e.g. in case of 1,2 or 3) occur all the time!

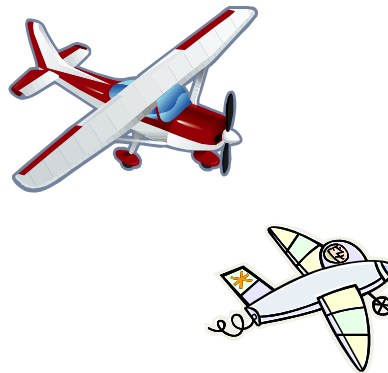
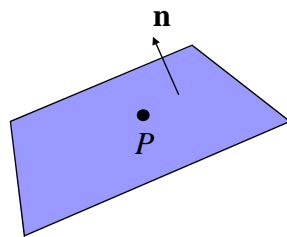
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## A Robust Normal Algorithm



NOTE: The orientation of the resulting normal is such that the vertices are listed in counterclockwise order around it.

- Just add together all the cross products of adjacent edge vectors  
i.e.  $(B-A) \times (E-A) + (C-B) \times (A-B) +$   
 $(D-C) \times (B-C) + (E-D) \times (C-D) + (A-E) \times (D-E)$
- Normalize the result
- Robust:  $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$ 
  - Short edges or nearly co-linear vertex triples give negligible cross product contribution
  - Long nearly-perpendicular edges give biggest contribution

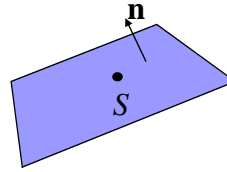


## THE GEOMETRY OF PLANES

## The Point-Normal Form of a Plane

Define plane by:

1. A **point**  $S$  on the plane
2. **Normal vector**  $\mathbf{n}$ :  
orthogonal to the plane (with  $|\mathbf{n}|=1$ )



- For any point  $P$  on the plane,  $(P-S)$  is orthogonal to  $\mathbf{n}$ :  
 $\mathbf{n} \cdot (P-S) = 0$  ("point-normal form" of plane equation)
- If  $\mathbf{p}$  and  $\mathbf{s}$  are the position vectors to  $P$  and  $S$ :  
 $\mathbf{n} \cdot (\mathbf{p} - \mathbf{s}) = 0$   
 $\Leftrightarrow \mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{s}$   
 $\Leftrightarrow \mathbf{n} \cdot \mathbf{p} = d$  where  $d = \mathbf{n} \cdot \mathbf{s}$
- If  $\mathbf{n}=(a,b,c)^T$  and  $\mathbf{p}=(x,y,z)^T$ , then this is the familiar equation  
 $ax + by + cz = d$

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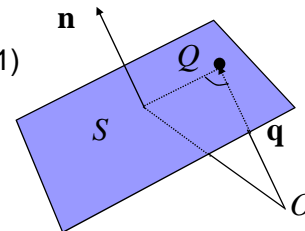
## Distance of a Plane to the Origin

- Let  $Q$  be a point on the plane such that  $\mathbf{q}$  is parallel to  $\mathbf{n}$
- $|\mathbf{q}|$  is the "shortest distance" to the plane from the origin
- Plane equation  $\mathbf{n} \cdot \mathbf{p} = d$  valid for every point  $P$  on plane:

$$\begin{aligned} \mathbf{n} \cdot \mathbf{q} &= d && \text{(Q is on the plane)} \\ \mathbf{n} \cdot \mathbf{q} &= |\mathbf{n}| |\mathbf{q}| \cos(0^\circ) && \text{(\mathbf{n} is parallel to \mathbf{q})} \\ &= |\mathbf{q}| && \text{(|\mathbf{n}|=1 and \cos(0^\circ)=1)} \end{aligned}$$

$$\Rightarrow |\mathbf{q}| = d$$

- **Conclusion:**  
In the plane equations  
 $\mathbf{n} \cdot \mathbf{p} = d$  and  $ax + by + cz = d$   
 $d$  is the distance to the plane from the origin  
provided  $\mathbf{n} = (a,b,c)^T$  is a unit vector

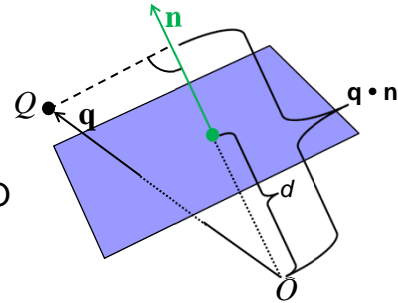


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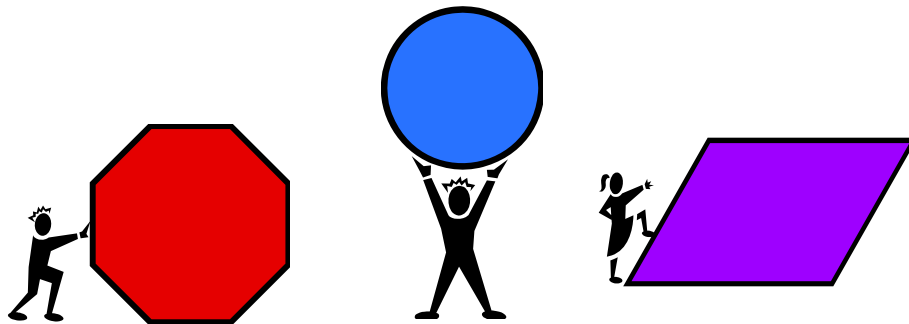
## Distance of a Point to a Plane

How far is a point  $Q$  from the plane  $\mathbf{n} \cdot \mathbf{p} = d$ ?

- The shortest line from  $Q$  to the plane is parallel to the normal  $\mathbf{n}$
- Project  $\mathbf{q}$  (position vector of  $Q$ ) onto the normal  $\mathbf{n}$ :  
 $\mathbf{q} \cdot \mathbf{n} =$  distance from  $Q$  to the origin  $O$  along normal  $\mathbf{n}$
- We only want distance to plane, so subtract distance of plane to origin  $d$ :  
 $\mathbf{q} \cdot \mathbf{n} - d =$  distance from  $Q$  to plane along normal  $\mathbf{n}$  (for  $|\mathbf{n}|=1$ )



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## 2D AFFINE TRANSFORMATIONS

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## 2D Affine Transformations

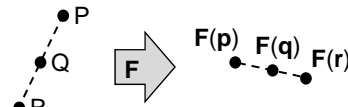
- Function  $F$  that gets a vector  $\mathbf{p}$  and produces a vector  $F(\mathbf{p})$
- $F$  consists of a linear transformation and a translation:  

$$F(\mathbf{p}) = \mathbf{M} \mathbf{p} + \mathbf{t}$$
- The linear transformation is a matrix multiplication:  $\mathbf{M} \mathbf{p}$
- The translation is a vector addition:  $\dots + \mathbf{t}$

### Properties:

- $F$  preserves **collinearity**:  
 If  $P, Q, R$  are on a straight line, then also  $F(\mathbf{p}), F(\mathbf{q}), F(\mathbf{r})$
- $F$  preserves **ratios of distances** along a line:  
 If  $P, Q, R$  are on a straight line, then  

$$|Q-P| / |R-Q| = |F(\mathbf{q}) - F(\mathbf{p})| / |F(\mathbf{r}) - F(\mathbf{q})|$$

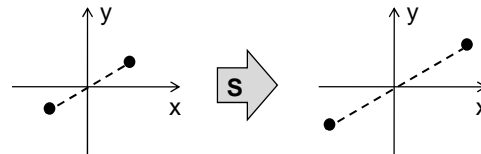


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## Scaling and Translation

### Scaling $S$ (about the origin)

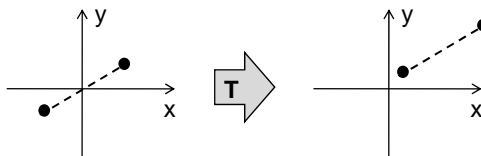
- Squeezing and stretching along the x- and y-axis
- $s_x / s_y$  is scaling factor along x/y-axis
- Scaling factor  $< 1$  means squeezing
- Scaling factor  $> 1$  means stretching



$$S(\mathbf{p}) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \end{pmatrix}$$

### Translation $T$

- Moving along x- and y-axes
- $t_x / t_y$  is distance along x/y-axis



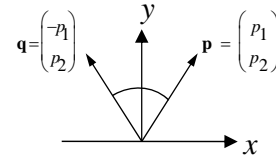
$$T(\mathbf{p}) = \mathbf{I} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

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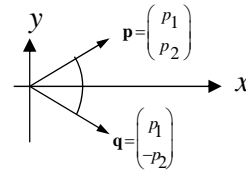
## Reflections at Axes and Origin

Special cases of scaling

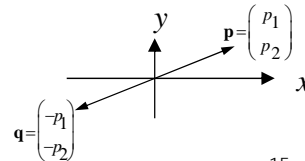
Reflection at the y-axis:  $\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{p}$



Reflection at the x-axis:  $\mathbf{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{p}$



Reflection at the origin:  $\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{p}$



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## Rotation

Rotation  $\mathbf{R}$  about origin anti-clockwise by angle  $\beta$

$\alpha$  = initial angle of point P

$\beta$  = angle of rotation so that P becomes P'

1. P has the following coordinates:

$$x = r \cos(\alpha) \quad y = r \sin(\alpha)$$

2. P' has the following coordinates:

$$x' = r \cos(\alpha + \beta) = r \cos(\alpha) \cos(\beta) - r \sin(\alpha) \sin(\beta)$$

$$y' = r \sin(\alpha + \beta) = r \sin(\alpha) \cos(\beta) + r \cos(\alpha) \sin(\beta)$$

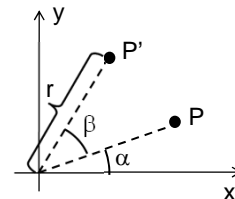
3. Substitute formula for x and y in x' and y':

$$x' = x \cos(\beta) - y \sin(\beta)$$

$$y' = y \cos(\beta) + x \sin(\beta)$$

$$\mathbf{R}(\mathbf{p}) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

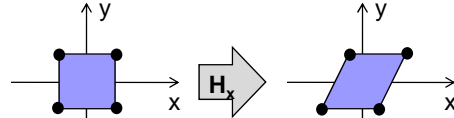
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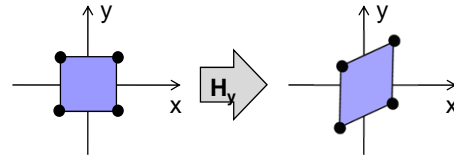
# Shearing

## Horizontal Shear $H_x$ :

- Shifts points parallel to the x-axis proportionally to their y-coordinate
- The further up a point, the more it is shifted to the right (or left)



Analogously: **vertical shear  $H_y$**



## General Shear $H$ :

- The greater shearing factor  $s_x$  or  $s_y$  the stronger the horizontal or vertical shearing
- Horizontal shearing:  $s_x > 0$  and  $s_y = 0$
- Vertical shearing:  $s_x = 0$  and  $s_y > 0$
- Shearing preserves the area of a shape

$$\mathbf{H}(\mathbf{p}) = \begin{pmatrix} 1 & s_x \\ s_y & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p_x + s_x p_y \\ p_y + s_y p_x \end{pmatrix}$$

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# Affine Transformation Properties

- Straight lines are preserved
- Parallel lines remain parallel
- Proportional distances are preserved
- Any arbitrary affine transformation can be represented as a sequence of shearing, scaling, rotation and translation
- **Affine transformations in general do not commute**  
i.e.  $\mathbf{T}_1 \mathbf{T}_2 \neq \mathbf{T}_2 \mathbf{T}_1$
- Transformations *are* associative:  $\mathbf{T}_1(\mathbf{T}_2 \mathbf{T}_3) = (\mathbf{T}_1 \mathbf{T}_2) \mathbf{T}_3$

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## SUMMARY

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## Summary

1. Applications of  $\cdot$  and  $\times$ :  
areas and volumes, coordinate transformations, normals
2. Planes
  1. Point-Normal Form:  $\mathbf{n} \cdot \mathbf{p} = d$  with  $d = \text{distance to origin}$
  2. Distance from Q to plane:  $\mathbf{q} \cdot \mathbf{n} - d$
3. 2D Affine Transformations:  $\mathbf{F}(\mathbf{p}) = \mathbf{M} \mathbf{p} + \mathbf{t}$   
scaling, translation, rotation, shearing

### References:

- Dot Product: Hill, Chapter 4.3
- Cross Product: Hill, Chapter 4.4
- Introduction to Affine Transformations: Hill, Chapter 5.2

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# Quiz

1. Transform  $P=(2, 2, -1)$  to the new coordinate system with axis vectors  $u=(0,1,0)$ ,  $v=(0,0,-1)$ ,  $n=(-1,0,0)$  and origin  $E=(0,2,0)$ .
2. How far is the plane  $3x + y - 2z = 5$  from the origin?
3. How far is point  $Q=(3,4,2)$  from the plane?
4. Do the following to point  $R=(1,2)$ : scale it along the y-axis with factor 0.5; move it up the y-axis by 4; shear it vertically by 2.

Given:  $a = b$   
 $a^2 = ab$   
 $a^2 - b^2 = ab - b^2$   
 $(a+b)(a-b) = b(a-b)$   
 $(a+b) = b$   
 $a+a = a$   
 $2a = a$   
 $2 = 1 !!!$

