Computer Graphics and Image Processing Geometry II

Part 1 – Lecture 3

Today’s Outline

- Applications of Dot and Cross Products
- The Geometry of Planes
- 2D Affine Transformations

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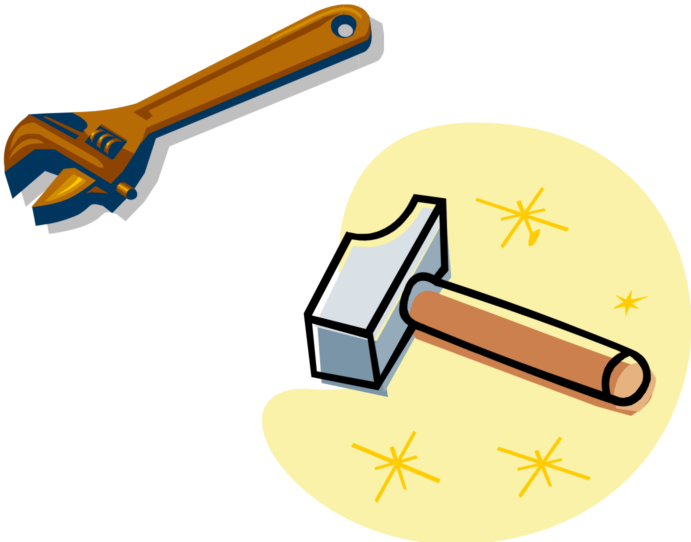
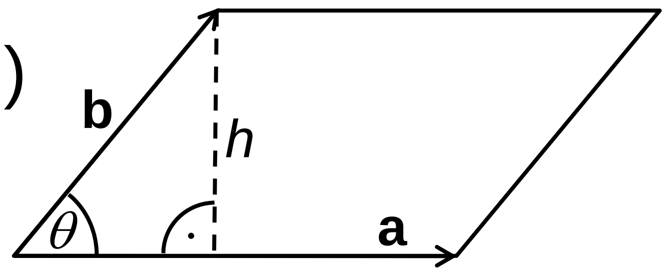


APPLICATIONS OF DOT AND CROSS PRODUCTS

Areas and Volumes

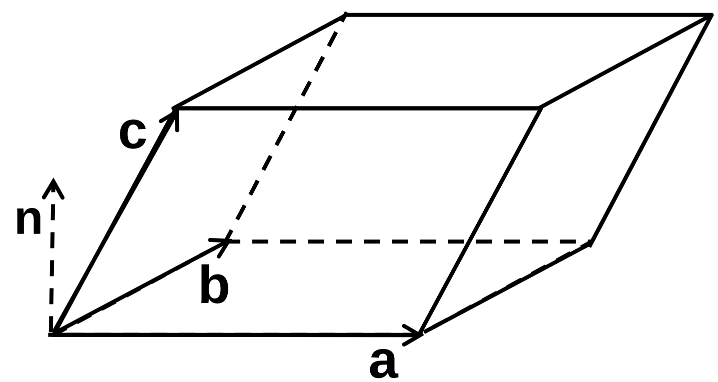
- $|\mathbf{a} \times \mathbf{b}|$ is area of parallelogram:

  $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\,|\mathbf{n}| \quad (|\mathbf{n}|=1)$$
  $$= |\mathbf{a}|\,|\mathbf{b}|\sin(\theta) \quad (h=\sin(\theta)\,|\mathbf{b}|)$$
  $$= |\mathbf{a}|\,h$$



- $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}$ is the volume of a parallelepiped:

  $$(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = (|\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\,\mathbf{n})\cdot\mathbf{c}$$
  $$= (\textit{area of bottom})\ \mathbf{n}\cdot\mathbf{c}$$
  $$= (\textit{area of bottom})\ \textit{height}$$



Reminder: $\mathbf{n}\cdot\mathbf{c} = |\mathbf{n}|\;|\mathbf{c}|\;\cos(\theta)$

Coordinate Transformations

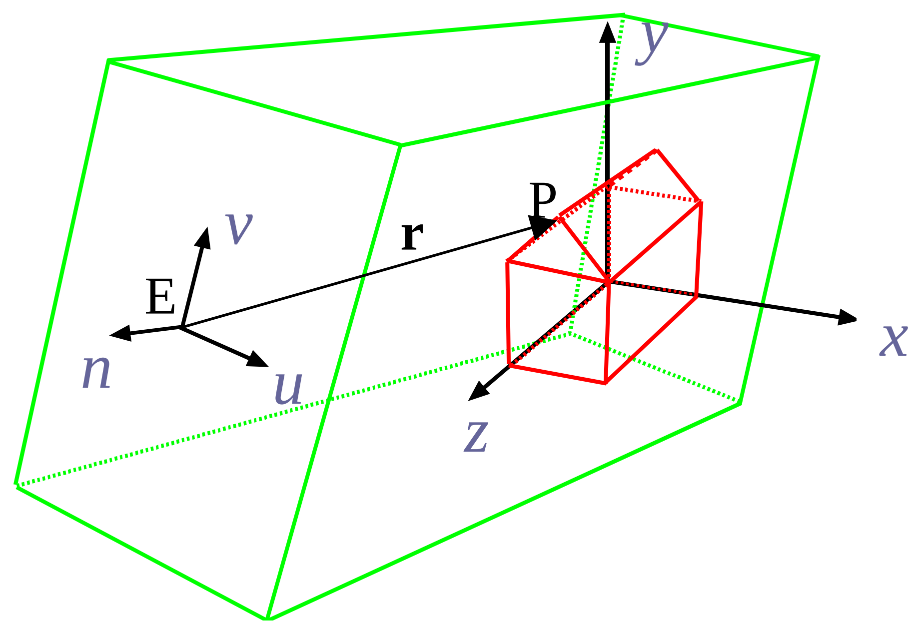
Given:

New coordinate system with location E and axis unit vectors **u**, **v**, **n**

Wanted: Coordinates P' of a point P in the new coordinate system

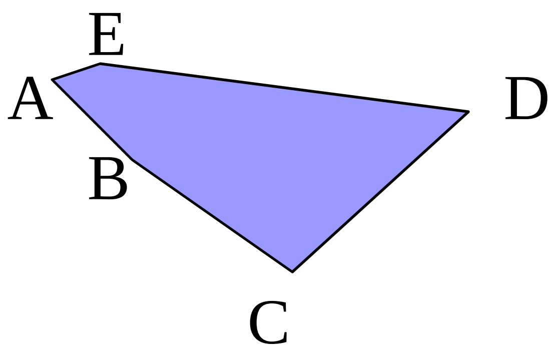
Idea:

1. Find position vector **r** expressing P relative to E: **r** = P – E
2. Project **r** onto each of the axis unit vectors to get the new coordinates: P' = (**r • u**, **r • v, r • n**)



The Normal of a Polygon

- In principle, get normal **n** from the cross product of any two adjacent edge vectors, e.g. **n** = (D-C)×(B-C)
- But this is *non-robust* - gives erroneous or unrepresentative value when:
  1. 3 vertices co-linear (on a straight line)
  2. 2 adjacent vertices very close together
  3. Polygon not coplanar (i.e. not all points on a plane)

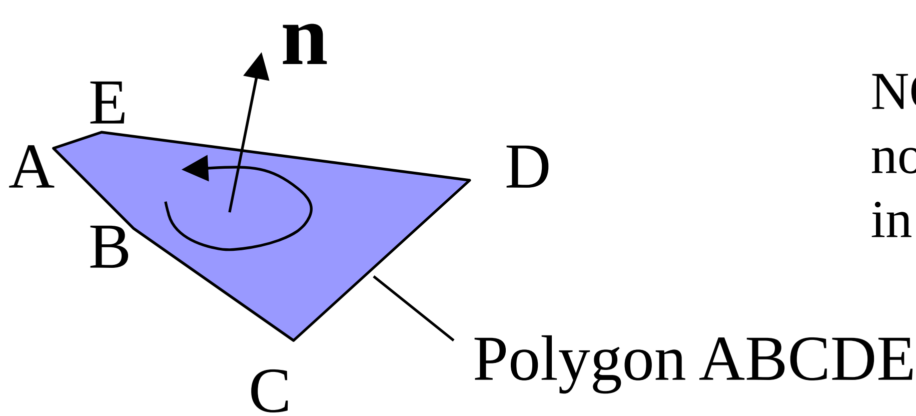


$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

→ Magnitude of cross product tends to zero and direction is sensitive to slight movement in either point

Warning: In computer graphics exceptional conditions (e.g. in case of 1,2 or 3) occur all the time!

A Robust Normal Algorithm

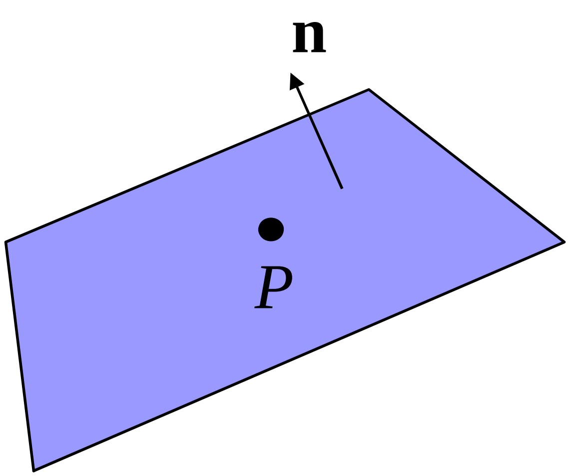


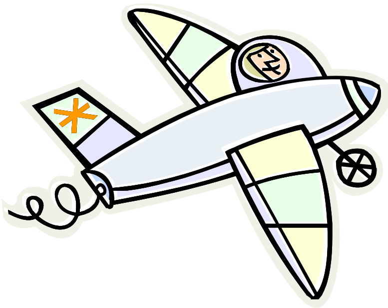
NOTE: The orientation of the resulting normal is such that the vertices are listed in counterclockwise order around it.

- Just add together all the cross products of adjacent edge vectors
  i.e. (B–A) × (E–A) + (C–B) × (A–B) +
  (D–C) × (B–C) + (E–D) × (C–D) + (A-E) × (D–E)
- Normalize the result
- Robust:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

  - Short edges or nearly co-linear vertex triples give negligible cross product contribution
  - Long nearly-perpendicular edges give biggest contribution



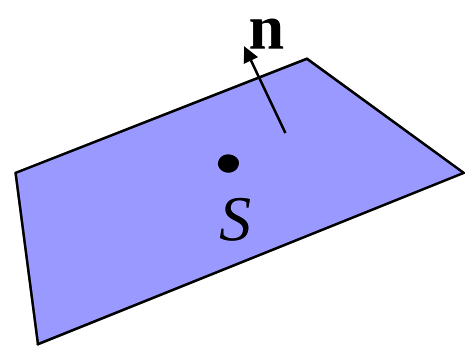


THE GEOMETRY OF PLANES

The Point-Normal Form of a Plane

Define plane by:

1. A **point** S on the plane
2. **Normal vector n**:
   orthogonal to the plane (with |**n**|=1)



- For any point P on the plane, (P-S) is orthogonal to **n**:

  $\mathbf{n}\bullet(P-S) = 0$ ("point-normal form" of plane equation)

- If **p** and **s** are the position vectors to P and S:

  $\mathbf{n}\bullet(\mathbf{p} - \mathbf{s}) = 0$

  $\Leftrightarrow \mathbf{n} \bullet \mathbf{p} = \mathbf{n} \bullet \mathbf{s}$

  $\Leftrightarrow \mathbf{n} \bullet \mathbf{p} = \mathrm{d}$ where $\mathrm{d} = \mathbf{n} \bullet \mathbf{s}$

- If $\mathbf{n}=(a,b,c)^T$ and $\mathbf{p}=(x,y,z)^T$, then this is the familiar equation

  $ax + by + cz = \mathrm{d}$

Distance of a Plane to the Origin

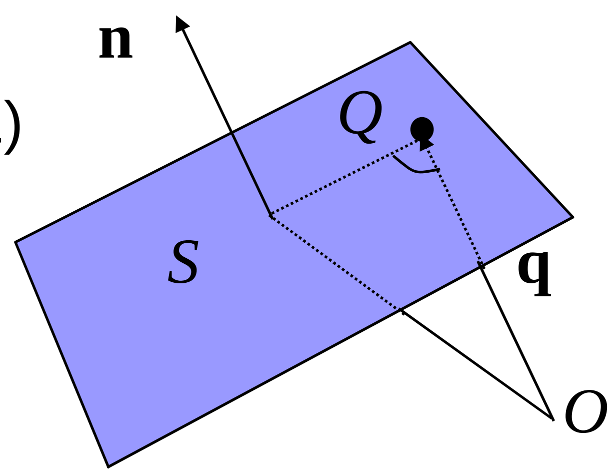
- Let Q be a point on the plane such that **q** is parallel to **n**
- |**q**| is the "shortest distance" to the plane from the origin
- Plane equation $\mathbf{n} \cdot \mathbf{p} = d$ valid for every point P on plane:

$$\mathbf{n} \cdot \mathbf{q} = d \qquad \text{(Q is on the plane)}$$

$$\mathbf{n} \cdot \mathbf{q} = |\mathbf{n}|\,|\mathbf{q}| \cos(0°) \qquad (\mathbf{n} \text{ is parallel to } \mathbf{q})$$

$$= |\mathbf{q}| \qquad (|\mathbf{n}|=1 \text{ and } \cos(0°)=1)$$

$$\Rightarrow \quad |\mathbf{q}| = d$$



- **Conclusion**:
  In the plane equations

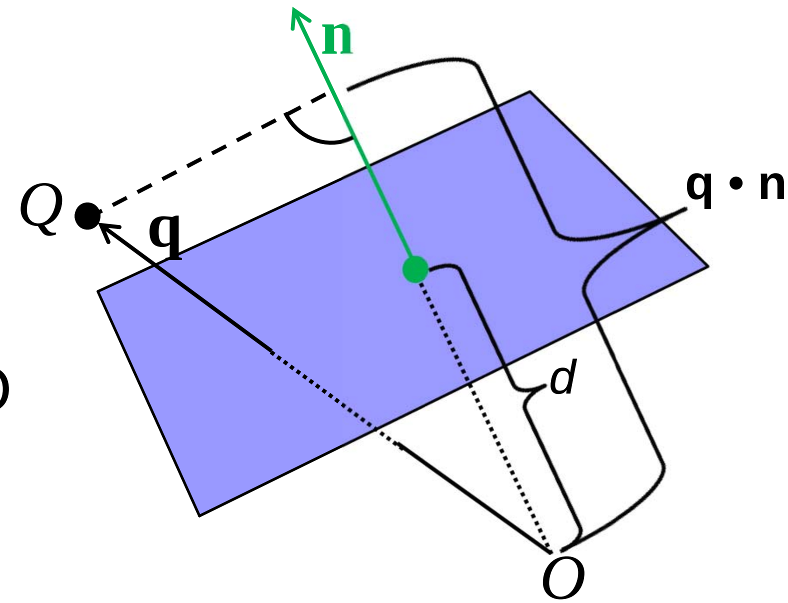
$$\mathbf{n} \cdot \mathbf{p} = d \quad \text{and} \quad ax + by + cz = d$$

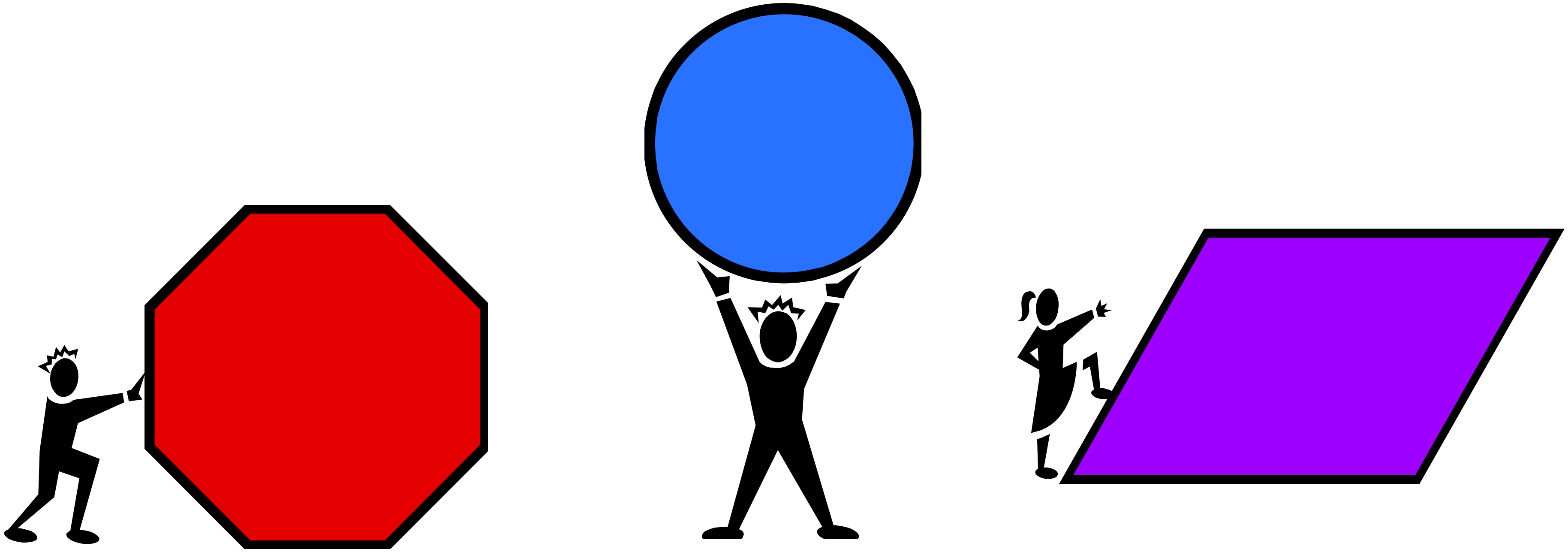
  d is the distance to the plane from the origin
  provided $\mathbf{n} = (a,b,c)^T$ is a unit vector

Distance of a Point to a Plane

How far is a point Q from the plane $\mathbf{n} \bullet \mathbf{p} = d$?

- The shortest line from Q to the plane is parallel to the normal **n**
- Project **q** (position vector of Q) onto the normal **n**:
  $\mathbf{q} \bullet \mathbf{n}$ = distance from Q to the origin O along normal n
- We only want distance to plane, so subtract distance of plane to origin d:
  $\mathbf{q} \bullet \mathbf{n} - d$ = distance from Q to plane along normal **n** (for $|\mathbf{n}|=1$)



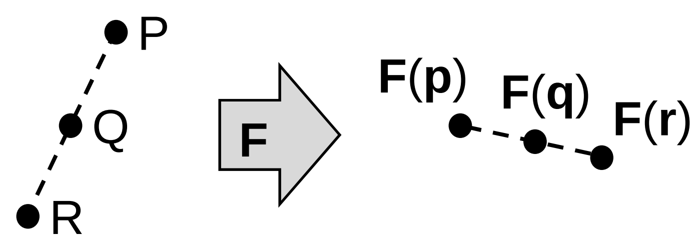


2D AFFINE TRANSFORMATIONS

2D Affine Transformations

- Function **F** that gets a vector **p** and produces a vector **F**(**p**)
- **F** consists of a linear transformation and a translation:
  F(**p**) = **M p** + **t**
- The linear transformation is a matrix multiplication: **M p**
- The translation is a vector addition: … + **t**

Properties:

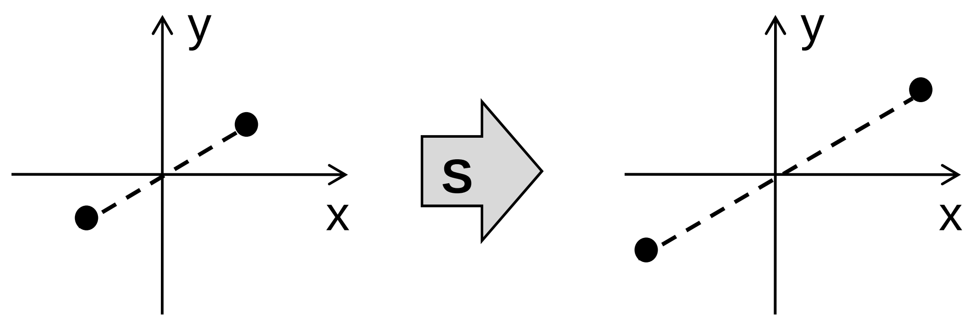


1. F preserves **collinearity**:
   If P, Q, R are on a straight line, then also **F**(**p**), **F**(**q**), **F**(**r**)
2. F preserves **ratios of distances** along a line:
   If P, Q, R are on a straight line, then
   |Q-P| / |R-Q| = |**F**(**q**) - **F**(**p**)| / |**F**(**r**) - **F**(**q**)|

Scaling and Translation

Scaling S (about the origin)

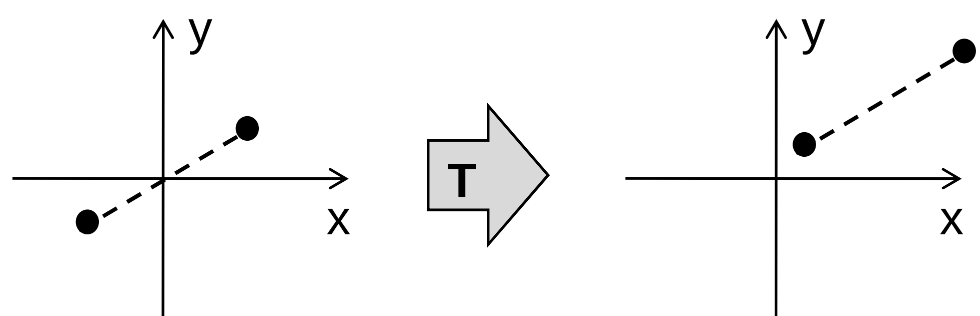
- Squeezing and stretching along the x- and y-axis
- s_x / s_y is scaling factor along x/y-axis
- Scaling factor < 1 means squeezing
- Scaling factor > 1 means stretching



$$\mathbf{S}(\mathbf{p}) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \end{pmatrix}$$

Translation T

- Moving along x- and y-axes
- t_x / t_y is distance along x/y-axis



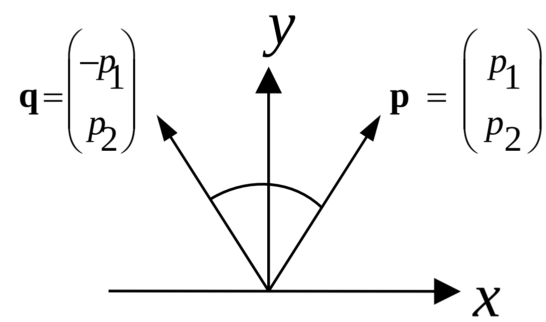
$$\mathbf{T}(\mathbf{p}) = \mathbf{I} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Reflections at Axes and Origin

Special cases of scaling

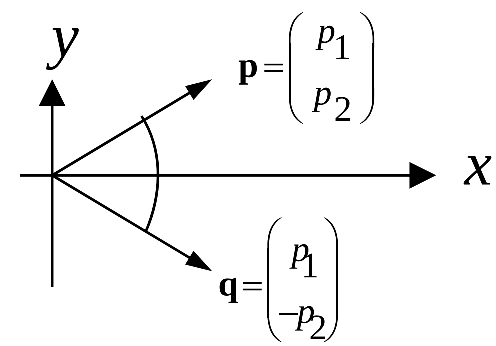
Reflection at the y-axis:

$$\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{p}$$



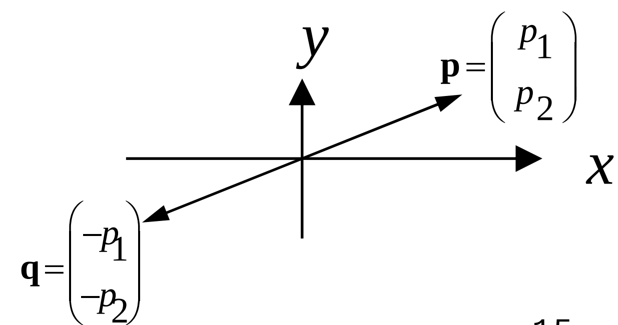
Reflection at the x-axis:

$$\mathbf{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{p}$$



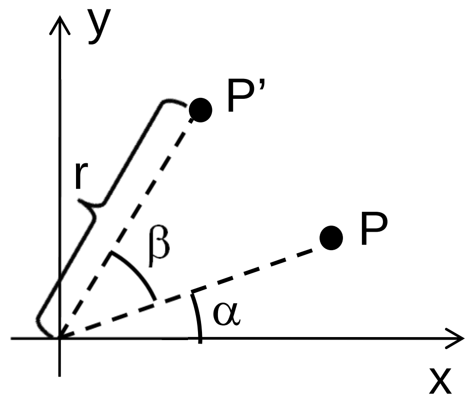
Reflection at the origin:

$$\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{p}$$



Rotation

Rotation **R** about origin anti-clockwise by angle β



α = initial angle of point P

β = angle of rotation so that P becomes P'

1. P has the following coordinates:
   $x = r\cos(\alpha)$ $\quad y = r\sin(\alpha)$
2. P' has the following coordinates:
   $x' = r\cos(\alpha + \beta) = r\cos(\alpha)\cos(\beta) - r\sin(\alpha)\sin(\beta)$
   $y' = r\sin(\alpha + \beta) = r\sin(\alpha)\cos(\beta) + r\cos(\alpha)\sin(\beta)$
3. Substitute formula for x and y in x' and y':
   $x' = x\cos(\beta) - y\sin(\beta)$
   $y' = y\cos(\beta) + x\sin(\beta)$

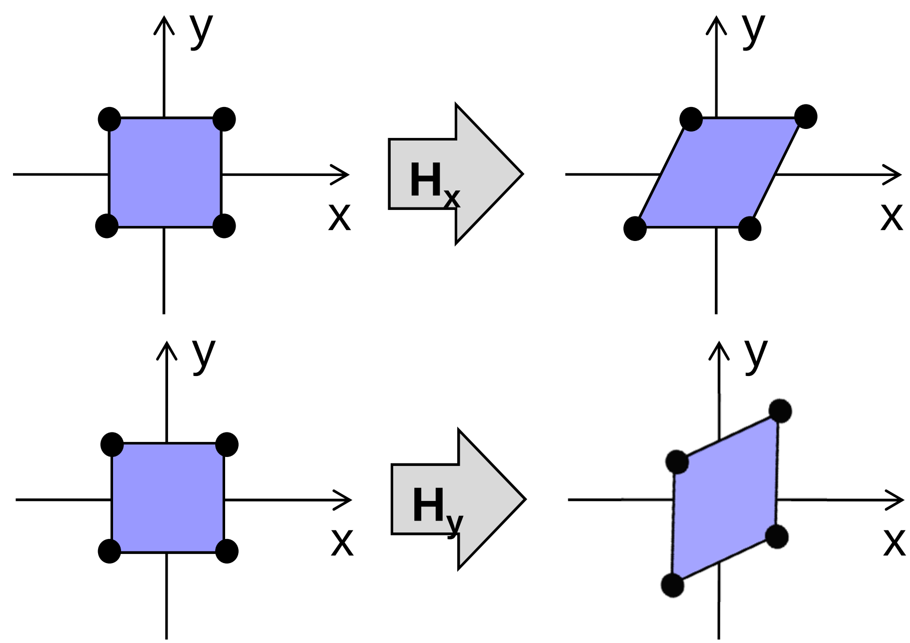
$$\mathbf{R}(\mathbf{p}) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

Shearing

Horizontal Shear H_x:

- Shifts points parallel to the x-axis proportionally to their y-coordinate
- The further up a point, the more it is shifted to the right (or left)

Analogously: **vertical shear H_y**



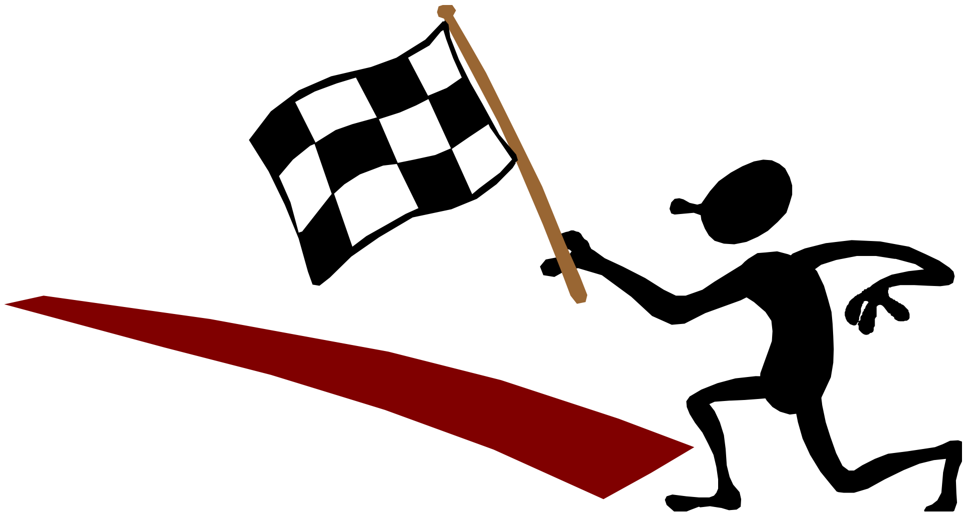
General Shear H:

- The greater shearing factor s_x or s_y the stronger the horizontal or vertical shearing
- Horizontal shearing: $s_x>0$ and $s_y=0$
- Vertical shearing: $s_x=0$ and $s_y>0$
- Shearing preserves the area of a shape

$$\mathbf{H}(\mathbf{p}) = \begin{pmatrix} 1 & s_x \\ s_y & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p_x + s_x p_y \\ p_y + s_y p_x \end{pmatrix}$$

Affine Transformation Properties

- Straight lines are preserved
- Parallel lines remain parallel
- Proportional distances are preserved
- Any arbitrary affine transformation can be represented as a sequence of shearing, scaling, rotation and translation
- **Affine transformations in general do not commute i.e. $\mathbf{T_1 T_2 \neq T_2 T_1}$**
- Transformations *are* associative: $\mathbf{T}_1(\mathbf{T}_2\mathbf{T}_3) = (\mathbf{T}_1\mathbf{T}_2)\,\mathbf{T}_3$



SUMMARY

Summary

1. Applications of · and ×:
   areas and volumes, coordinate transformations, normals
2. Planes
   1. Point-Normal Form: $\mathbf{n} \bullet \mathbf{p} = d$ with d = *distance to origin*
   2. Distance from Q to plane: $\mathbf{q} \bullet \mathbf{n} - d$
3. 2D Affine Transformations: $\mathbf{F}(\mathbf{p}) = \mathbf{M}\,\mathbf{p} + \mathbf{t}$
   scaling, translation, rotation, shearing

References:

- Dot Product: Hill, Chapter 4.3
- Cross Product: Hill, Chapter 4.4
- Introduction to Affine Transformations: Hill, Chapter 5.2

Quiz

1. Transform P=(2, 2, -1) to the new coordinate system with axis vectors u=(0,1,0), v=(0,0,-1), n=(-1,0,0) and origin E=(0,2,0).
2. How far is the plane $3x + y - 2z = 5$ from the origin?
3. How far is point Q=(3,4,2) from the plane?
4. Do the following to point R=(1,2): scale it along the y-axis with factor 0.5; move it up the y-axis by 4; shear it vertically by 2.