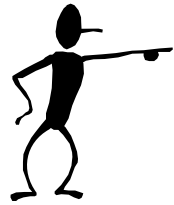
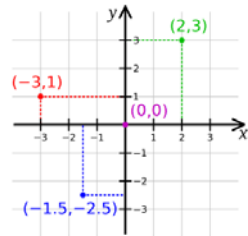


Computer Graphics and Image Processing Geometry I

Part 1 – Lecture 2

Today's Outline

- Points, Vectors and Matrices
- The Dot Product ·
- The Cross Product \times



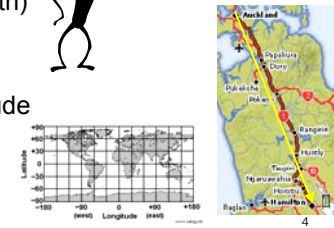
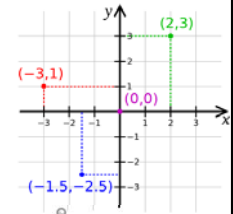
$$\begin{pmatrix} x & u \\ y & v \end{pmatrix}$$

POINTS, VECTORS AND MATRICES

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Points and Vectors

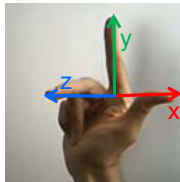
- Cartesian **coordinate system**: orthogonal axes with numbers (coordinates)
Origin: where all coordinates are zero (center)
- **Point**: a position in space, e.g. Auckland
 Given as pair (x,y) of coordinate values
- **Vector**: a displacement / difference between two points (direction + length)
- Example: Where is Hamilton?
 - Point:
37.43S Latitude, 175.19E Longitude
 - Vector:
120km to the south-south-west of Auckland



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Representing Points and Vectors

- **Points** are represented as tuples
 - 2D: 2-tuples (x, y) with x and y coordinates
 - 3D: 3-tuples (x, y, z) with x, y, z coordinates
- **Vectors** are also represented as tuples, but usually written as a column instead of a row



Right-handed 3D coordinate system

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{array}{l} x \text{ component} \\ \text{and } y \text{ component} \end{array} \quad (\text{in 3D also } z \text{ component})$$

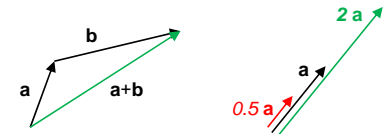
- **Position vector** of a point: vector from origin to the point (often convenient to use vectors instead of points)
- Our notation:
 - Points are written in capital letters, e.g. P
 - Vectors in small bold letters, e.g. position vector of P is \mathbf{p}

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Operations on Points and Vectors

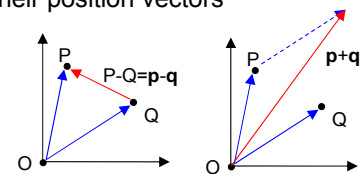
Vectors

- Add, subtract
- Scale (change length)



Points

- Subtracting one point from another gives a vector (displacement)
- **Cannot** add two points (Auckland + Hamilton = ???), but can add and subtract their position vectors



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Basic Operations on Vectors

- **Addition**

- Represents the combined displacement
- Add corresponding components

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

- **Subtraction**

- Subtract the corresponding components
- Same as addition of a negated vector, i.e. one in opposite direction

$$\mathbf{u} - \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix}$$


- **Scaling**

- Changing the length
- Defined such that $\mathbf{v} + \mathbf{v} = 2\mathbf{v}$
- Multiply all components by the scalar

$$s \mathbf{u} = s \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s u_1 \\ s u_2 \end{pmatrix}$$

- **Magnitude** of a vector

- Calculating its "length" or 2-norm



$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}, \quad |s\mathbf{u}| = |s| |\mathbf{u}|$$

- **Normalization**

- Scaling vector so that it has length 1 (unit vector)
- Scale by reciprocal of magnitude

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

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Matrices

Matrix: several vectors stuck together

- $m \times n$ matrix has m rows and n columns
- Like m row vectors or n column vectors

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{pmatrix}$$

row

column

- **Operations**

- **Addition/Subtraction:**

like adding/subtracting several vectors at the same time

$$\mathbf{M} \pm \mathbf{N} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \pm \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} m_{11} \pm n_{11} & m_{12} \pm n_{12} \\ m_{21} \pm n_{21} & m_{22} \pm n_{22} \end{pmatrix}$$



- **Scaling:** like scaling several vectors at the same time

$$s\mathbf{M} = \begin{pmatrix} s m_{11} & s m_{12} \\ s m_{21} & s m_{22} \end{pmatrix}$$

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Matrix Multiplication

Multiplying an $l \times m$ matrix **B** and an $m \times n$ matrix **C** gives an $l \times n$ matrix **A** with $a_{ij} = b_{i1}c_{1j} + \dots + b_{im}c_{mj} = \sum_{k=1}^m b_{ik}c_{kj}$

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

- “Rows times columns” with the products summed up
- Elements of **A** are the dot products of the row vectors of **B** and column vectors of **C**
- Can be used to transform several vectors simultaneously

$$\mathbf{B} \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad \mathbf{B} \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

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Identity and Inverse

Identity matrix I: $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Neutral element of matrix multiplication, i.e. for all **M**:
I M = M I = M

Inverse matrix \mathbf{M}^{-1} of a square matrix **M** (does not always exist):

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I} \quad \text{and} \quad (\mathbf{M}^{-1})^{-1} = \mathbf{M}$$

Inverse of a 2x2 matrix:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{-1} = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

Exists only if $m_{11}m_{22} - m_{12}m_{21} \neq 0$

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The Transpose Operation T

Make rows out of columns (or vice versa)

- Transpose of a row vector is a column vector (and vice versa)

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \mathbf{u}^T = (u_1 \ u_2)$$

- For matrix \mathbf{M} :

m_{ij} and m_{ji} are swapped
for all $i=1..m, j=1..n$

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \Rightarrow \mathbf{M}^T = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \\ m_{13} & m_{23} \end{pmatrix}$$

Rules:

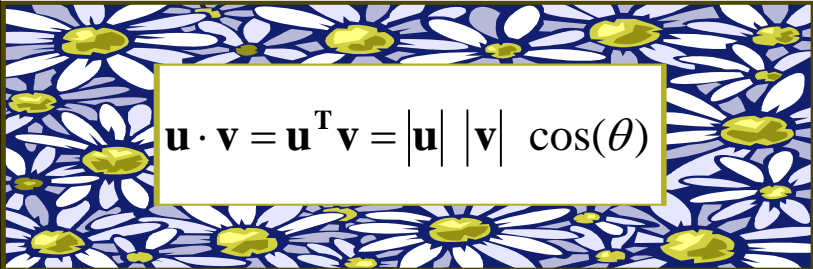
$$(\mathbf{M}^T)^T = \mathbf{M}$$

$$(\mathbf{sM})^T = \mathbf{s}(\mathbf{M}^T)$$

$$(\mathbf{M} + \mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$$

$$(\mathbf{MN})^T = \mathbf{N}^T \mathbf{M}^T$$

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$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta)$$

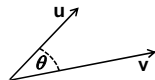
THE DOT PRODUCT .

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The Dot Product •

Takes two vectors \mathbf{u} and \mathbf{v} , the result is a scalar (a single number)
(therefore also known as scalar product)

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2 = \mathbf{u}^T \mathbf{v}$$



$$= |\mathbf{u}| |\mathbf{v}| \cos(\theta) \quad (\theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v})$$

Rules:

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	(Symmetry)
$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$	(Linearity)
$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$	(Homogeneity)
$ \mathbf{b} ^2 = \mathbf{b} \cdot \mathbf{b}$	

Example: $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$

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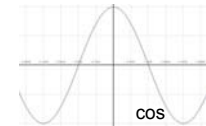
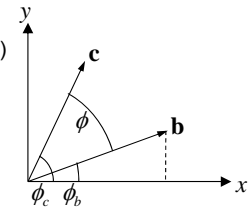
The Angle between Two Vectors

Most important application of dot product:
find angle between two vectors (or two intersecting lines)

$$\mathbf{b} = \begin{pmatrix} |b| \cos \phi_b \\ |b| \sin \phi_b \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} |c| \cos \phi_c \\ |c| \sin \phi_c \end{pmatrix}$$

hence

$$\begin{aligned} \mathbf{b} \cdot \mathbf{c} &= |b||c| \cos \phi_b \cos \phi_c + |b||c| \sin \phi_b \sin \phi_c \\ &= |b||c| \cos(\phi_c - \phi_b) \\ &= |b||c| \cos \phi \end{aligned}$$



Two non-zero vectors \mathbf{b} and \mathbf{c} with common start point are:

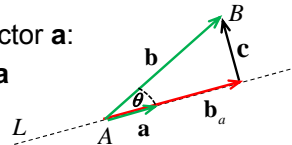
less than 90° apart	if $\mathbf{b} \cdot \mathbf{c} > 0$
exactly 90° apart	if $\mathbf{b} \cdot \mathbf{c} = 0$ [\mathbf{b} and \mathbf{c} are <i>orthogonal (perpendicular)</i>]
more than 90° apart	if $\mathbf{b} \cdot \mathbf{c} < 0$

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Orthogonal Projection of a Vector

Projecting a vector \mathbf{b} onto another vector \mathbf{a} :

- L is a line through A with direction \mathbf{a}
- \mathbf{b} is the vector from A to B



Given: \mathbf{a} and \mathbf{b}

Wanted: \mathbf{b}_a (orthogonal projection of \mathbf{b} onto \mathbf{a})

Solution:

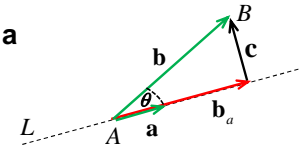
1. Length of \mathbf{b}_a : $|\mathbf{b}_a| = \cos(\theta) |\mathbf{b}| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$ because of def. •
 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$

2. Vector \mathbf{b}_a : $\mathbf{b}_a = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{a}|} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$ because $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

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Distance from a Line to a Point

- L is a line through A with direction \mathbf{a}
- \mathbf{b} is the vector from A to B



Given: \mathbf{a} and \mathbf{b}

Wanted: $|\mathbf{c}|$

Solution:

$$|\mathbf{c}| = |\mathbf{b} - \mathbf{b}_a| = \left| \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} \right|$$

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$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$$

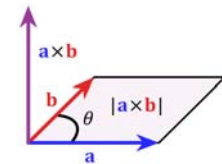
THE CROSS PRODUCT \times

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The Cross Product \times

Takes two 3D vectors \mathbf{u} and \mathbf{v} , the result is a 3D vector
(therefore also known as vector product)

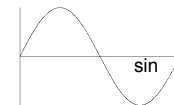
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$$



- θ is the angle between \mathbf{a} and \mathbf{b}
- \mathbf{n} is unit normal vector ($|\mathbf{n}|=1$) orthogonal to \mathbf{a} and \mathbf{b}
- *Hard to remember?* \rightarrow

Rules:

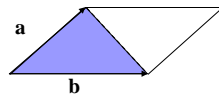
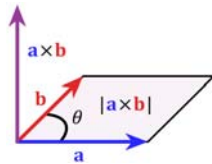
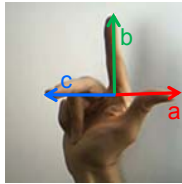
- ($\mathbf{a} + \mathbf{b}$) \times $\mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (Linearity)
- ($s\mathbf{a}$) \times $\mathbf{b} = s(\mathbf{a} \times \mathbf{b})$ (Homogeneity)
- (no symmetry)



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Properties of the Cross Product

1. $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular/orthogonal to both \mathbf{a} and \mathbf{b}
2. Direction of $\mathbf{a} \times \mathbf{b}$ is given by "right-hand rule"
3. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
4. $|\mathbf{a} \times \mathbf{b}|$ is area of parallelogram defined by \mathbf{a} and \mathbf{b}
5. Area of triangle defined by \mathbf{a} and \mathbf{b} is $0.5 * |\mathbf{a} \times \mathbf{b}|$



$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$$

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SUMMARY

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Summary

1. **Vectors:**
addition, subtraction, scaling, magnitude, normalization
2. **Matrices:**
addition, subtraction, scaling, multiplication
3. **Dot product:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta)$
4. **Cross product:**
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{n}$$

References:

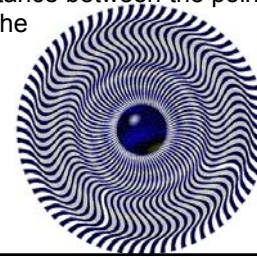
- Vectors: Hill, Chapter 4.2
- Dot Product: Hill, Chapter 4.3
- Cross Product: Hill, Chapter 4.4

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Quiz

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 3 & -1 \end{pmatrix}$$

1. Calculate: $\mathbf{a} + \mathbf{b}$, $|\mathbf{b}|$, $\mathbf{a} \cdot \mathbf{a}$, $\mathbf{M}\mathbf{a}$, $\mathbf{M}\mathbf{N}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$
2. What can you tell about the angle between \mathbf{a} and \mathbf{b} ?
3. What is the projection of \mathbf{b} onto \mathbf{a} ?
4. What is the distance between the point given by \mathbf{b} and the line going through the origin along \mathbf{a} ?



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