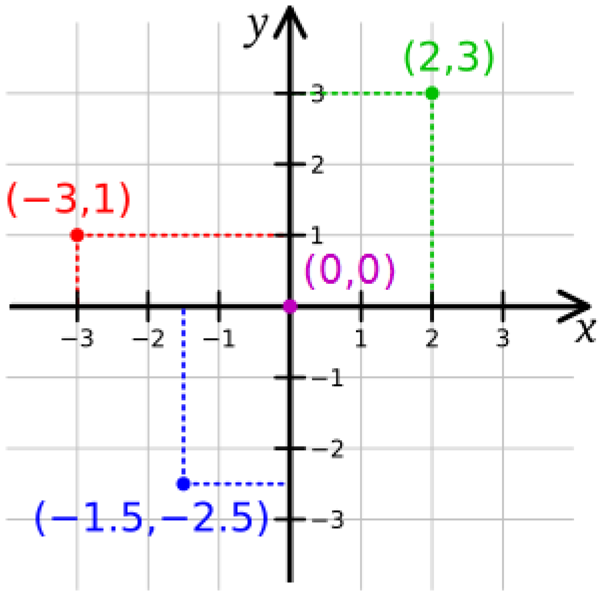
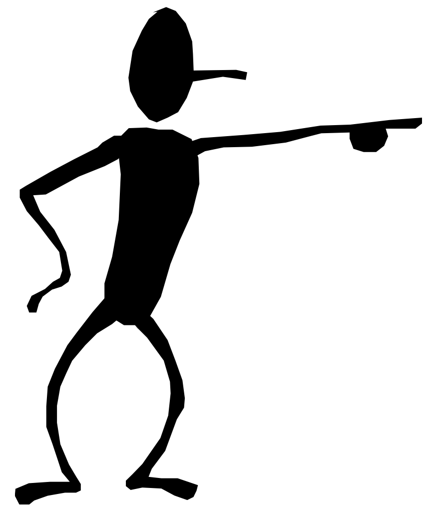
Computer Graphics and Image Processing Geometry I

Part 1 – Lecture 2

Today's Outline

- Points, Vectors and Matrices
- The Dot Product $\cdot$
- The Cross Product $\times$

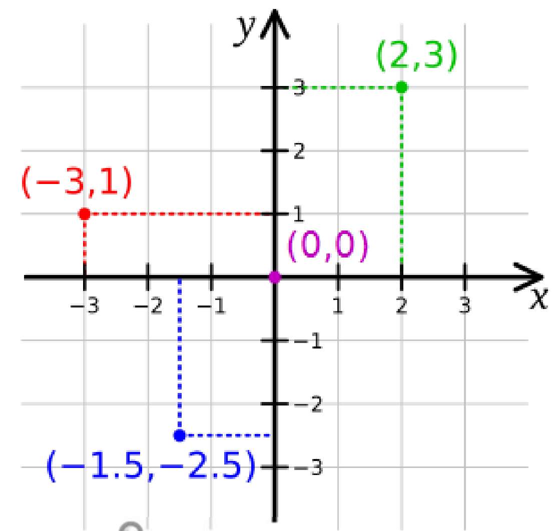




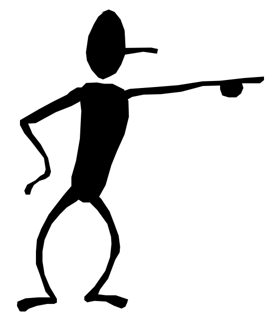
$$\begin{pmatrix} x & u \\ y & v \end{pmatrix}$$

POINTS, VECTORS AND MATRICES

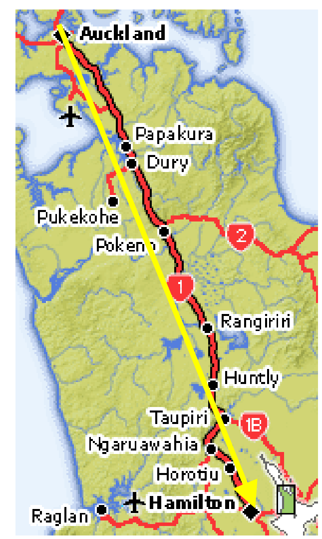
Points and Vectors

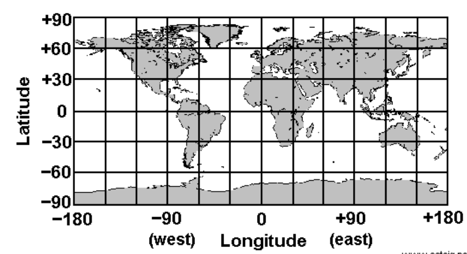


- Cartesian **coordinate system**:
  orthogonal axes with numbers (coordinates)
  Origin: where all coordinates are zero (center)
- **Point**: a position in space, e.g. Auckland
  Given as pair (x,y) of coordinate values
- **Vector**: a displacement / difference
  between two points (direction + length)



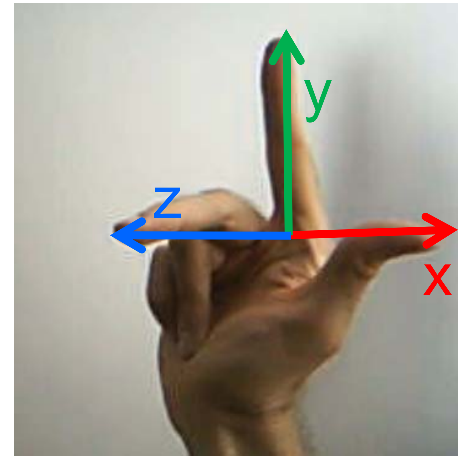
- Example: Where is Hamilton?
  - Point:
    37.43S Latitude, 175.19E Longitude
  - Vector:
    120km to the south-south-west
    of Auckland





Representing Points and Vectors

- **Points** are represented as tuples
  - 2D: 2-tuples *(x, y)* with x and y coordinates
  - 3D: 3-tuples *(x, y, z)* with x, y, z coordinates
- **Vectors** are also represented as tuples,
  but usually written as a column instead of a row



Right-handed 3D coordinate system

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

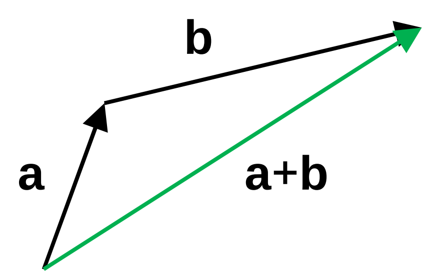
x component
and *y* component (in 3D also *z* component)

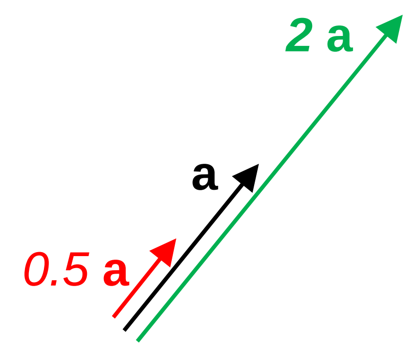
- **Position vector** of a point: vector from origin to the point
  (often convenient to use vectors instead of points)
- Our notation:
  - Points are written in capital letters, e.g. P
  - Vectors in small bold letters, e.g. position vector of P is **p**

Operations on Points and Vectors

Vectors

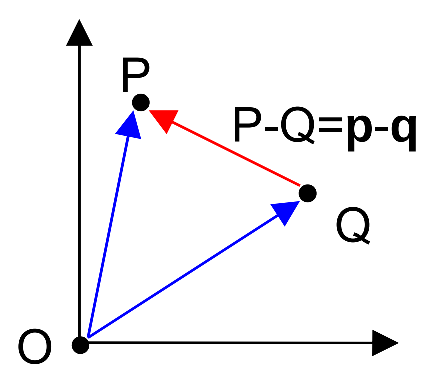
- Add, subtract
- Scale (change length)

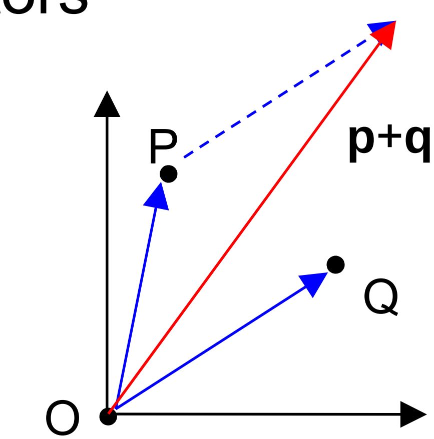




Points

- Subtracting one point from another gives a vector (displacement)
- **Cannot** add two points (Auckland + Hamilton = ???), but can add and subtract their position vectors





Basic Operations on Vectors

- **Addition**
  - Represents the combined displacement
  - Add corresponding components

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

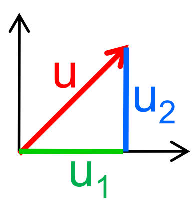
- **Subtraction**
  - Subtract the corresponding components
  - Same as addition of a negated vector, i.e. one in opposite direction

$$\mathbf{u} - \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix}$$

- **Scaling**
  - Changing the length
  - Defined such that **v** + **v** = 2**v**
  - Multiply all components by the scalar

$$s\ \mathbf{u} = s \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s\ u_1 \\ s\ u_2 \end{pmatrix}$$

- **Magnitude** of a vector
  - Calculating its "length" or 2-norm



$$|\mathbf{u}| = \sqrt{u_1^{\ 2} + u_2^{\ 2}} \quad , \quad |s\mathbf{u}| = |s||\mathbf{u}|$$

- **Normalization**
  - Scaling vector so that it has length 1 (unit vector)
  - Scale by reciprocal of magnitude

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

Matrices

Matrix: several vectors stuck together

- $m \times n$ matrix has m rows and n columns
- Like m row vectors or n column vectors

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{pmatrix}$$

row

column

Operations

- **Addition/Subtraction**:
  like adding/subtracting several vectors at the same time

$$\mathbf{M} \pm \mathbf{N} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \pm \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} m_{11} \pm n_{11} & m_{12} \pm n_{12} \\ m_{21} \pm n_{21} & m_{22} \pm n_{22} \end{pmatrix}$$

- **Scaling**: like scaling several vectors at the same time

$$s\mathbf{M} = \begin{pmatrix} s\ m_{11} & s\ m_{12} \\ s\ m_{21} & s\ m_{22} \end{pmatrix}$$

Matrix Multiplication

Multiplying an $l \times m$ matrix **B** and an $m \times n$ matrix **C** gives an $l \times n$ matrix **A** with $a_{ij} = b_{i1}c_{1j} + \ldots + b_{im}c_{mj} = \sum_{k=1}^{m} b_{ik}c_{kj}$

$$\mathbf{A} = \mathbf{B}\,\mathbf{C} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

- “Rows times columns” with the products summed up
- Elements of **A** are the dot products of the row vectors of **B** and column vectors of **C**
- Can be used to transform several vectors simultaneously

$$\mathrm{B}\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \qquad \mathrm{B}\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

Identity and Inverse

Identity matrix I:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Neutral element of matrix multiplication, i.e. for all M:
I M = M I = M

Inverse matrix $\mathbf{M}^{-1}$ of a square matrix M (does not always exist):

$\mathbf{M}\,\mathbf{M}^{-1} = \mathbf{M}^{-1}\,\mathbf{M} = \mathbf{I}$ and **$(\mathbf{M}^{-1})^{-1} = \mathbf{M}$**

Inverse of a 2x2 matrix:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{-1} = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

Exists only if $m_{11}\, m_{22} - m_{12}\, m_{21} \neq 0$

The Transpose Operation ^{T}

Make rows out of columns (or vice versa)

- Transpose of a row vector is a column vector (and vice versa)

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \mathbf{u}^T = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$$

- For matrix **M**:
  m_{ij} and m_{ji} are swapped
  for all $i=1..m$, $j=1..n$

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \Rightarrow \mathbf{M}^T = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \\ m_{13} & m_{23} \end{pmatrix}$$

Rules:

$(\mathbf{M}^T)^T = \mathbf{M}$

$(s\mathbf{M})^T = s(\mathbf{M}^T)$

$(\mathbf{M}+\mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$

$(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T$

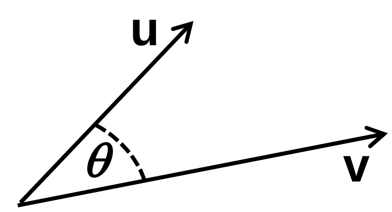
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}} \mathbf{v} = |\mathbf{u}| \; |\mathbf{v}| \; \cos(\theta)$$

THE DOT PRODUCT ·

The Dot Product •

Takes two vectors u and v, the result is a scalar (a single number) (therefore also known as scalar product)

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}u_1\\u_2\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}=u_1v_1+u_2v_2=\mathbf{u}^{\mathrm{T}}\mathbf{v}$$



$$=|\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta)$$ (θ is the angle between **u** and **v**)

Rules:

a•b=b•a (Symmetry)

(**a**+**b**)•**c**=**a**•**c**+**b**•**c** (Linearity)

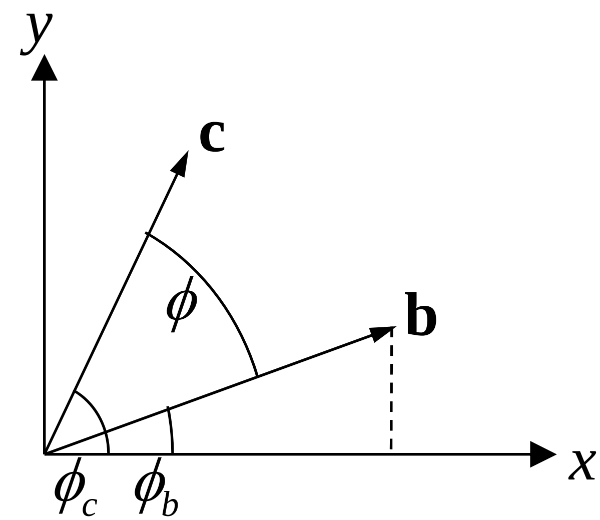
(s**a**)•**b**=s(**a**•**b**) (Homogeneity)

|**b**|^2= **b**•**b**

Example: |**a** - **b**|^2 = (**a** - **b**) • (**a** - **b**) = **a** • **a** - 2**a** • **b** + **b** • **b**

The Angle between Two Vectors

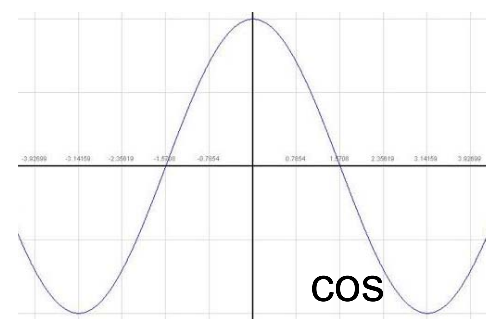
Most important application of dot product:
find angle between two vectors (or two intersecting lines)



$$\mathbf{b} = \begin{pmatrix} |b|\cos\phi_b \\ |b|\sin\phi_b \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} |c|\cos\phi_c \\ |c|\sin\phi_c \end{pmatrix}$$

hence

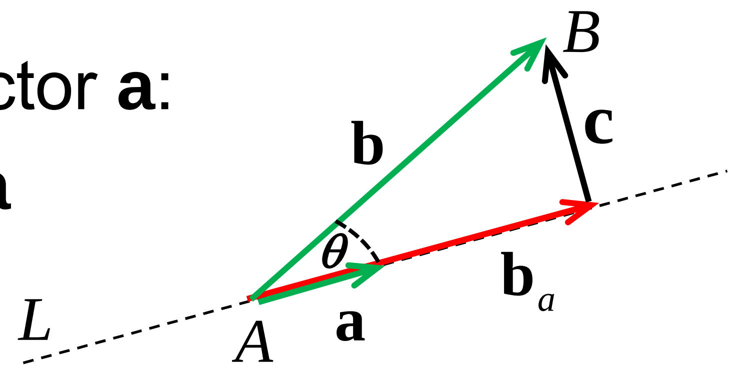
$$\begin{aligned} \mathbf{b} \bullet \mathbf{c} &= |b||c|\cos\phi_b \cos\phi_c + |b||c|\sin\phi_b \sin\phi_c \\ &= |b||c|\cos(\phi_c - \phi_b) \\ &= |b||c|\cos\phi \end{aligned}$$



Two non-zero vectors **b** and **c** with common start point are:

less than	90^{o} apart	if **b•c**>0	
exactly	90^{o} apart	if **b•c**=0	[**b** and **c** are *orthogonal* (*perpendicular*)]
more than	90^{o} apart	if **b•c**<0	

Orthogonal Projection of a Vector



Projecting a vector **b** onto another vector **a**:

- *L* is a line through *A* with direction **a**
- **b** is the vector from *A* to *B*

Given: **a** and **b**

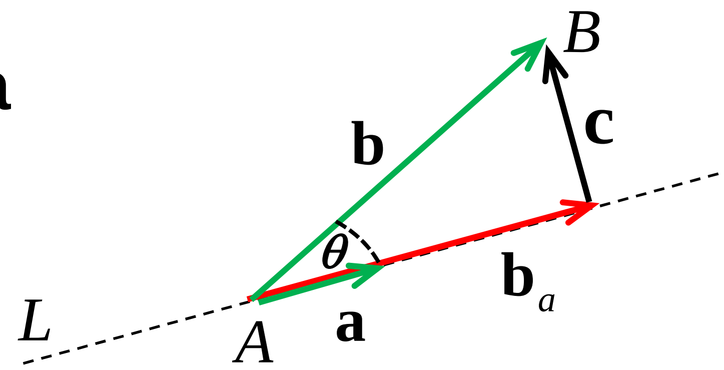
Wanted: $\mathbf{b_a}$ (orthogonal projection of **b** onto **a**)

Solution:

1. Length of $\mathbf{b_a}$: $$|\mathbf{b}_\mathbf{a}| = \cos(\theta)\ |\mathbf{b}| = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$ because of def. • $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos(\theta)$
2. Vector $\mathbf{b_a}$: $$\mathbf{b}_\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$$ because $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

Distance from a Line to a Point

- L is a line through A with direction **a**
- **b** is the vector from A to B



Given: **a** and **b**

Wanted: $|\mathbf{c}|$

Solution:

$$|\mathbf{c}| = |\mathbf{b} - \mathbf{b}_{\mathbf{a}}| = \left|\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\right|$$

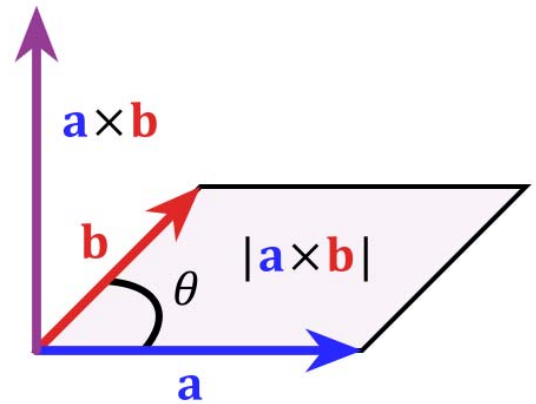
$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}=|\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

THE CROSS PRODUCT ×

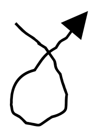
The Cross Product ×

Takes two 3D vectors u and v, the result is a 3D vector (therefore also known as vector product)

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}=|\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$



- θ is the angle between **a** and **b**
- **n** is unit normal vector (|**n**|=1) orthogonal to **a** and **b**
- *Hard to remember?* ⟹

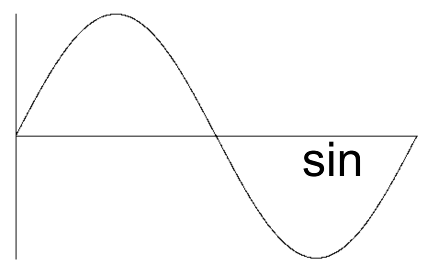


Rules:

(**a**+**b**)×**c** = **a**×**c** + **b**×**c** (Linearity)

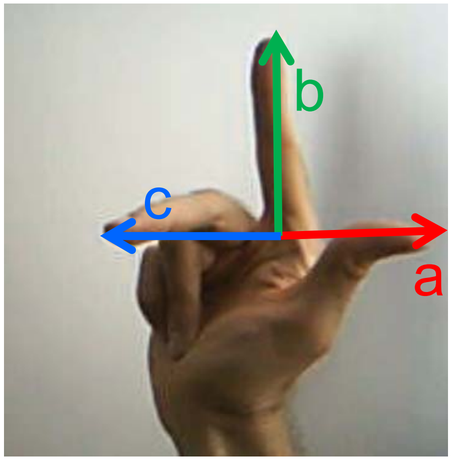
(s**a**)×**b** = s(**a**×**b**) (Homogeneity)

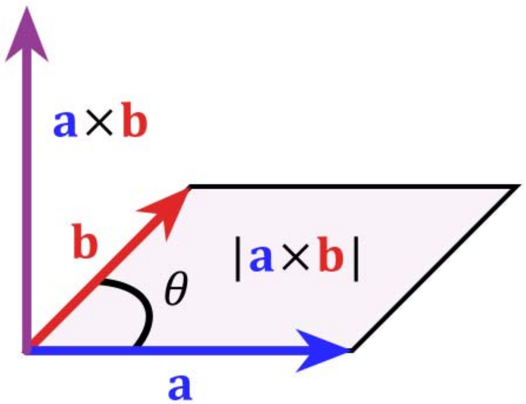
(no symmetry)

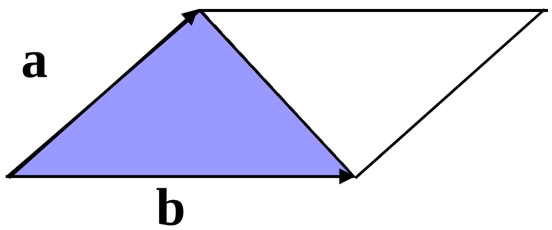


Properties of the Cross Product

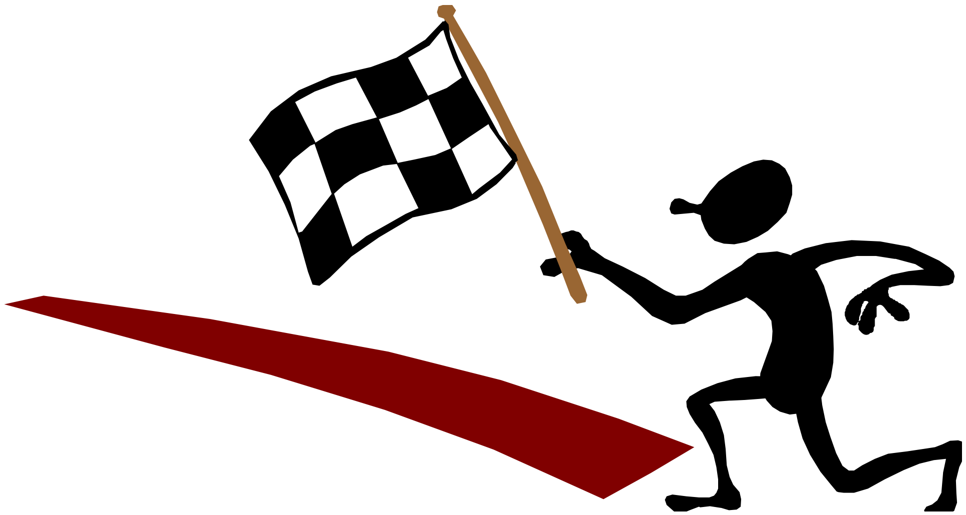
1. **a** × **b** is a vector perpendicular/orthogonal to both **a** and **b**
2. Direction of **a** × **b** is given by "right-hand rule"
3. **a** × **b** = – **b** × **a**
4. |**a** × **b**| is area of parallelogram defined by **a** and **b**
5. Area of triangle defined by **a** and **b** is 0.5 * |**a** × **b**|







$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$



SUMMARY

Summary

1. **Vectors**:
   addition, subtraction, scaling, magnitude, normalization
2. **Matrices**:
   addition, subtraction, scaling, multiplication
3. **Dot product**: $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta)$
4. **Cross product**:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

References:

- Vectors: Hill, Chapter 4.2
- Dot Product: Hill, Chapter 4.3
- Cross Product: Hill, Chapter 4.4

Quiz

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 3 & -1 \end{pmatrix}$$

1. Calculate: **a+b**, |**b**| **a**, **Ma**, **MN**, **a**·**b**, **a**×**b**
2. What can you tell about the angle between **a** and **b**?
3. What is the projection of **b** onto **a**?
4. What is the distance between the point given by **b** and the line going through the origin along **a**?