

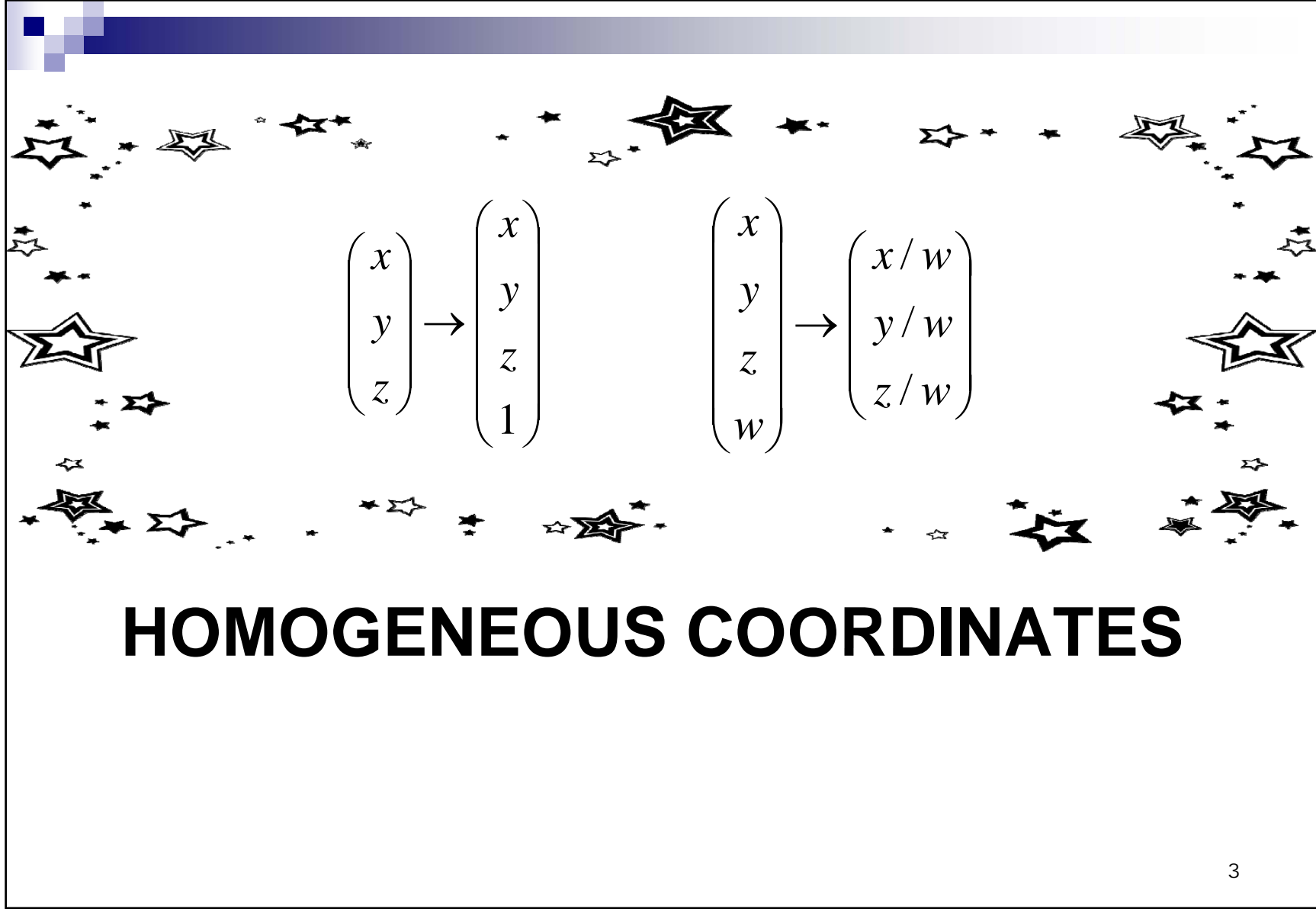
Computer Graphics and Image Processing Geometry III

Part 1 – Lecture 4



Today's Outline

- Homogeneous Coordinates
- 3D Affine Transformations
- Examples



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \rightarrow \begin{pmatrix} x/w \\ y/w \\ z/w \end{pmatrix}$$

HOMOGENEOUS COORDINATES

Homogeneous Coordinates

- Affine transformation \mathbf{F} consists of a linear transformation and a translation:

$$\mathbf{F}(\mathbf{p}) = \mathbf{M} \mathbf{p} + \mathbf{t}$$

- Can we get rid of the vector addition so that $\mathbf{F}(\mathbf{p}) = \mathbf{M} \mathbf{p}$ works even for translations?

Wanted: Representation of translations as matrices

Solution: Homogeneous Coordinates

- Add an additional coordinate to every vector, which is set to 1 if the vector can be translated

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Normally we just ignore the additional coordinate

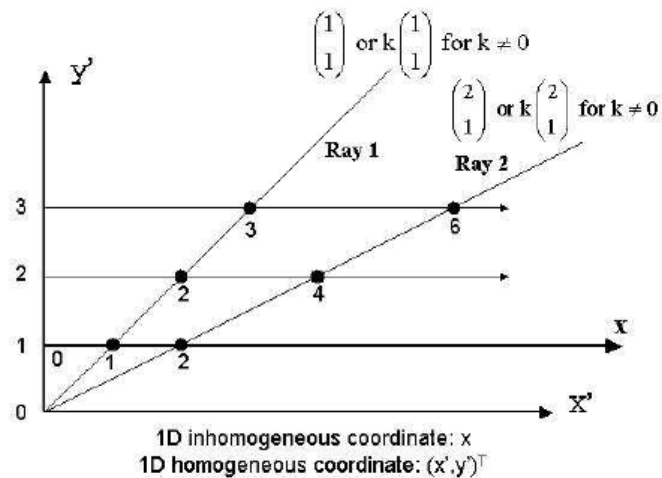
- Also add another row and column to our matrices, e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

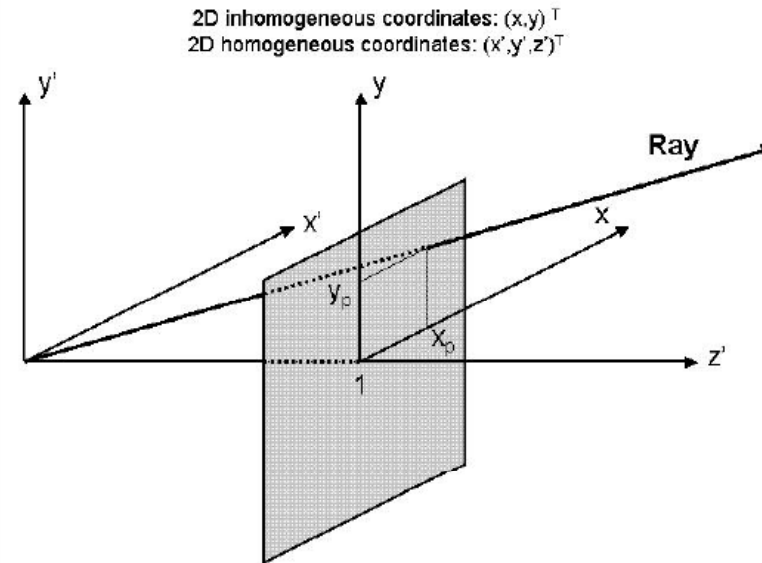
Homogeneous coordinate 1D-2D

In inhomogeneous 1D (cartesian) coordinates, a point is represented by a single value (for example, $x=1$).

In homogeneous coordinates, a 1D point is represented by a 2D vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ or $\begin{pmatrix} x' \\ 1 \end{pmatrix}$, which defines a ray:



In inhomogeneous 2D (cartesian) coordinates, a point is represented by a 2D vector $(x, y)^T$. In homogeneous coordinates, it is viewed as a 3D vector $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ or multiple of the vector $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$.



Using Homogeneous Coordinates

1. Every vector gets an additional coordinate with value 1 (if translations should affect the vector, otherwise 0)
2. Every matrix gets an additional row and column (0, ..., 0, 1)

For transformations other than translations: no difference

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \\ 1 \end{pmatrix}$$

Now we can formulate translations as matrices **T**:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ 0 \end{pmatrix} = \mathbf{T} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

Converting Coordinates

Ordinary to Homogeneous Coordinates:

- Just add another coordinate (often called w-coordinate)
- $w=1$ if translations are possible, $w=0$ if not

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Homogeneous to Ordinary Coordinates:

- Divide all other coordinates by w-coordinate (if $w \neq 0$)
- All homogenous 2D coordinate points $(w p_1, w p_2, w)^T$ with $w \neq 0$ represent the same ordinary coordinate point $(p_1, p_2)^T$
- Usually $w=1$, so conversion means just omitting the w-coordinate

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \rightarrow \begin{pmatrix} x/w \\ y/w \\ z/w \end{pmatrix}$$

Conversion example

- **Example:** the 3D point $(5, 3, 2)^T$ has the *homogeneous representation* $(5\lambda, 3\lambda, 2\lambda, \lambda)^T$ with arbitrary factor $\lambda \neq 0$, e.g., $(5, 3, 2, 1)^T$, or $(15, 9, 6, 3)^T$, or $(-55, -33, -22, -11)^T$ and so on.
- Conversely, the homogeneous vector $(30, 10, 15, 5)^T$ represents the point $(6, 2, 3)$.
- In homogeneous coordinates projective transformations as well as affine transformations (e.g. translations, rotations, scaling) are specified by linear equations.



3D AFFINE TRANSFORMATIONS

3D Affine Transformations

- Mostly analogous to 2D
- Now we use homogeneous coordinates, i.e. all transformations are represented as a matrix \mathbf{M} that can be left-multiplied: $\mathbf{M} \mathbf{v}$

Translation \mathbf{T} by a vector $\mathbf{t} = (t_x, t_y, t_z)^T$

- Similar to identity matrix
- Rightmost column contains \mathbf{t}

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling \mathbf{S} about origin with scaling factors s_x, s_y, s_z

- Similar to identity matrix
- Diagonal contains scaling factors
- A negative s_x, s_y or s_z causes reflection on the $x=0, y=0, \text{ or } z=0$ plane

$$\mathbf{S} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

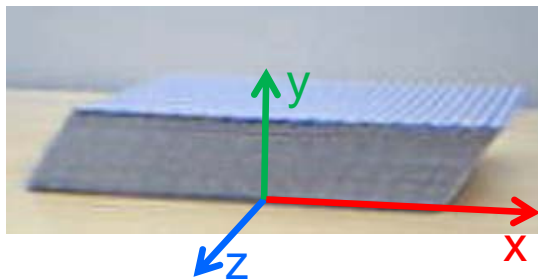
Shearing

General **shearing H**:

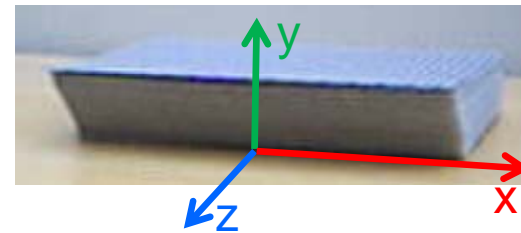
- Any coordinate (x/y/z) can linearly influence any other coordinate
- h_{yx} expresses how much y influences x

$$\mathbf{H} = \begin{pmatrix} 1 & h_{yx} & h_{zx} & 0 \\ h_{xy} & 1 & h_{zy} & 0 \\ h_{xz} & h_{yz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

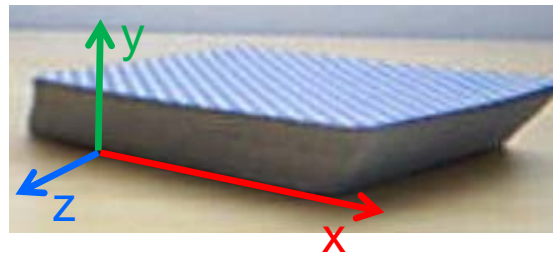
Examples:



$h_{yx} > 0$
and all
others 0



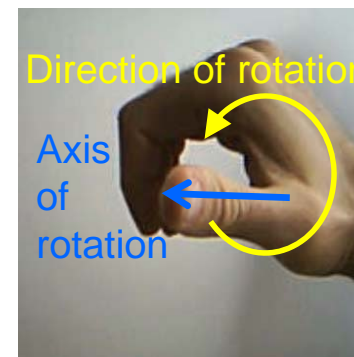
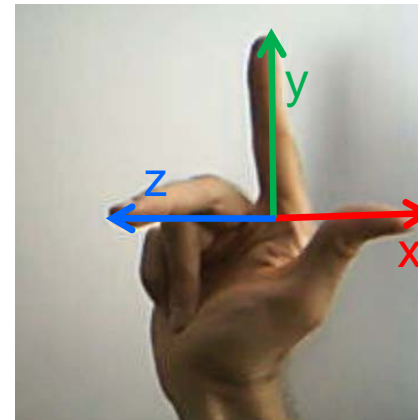
$h_{yz} > 0$
and all
others 0



Both together, i.e.
 $h_{yx} > 0$ and $h_{yz} > 0$
and all others 0

Rotation

- Rotations are the most difficult transformations
- We will consider three different rotation situations:
 1. Rotation around the three coordinate axes (x, y, z)
 2. Rotation to align an object with a new coordinate system
 3. Rotation around an arbitrary axis
- We use a right-handed coordinate system
- We use positive (right-handed) rotation, i.e. counterclockwise when looking into an axis



Rotation around Coordinate Axes

- Three axes (x, y, z) to rotate about, so three different matrices
- Let $C = \cos \theta$ and $S = \sin \theta$, then the matrices for positive (right handed) rotation are:

Rotation about x-axis: $\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C & -S & 0 \\ 0 & S & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Rotation about y-axis: $\mathbf{R}_y = \begin{pmatrix} C & 0 & S & 0 \\ 0 & 1 & 0 & 0 \\ -S & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Rotation about z-axis: $\mathbf{R}_z = \begin{pmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note on 3 × 3 rotation matrices:

Row and column corresponding to axis of rotation are as for identity **I**

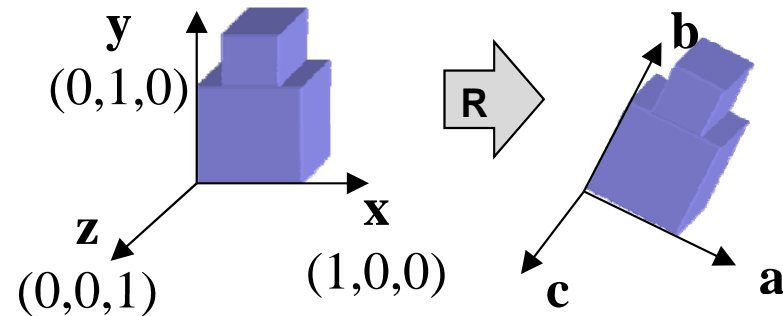
Other elements are C on diagonal, $\pm S$ off diagonal, so that $\mathbf{R} = \mathbf{I}$ if $\theta = 0$.

Sign of S can be inferred from the fact that rotation around x,y,z by $\theta=90^\circ$ transforms $y \rightarrow z$, $z \rightarrow x$, $x \rightarrow y$, respectively.

Rotating to Align with New Coordinate Axes

Wanted: matrix \mathbf{R} that rotates the coordinate system to align with a new coordinate system $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ with the same origin

- $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are the unit vectors of our normal coordinate system
- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors along the axes of the new system



Solution:

\mathbf{R} should do the following:

$$\mathbf{R} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T = \mathbf{a}$$

$$\mathbf{R} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T = \mathbf{b}$$

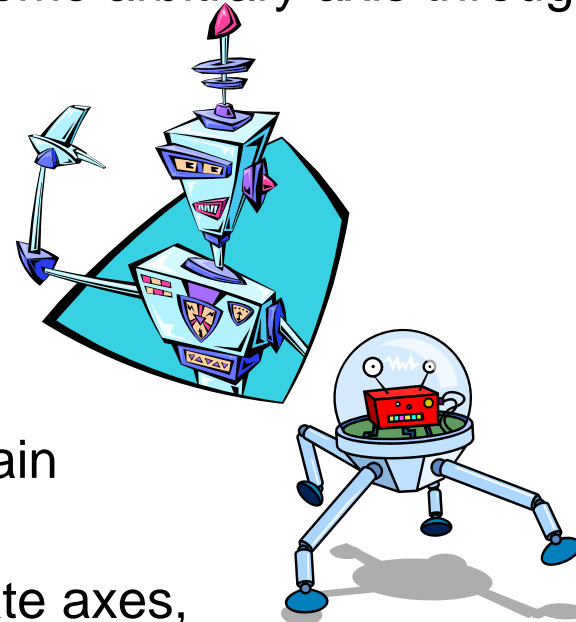
$$\mathbf{R} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T = \mathbf{c}$$

Using homogeneous coordinates:

$$\mathbf{R} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x & 0 \\ a_y & b_y & c_y & 0 \\ a_z & b_z & c_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

Rotation about an Arbitrary Axis

- Often need to rotate an object about some arbitrary axis through a reference point on it
- E.g. forearm of robot rotating around an axis through the elbow
- Involves three steps:
 1. Translate reference point to origin
 2. Do the rotation
 3. Translate reference point back again
- Translation is easy (steps 1 and 3)
- We know how to rotate about coordinate axes, but how about an arbitrary axis through the origin?
 1. Textbook method: decompose rotation into primitive rotations about x,y and z axes
 2. Coordinate system alignment method

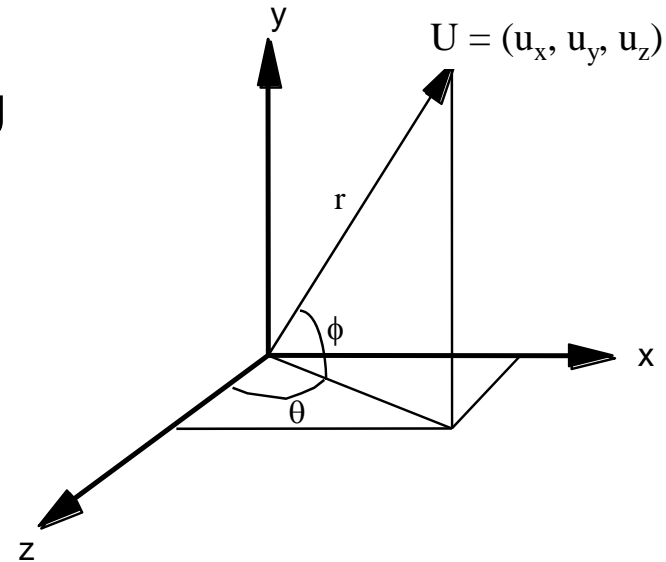


Arbitrary Axis Rotation: Textbook

1. Rotate the object so that the required axis of rotation \mathbf{r} lies along the z-axis ($\mathbf{R}_{\text{alignZ}}$)
2. Do the rotation about z-axis
3. Undo original rotation ($\mathbf{R}_{\text{alignZ}}^{-1}$)

How to get $\mathbf{R}_{\text{alignZ}}$?

1. Measure azimuth θ as a right handed rotation about the y-axis, starting at the z-axis
2. Measure elevation ϕ (or "latitude") as angle above plane $y=0$
3. $\mathbf{R}_{\text{alignZ}} = \mathbf{R}_x(\phi) \mathbf{R}_y(-\theta)$



$$\phi = \tan^{-1} \frac{u_y}{\sqrt{u_x^2 + u_z^2}}$$

$$\theta = \text{atan2}(u_x, u_z)$$

(i.e. a four quadrant $\tan^{-1} \frac{u_x}{u_z}$)

Arbitrary Axis Rotation: Alignment

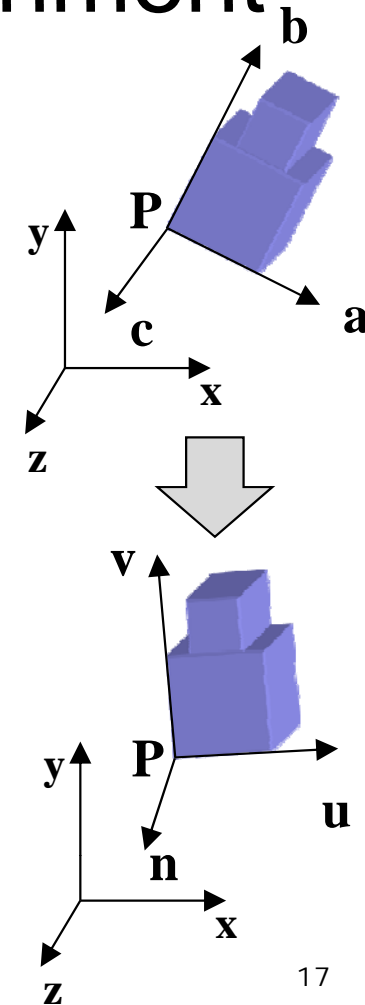
Given:

- Coordinate system **a,b,c** attached to the object we want to rotate
- Position **P** of the object's coordinate system
- New coordinate system **u,v,n** to rotate object to

Solution:

1. Translate object to origin (\mathbf{T}_P^{-1})
2. Rotate **a,b,c** to align with world coord. axes (inverse of the "rotate to align" case: \mathbf{R}_{abc}^{-1})
3. Rotate coord. axes to align with **u,v,n** (\mathbf{R}_{uvn})
4. Translate object back to original position (\mathbf{T}_P)

Full matrix is: $\mathbf{T}_P \mathbf{R}_{uvn} \mathbf{R}_{abc}^{-1} \mathbf{T}_P^{-1}$



The Inverse of a Rotation Matrix

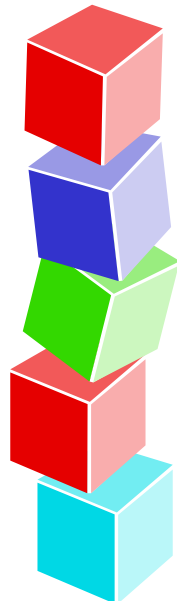
- Columns of a rotation matrix are unit vectors along the rotated coordinate axis directions
- So columns are orthogonal, i.e. dot products = 0

$$\begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}^T \mathbf{R} = \mathbf{I}$$

$$\text{therefore } \mathbf{R}^{-1} = \mathbf{R}^T$$

- The inverse of a rotation matrix is its transpose (matrices with this property are called *orthogonal*)



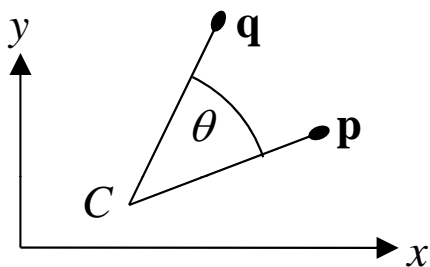
EXAMPLES

Composition of Transformations

- All transformations can be represented as matrix \mathbf{M}
- Combine several transformations into a single matrix by multiplying all transformation matrixes: $\mathbf{M}_n \mathbf{M}_{n-1} \dots \mathbf{M}_1 = \mathbf{M}$
- Transformation of rightmost matrix is applied first (i.e. \mathbf{M}_1)

Example: rotating an object about its centre point C

1. Translate the object so that its centre is at the origin
2. Rotate about the origin
3. Translate object back to its original position



$$(q_1 \ q_2 \ 1)^T = \mathbf{M}(p_1 \ p_2 \ 1)^T$$

where

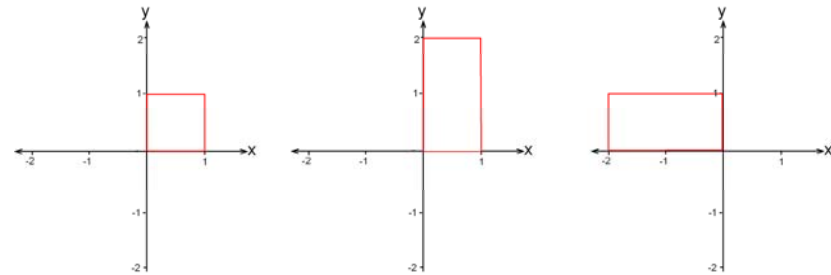
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

The Order of Transformations Matters!

In general affine transformations do not commute, i.e. $\mathbf{M N} \neq \mathbf{N M}$

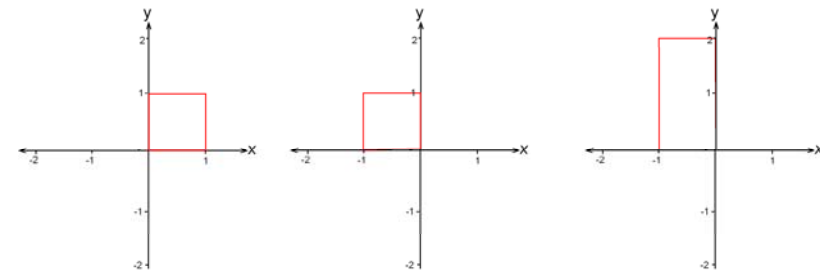
a) First scale by (1,2), then rotate 90°

$$\mathbf{M} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



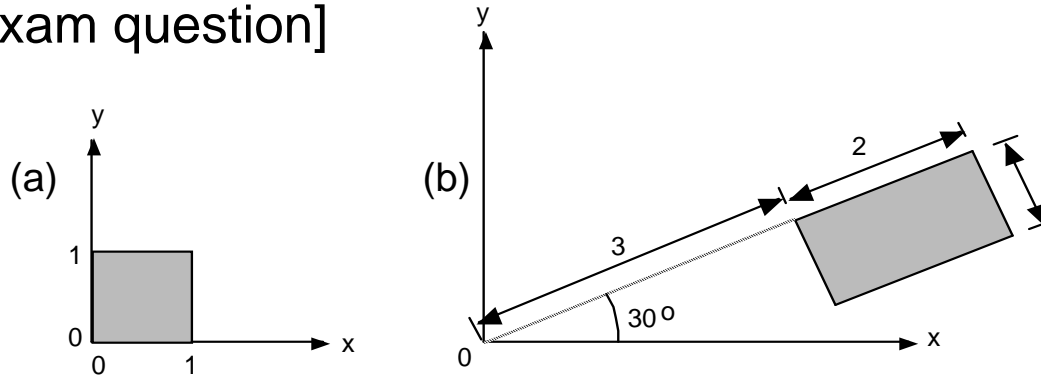
b) First rotate 90° then scale by (1,2)

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Exam Question 1

Which homogeneous 2D matrix \mathbf{M} transforms (a) to (b)?
[1996 exam question]

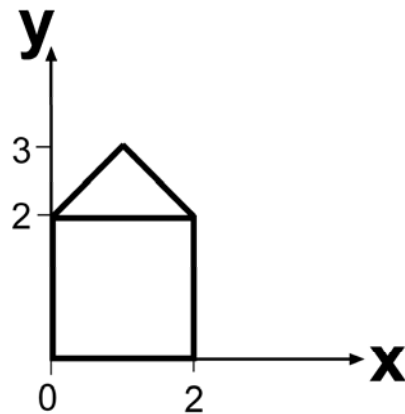


- Sometimes easier to do these backwards, then take inverse, i.e. starting with figure (b):
Rotate -30° , shift by $(-3, 1)$, scale by $(1/2, 1)$
- Hence the required transformation is:
 $R(30^\circ) T(3, -1) S(2, 1)$
(first scaling, then translation, finally rotation)
- Don't forget to use homogeneous matrices

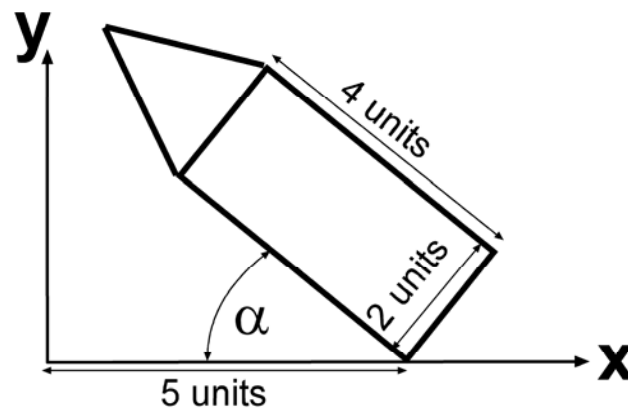
Exam Question 2

Which homogeneous 2D matrix \mathbf{M} transforms (a) to (b)?
You are allowed to write \mathbf{M} as a product of simpler matrices
(i.e. you need not multiply the matrices) [2003 exam question]

(a)



(b)





SUMMARY

Summary

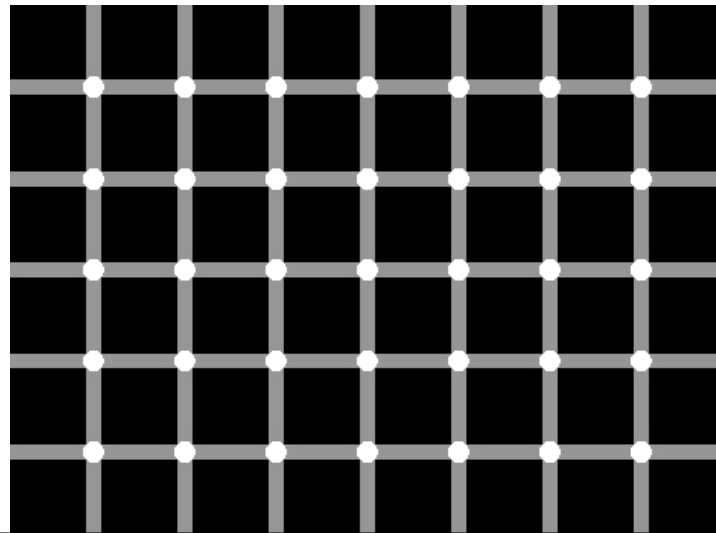
1. **Homogeneous coordinates** make it possible to represent translation as a matrix
2. **3D Affine Transformations** similar to 2D: translation, scaling, shearing and rotation
 - Column vectors of rotation matrix \mathbf{R} are axis unit vectors of new coord. system to align x,y,z with
 - $\mathbf{R}^{-1} = \mathbf{R}^T$
3. Transformations are applied from right to left

References:

- Homogeneous Coordinates: Hill, Chapter 4.5.1
- 3D Affine Transformations: Hill, Chapter 5.3

Quiz

1. An object has a local coordinate system $\mathbf{a}=(1,0,0)$, $\mathbf{b}=(0,0,-1)$, $\mathbf{c}=(0,1,0)$ at position $(-10,2,5)$. Which homogeneous matrix rotates the object into the new coordinate system $\mathbf{u}=(0,-1,0)$, $\mathbf{v}=(0,0,-1)$, $\mathbf{n}=(1,0,0)$?
2. Solve the sample exam questions 1 and 2.
3. Create your own variation of the exam questions and solve it.



Count the
black dots!