Some Examples, Questions, and Answers: Lecture 8 – COMPSCI 369

Georgy Gimel'farb

Constrained Optimisation: The Method of Lagrange

Example 8.1: Minimise $f(x,y,z) = x^2 + 2y^2 + 3z^2$ subject to $2x + 4y + 6z = 12$.

Each stationary point (x^*, y^*, z^*) of the Lagrangian $F(x,y,z,\lambda) = x^2 + 2y^2 + 3z^2 - \lambda(2x + 4y + 6z - 1)$ is the root of its gradient, $\nabla F(x,y,z,\lambda) = \mathbf{0}$, i.e.

$$\begin{bmatrix} 2x - 2\lambda \\ 4y - 4\lambda \\ 6z - 6\lambda \\ -2x - 4y - 6z + 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ or } \left\{ \begin{array}{rcl} x & = & \lambda \\ y & = & \lambda \\ z & = & \lambda \\ -12\lambda & = & 0 \end{array} \right.$$

or $\lambda^* = 1$, and $x^* = y^* = z^* = 1$. The Lagrangian for this quadratic function with a linear constraint has one stationary point: $F(1,1,1) = 6$, corresponding to the constrained minimum $f(1,1) = 6$.

Example 8.2: Minimize $f(x,y,z) = x^2 + 2y^2 + 3z^2$ subject to $2x + 4y + 6z = 12$ and $x - y - z = c$.

The Lagrangian $F(x,y,z,\lambda_1,\lambda_2) = x^2+2y^2+3z^2-\lambda_1(2x+4y+6z-12)-\lambda_2(x-y-z-6)$. yields the system of equations for the roots of its gradient, $\nabla F(x,y,z,\lambda_1,\lambda_2) = \mathbf{0}$:

$$\left\{ \begin{array}{rcl} x & = & \lambda_1 + \lambda_2 \\ y & = & \lambda_1 - \lambda_2 \\ z & = & \lambda_1 - \lambda_2 \\ 2x + 4y + 6z & = & 12 \\ x - y - z & = & 6 \end{array} \right. \text{ or } \left\{ \begin{array}{rcl} 12\lambda_1 - 8\lambda_2 & = & 12 \\ -\lambda_1 + 3\lambda_2 & = & 6 \end{array} \right., \text{ or } \left\{ \begin{array}{rcl} \lambda_1^* & = & 3 \\ \lambda_2^* & = & 3 \\ x^* & = & 6 \\ y^* & = & 0 \\ z^* & = & 0 \end{array} \right.$$

Therefore, the Lagrangian for this quadratic function with two linear constraints has one stationary point: $F(6,0,0,3,3) = 36$, corresponding to the constrained minimum $f(6,0,0) = 36$.

Question 8.1: Minimise $f(x,y) = x^4 + y^4$ subject to $x^2y^2 = 1$.

Question 8.2: State and solve the dual (to Question 8.1) optimisation problem.

Nonlinear Systems of Equations

Example 8.3: Search for a fixed point of the nonlinear system $\mathbf{g}(\mathbf{u})$ of two equations with two unknowns $\mathbf{u} = [x, y]^\mathsf{T}$:

$$\mathbf{g}(\mathbf{u}) = \left[\begin{array}{lclcl} v(x,y) & = & x^2 + xy + 2y^2 - 4 & = & 0 \\ w(x,y) & = & 2x^2 - 3xy + 3y^2 - 2 & = & 0 \end{array} \right]$$

with the Newton-Raphson method that starts from the point $\mathbf{u}_0 = [x_0, y_0]^\mathsf{T}$.

Linear approximation of this system around $\mathbf{u}_n = [x_n, y_n]^\mathsf{T}$ involves the Jacobian 2×2 matrix of the first derivatives:

$$\mathbf{J}(\mathbf{u}_n) = \left[\begin{array}{ll} \frac{\partial}{\partial x} v(x_n, y_n) = 2x_n + y_n & \frac{\partial}{\partial y} v(x_n, y_n) = x_n + 4y_n \\ \frac{\partial}{\partial x} w(x_n, y_n) = 4x_n - 3y_n & \frac{\partial}{\partial y} w(x_n, y_n) = -3x_n + 6y_n \end{array} \right]$$

Setting, at each iteration $n = 0, 1, \ldots$, this linear approximation to zero (see lecture notes `CS369-)ptimisation-2.pdf`, Slide 13) results in the following sequence of steps towards the goal solution:

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array} \right] = \mathbf{u}_n - \mathbf{J}^{-1}(\mathbf{u}_n)\mathbf{g}(\mathbf{u}_n) \equiv \left[\begin{array}{c} x_n \\ y_n \end{array} \right] - \left[\begin{array}{ll} 2x_n + y_n & x_n + 4y_n \\ 4x_n - 3y_n & -3x_n + 6y_n \end{array} \right]^{-1} \left[\begin{array}{c} v(x_n, y_n) \\ w(x_n, y_n) \end{array} \right]$$

Answers

To Question 8.1: Lagrangian $F(x, y, \lambda) = x^4 + y^4 - \lambda(x^2y^2 - 1)$ yields the system of equations for the roots of the gradient $\nabla F(x, y, \lambda) = \mathbf{0}$:

$$\left\{ \begin{array}{rcl} 4x^3 - 2\lambda xy^2 & = & 0 \\ 4y^3 - 2\lambda xy^2 & = & 0 \\ x^2y^2 & = & 1 \end{array} \right.$$

Because under this constraint, the goal x and y are non-zero, $x \neq 0$ and $y \neq 0$, the two above equations are reduced to $2x^2 = \lambda y^2$ and $2y^2 = \lambda x^2$, so that $\lambda^* = 2$; $x^ast = \pm 1$, $y^* = \pm 1$, and $f(\pm 1, \pm 1) = F(\pm 1, \pm 1, 2) = 2$.

To Question 8.2: The dual to Question 8.1 problem is as follows: Minimise the function

$$F(x, y, \lambda) = x^4 + y^4 - \lambda(x^2y^2 - 1)$$

by λ, subject to the system of constraints: $x \neq 0$, $y \neq 0$, and

$$\left\{ \begin{array}{l} 4x^3 - 2\lambda xy^2 = 0; \\ 4y^3 - 2\lambda x^2 y = 0 \end{array} \right.$$

These constraints yield $\lambda^* = 2$ and $x^* = y^*$. Then the root of the first derivative,

$$\frac{d}{d\lambda} F(x^*, y^*, \lambda) = -(x^*)^2(y^*)^2 + 1 = 0,$$

results in $x^* = y^* = \pm 1$.