COMPSCI 320SC 2004 Midterm Exam

Attempt *all* questions. Put the answers in the space below the questions. Write clearly! You may continue your answers onto the "overflow" page provided at the end of the test, if necessary.

Marks for each question are shown below and just before each answer box.

Use of calculators is NOT permitted.

This test is worth 15% of your final grade for the course.

Question #:	1	2	3	4	5	6	7	8	9	Total
Possible marks:	13	10	10	10	12	13	11	10	11	100
Awarded marks:										

University ID: ______________________

Student Name: ______________________

Student Signature: ______________________

Time Finished: ______________________

1. Prove the following statements relating to the asymptotic $Theta$ notation. All the functions involved take positive values on the natural numbers $n \geq 0$.

   (a) Let $f(n) = \log_2 \log_2 n$ and $g(n) = \log_2 n$. Prove that $f(n)$ is in $O(g(n))$ but not in $\Theta(g(n))$.

   (*4* ***marks***)

   Hint: Use the limit rule.

   (b) Give a direct proof that $f(n) = (0.999)^n + 0.001$ is in $\Theta(1)$ using the definition of Θ notation.

   (*3* ***marks***)

   (c) Prove formally that Θ notation is reflexive: $f(n)$ is in $\Theta(f(n))$

   (*3* ***marks***)

   (d) Prove formally that Θ notation is transitive: if $f(n)$ is in $\Theta(g(n))$ and $g(n)$ is in $\Theta(h(n))$, then $f(n)$ is in $\Theta(h(n))$

   (*3* ***marks***)

2. A function $f(n)$ is eventually nondecreasing if $f(n) \leq f(n+1)$ for $n \geq n_0$. A nondecreasing function $f(n)$ is b-smooth is $f(bn) \leq cf(n)$ for some constant $c > 0$ and $n \geq n_0$. A function is smooth if it is b-smooth for every integer $b \geq 2$. Using these definitions, solve the following problems.

   (a) Show that the function $f(n) = n \log_2 n$ is smooth.

   (***6 marks***)

   (b) Show that the function $f(n) = 3^n$ is not smooth.

   (***4 marks***)

3. Solve the recurrence $T(n) = \{T(\lfloor n/2 \rfloor) \text{ or } T(\lceil n/2 \rceil)\} + 1;\ \ T(1) = 0$ that specifies the time of a binary search in an ordered array of a size n where n is any natural number and find the function $f(n)$ such that $T(n) \in \Theta(f(n))$, that is, $T(n)$ is in the exact order of $f(n)$.

(***10 marks***)

Hint: use the conditional asymptotic notation and find out how to justify that the smoothness rule can be applied.

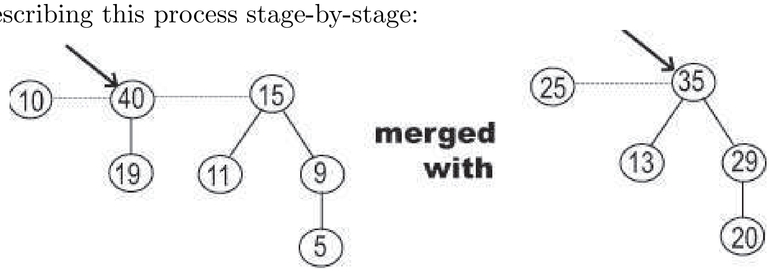
4. Solve the following homogeneous recurrence exactly by using the technique of the characteristic equation:

$$t_n = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1 \\ 3t_{n-1} - 2t_{n-2} & \text{otherwise} \end{cases}$$

Express your answer as simple as possible using the Θ notation.

(***10 marks***)

5. A binomial tree is defined recursively. The i-th binomial tree, B_i, $i \geq 0$ consists of a root node with i children, where the j-th child, $1 \leq j \leq i$, is in turn the root of a binomial tree B_{j-1}. A binomial heap consists of a collection of binomial trees arranged in increasing order of size. Describe basic stages of merging two binomial heaps and illustrate these stages by merging the two binomial heaps below and describing this process stage-by-stage:



(***12 marks***)

6. A single server has n customers to serve one-after-one, each customer i requiring the known individual time t_i, $1 \leq i \leq n$ for the service. If the customers are scheduled in the order $1, 2, \ldots, n$, then customer 1 is served immediately (that is, spends the time t_1 to get his service), customer 2 waits while customer 1 is served and then gets his turn (that is, he spends in total the time $t_1 + t_2$ to get his service), and so on, so that customer i waits while all the preceding customers j, $1 \leq j \leq i - 1$, are served and spends in total the time $t_1 + t_2 + \ldots + t_i$ to get his service.

   Formulate the greedy algorithm that builds the servicing schedule such that the total time spent by all the n customers to get their service is minimised and prove that the proposed algorithm is optimal.

   (***13 marks***)

7. (a) Illustrate the linear time algorithm for finding the median of an unsorted array on the input:

4	5	3	2	1	1	3	4	7	8	9	4	7	2	1	2	7	7	9

(***6 marks***)

(b) Give and explain the divide-and-conquer requrence equation for this algoirthm.

(***5 marks***)

8. Consider any integers x, y, and z such that z is positive. Prove the following:

$$xy \bmod z = [(x \bmod z)(y \bmod z)] \bmod z$$

(***5 marks***)

and

$$(x \bmod z)^y \bmod z = x^y \bmod z$$

(***5 marks***)

Hint: write x as $qz + r$, where $r = x \bmod z$ and $q = x \div z$.

9. Design an algorithm that will efficiently compute the following recursively defined function. Your algorithm should compute $f(n)$ in time $O(\log n)$. Don't worry about integer overflow.

   Hint: use the 'getting nasty' matrix method for computing the n-th Perrin number.

$$f(1) = 1, f(2) = 0, f(3) = -4, f(4) = 3, \text{ and}$$
$$f(n) = f(n-1) - 2f(n-2) + 3f(n-4) \text{ for } n \geq 5.$$

(***11 marks***)

Overflow Working Space

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