Fundamental Matrix / Image Rectification

COMPSCI 773 S1 T

VISION GUIDED CONTROL

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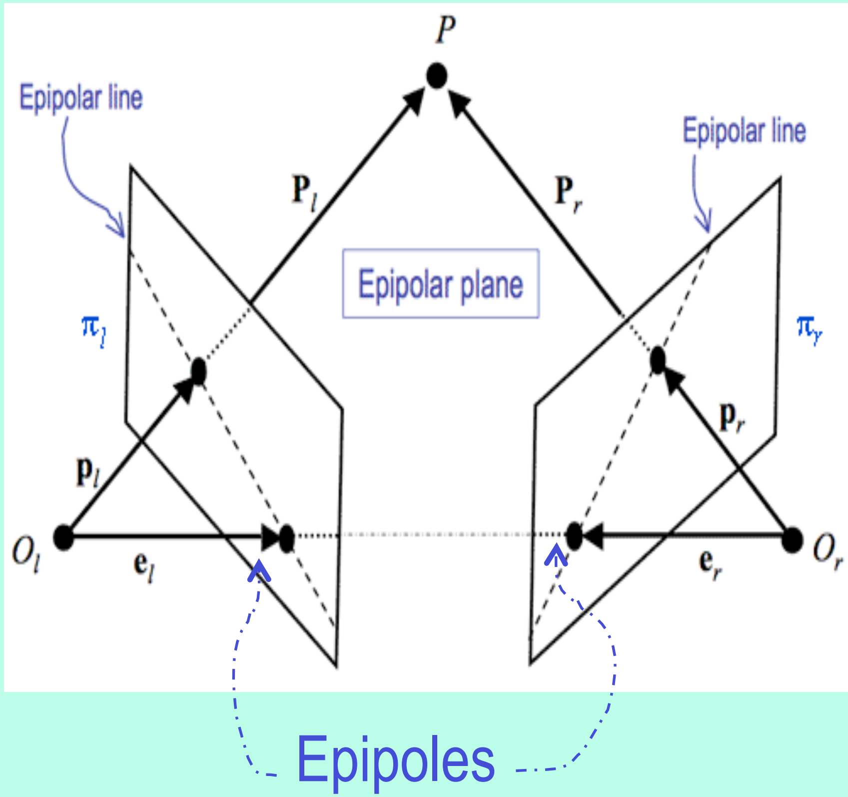
Epipolar Geometry

- O_l, O_r - projection centres
  - Origins of the reference frames
  - f_l, f_r - focal lengths of cameras
- $\pi_l,\ \pi_r$ - image planes
  - 3D reference frame for each camera: Z-axis = the optical axis

$\mathbf{P}_l=[X_l,Y_l,Z_l]^\mathsf{T}$, $\mathbf{P}_r=[X_r,Y_r,Z_r]^\mathsf{T}$ - the same 3D point P in the reference frames

$\mathbf{p}_l=[x_l, y_l, z_l=f_l]^\mathsf{T}$, $\mathbf{p}_r=[x_r, y_r, z_r=f_r]^\mathsf{T}$ - projections of P onto the image planes



Basics of Epipolar Geometry

- Reference frames of the left and right cameras - related via the **extrinsic parameters**
  - Translation vector $\mathbf{T} = (O_r - O_l)$ and a rotation matrix R defining a **rigid transformation** in 3-D space, given a 3-D point P, between $\mathbf{P}_l$ and $\mathbf{P}_r$: $\mathbf{P}_r = R(\mathbf{P}_l - \mathbf{T})$
- Epipoles $\mathbf{e}_l$ and $\mathbf{e}_r$ - the points at which the line through the centres of projection intersects the image planes
  - Left epipole - the image of the right projection centre
  - Right epipole - the image of the left projection centre
  - Canonical geometry: the epipole is at infinity of the baseline

Basics of Epipolar Geometry

- 3-D point $\mathbf{P} = [X, Y, Z]^\mathsf{T} \Leftrightarrow$ its projections $\mathbf{p}_l$ and $\mathbf{p}_r$:

$$\mathbf{p}_l = \frac{f_l \mathbf{P}_l}{Z_l}; \quad \mathbf{p}_r = \frac{f_r \mathbf{P}_r}{Z_r}$$

- **Epipolar plane**: the plane through P, O_l, and O_r
  - **Epipolar line**: its intersection with each image plane
  - **Conjugated lines**: both the lines for an epipolar plane
  - Given $\mathbf{p}_l$, the 3-D point P can lie anywhere on the ray $\mathbf{p}_l O_l$ depicted by the epipolar line through the corresponding $\mathbf{p}_r$
  - **Epipolar constraint**: the true match lies on the epipolar line

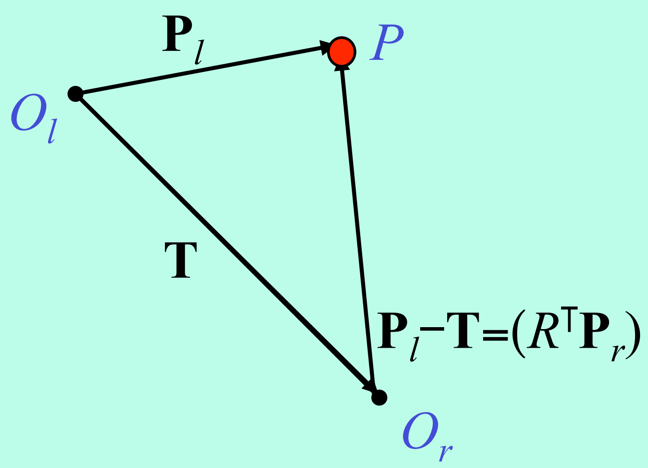
Basics of Epipolar Geometry

- All epipolar lines go through the epipole
  - With the exception of the epipole, only one epipolar line goes through any image point
  - Mapping between points on one image and corresponding epipolar lines on the other image $\Rightarrow$ the 1-D search region
  - Rejection of false matches due to occlusions
  - Corresponding points must lie on conjugated epipolar lines
- The obvious question: how to estimate the epipolar geometry, i.e. determine the ‘point-to-line’ mapping for images

Vector product $\mathbf{a}\times\mathbf{b} = \mathbf{n}\,|\mathbf{a}||\mathbf{b}|\sin\theta$ where the unit vector $\mathbf{n}$ is perpendicular to and θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$

The Essential Matrix, E

- Determining the mapping between points in one image and epipolar lines in the other image:
  - The equation of the epipolar plane through a 3-D point P as the **co-planarity** of the vectors $\mathbf{P}_l$, $\mathbf{T}$, and $\mathbf{P}_l-\mathbf{T}$:



$$(\mathbf{P}_l-\mathbf{T})^\mathsf{T}(\mathbf{T}\times\mathbf{P}_l)=0$$

$$\Rightarrow (R^\mathsf{T}\mathbf{P}_r)^\mathsf{T}(\mathbf{T}\times\mathbf{P}_l)=0 \Rightarrow \mathbf{P}_r^\mathsf{T}R(\mathbf{T}\times\mathbf{P}_l)=0$$

$$\Rightarrow \mathbf{T}\times\mathbf{P}_l \equiv \underbrace{\begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}}_{\text{Matrix } S \text{ of rank 2}}\mathbf{P}_l \Rightarrow \mathbf{P}_r^\mathsf{T}\underbrace{(RS)}_{\substack{\text{Matrix } E \\ \text{of rank 2}}}\mathbf{P}_l$$

Full rank matrix

Essential matrix

The Essential Matrix, E

- By construction, the matrix S (and thus E) are of rank 2
- The essential matrix gives a natural link between the epipolar constraint and the extrinsic parameters of the stereo system:

$$\mathbf{P}_l = \frac{Z_l \mathbf{p}_l}{f_l}; \quad \mathbf{P}_r = \frac{Z_r \mathbf{p}_r}{f_r} \quad \Rightarrow \quad \frac{Z_l Z_r}{f_l f_r} \mathbf{p}_r^{\mathsf{T}} E \mathbf{p}_l = 0 \quad \Rightarrow \quad \boxed{\mathbf{p}_r^{\mathsf{T}} E \mathbf{p}_l = 0}$$

Matrix E: the mapping between the points and epipolar lines

- Vector $\mathbf{a}_r = E\mathbf{p}_l \rightarrow$ parameters of the epipolar line $\mathbf{p}_r^{\mathsf{T}}\mathbf{a}_r = 0$ in the right image corresponding to the point $\mathbf{p}_l$ in the left image
- Vector $\mathbf{a}_l^{\mathsf{T}} = \mathbf{p}_r^{\mathsf{T}} E \rightarrow$ parameters of the epipolar line $\mathbf{a}_l^{\mathsf{T}}\mathbf{p}_l = 0$ in the left image corresponding to the point $\mathbf{p}_r$ in the right image

The Fundamental Matrix, F

- The mapping "points $\Leftrightarrow$ epipolar lines" can be obtained from corresponding points only
  - No prior information on the stereo system!
- Points $\bar{\mathbf{p}}_l$, $\bar{\mathbf{p}}_r$ in pixel and $\mathbf{p}_l$, $\mathbf{p}_r$ in camera coordinates:

$$\bar{\mathbf{p}}_l \equiv \begin{bmatrix} \bar{x}_l \\ \bar{y}_l \\ 1 \end{bmatrix} = M_l \mathbf{p}_l; \quad \bar{\mathbf{p}}_r \equiv \begin{bmatrix} \bar{x}_r \\ \bar{y}_r \\ 1 \end{bmatrix} = M_r \mathbf{p}_r \quad \Leftrightarrow \quad \mathbf{p}_l = M_l^{-1} \bar{\mathbf{p}}_l; \quad \mathbf{p}_r = M_r^{-1} \bar{\mathbf{p}}_r$$

$$\Rightarrow \quad \bar{\mathbf{p}}_r^{\top} \underbrace{M_r^{-\top} E M_l^{-1}}_{\text{Fundamental matrix } F} \bar{\mathbf{p}}_l \quad \Rightarrow \quad \bar{\mathbf{p}}_r^{\top} F \bar{\mathbf{p}}_l$$

M_l and M_r - matrices of the intrinsic camera parameters

The Fundamental Matrix, F

- Matrix F - the "pixels - epipolar lines" mapping:
  - Vector $\mathbf{a}_r = F\bar{\mathbf{p}}_l \rightarrow$ parameters of the epipolar line $\bar{\mathbf{p}}_r^\top \mathbf{a}_r = 0$ in the right image related to the pixel $\bar{\mathbf{p}}_l$ in the left image
  - Vector $\mathbf{a}_l^\top = \bar{\mathbf{p}}_r^\top F \rightarrow$ parameters of the epipolar line $\mathbf{a}_l^\top \bar{\mathbf{p}}_l = 0$ in the left image related to the pixel $\bar{\mathbf{p}}_r$ in the right image
  - Just as the matrix E, the fundamental matrix F has rank 2
  - F accounts for both the intrinsic and extrinsic parameters
- The epipolar constraint can be established with no prior knowledge of the stereo parameters!

The Eight-point Algorithm

- $n \geq 8$ corresponding points in the images are known
  - Each correspondence i - a homogeneous linear equation:

$$\overline{\mathbf{p}}_{r,i}^{\top} F \overline{\mathbf{p}}_{l,i} = 0 \Rightarrow \begin{bmatrix} \bar{x}_{r,i} & \bar{y}_{r,i} & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \bar{x}_{l,i} \\ \bar{y}_{l,i} \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \bar{x}_{r,i}\bar{x}_{l,i}F_{11} + \bar{x}_{r,i}\bar{y}_{l,i}F_{12} + \bar{x}_{r,i}F_{13} + \bar{y}_{r,i}\bar{x}_{l,i}F_{21} + \bar{y}_{r,i}\bar{y}_{l,i}F_{22} + \bar{y}_{r,i}F_{23} + \bar{x}_{l,i}F_{31} + \bar{y}_{l,i}F_{32} + F_{33} = 0$$

  - If the n points do not form a degenerate configuration, the 9 entries of F are given by the non-trivial solution of this homogeneous linear system

The Eight-point Algorithm

- Since the system is homogeneous, the solution is unique up to a signed scaling factor
- Typically, $n > 8$, so that the system is over-determined, and its solution is obtained by singular value decomposition (SVD) related techniques

- A - the system's matrix $n \times 9$:

$$A = \begin{bmatrix} \bar{x}_{r,1}\bar{x}_{l,1} & \bar{x}_{r,1}\bar{y}_{l,1} & \bar{x}_{r,1} & \bar{y}_{r,1}\bar{x}_{l,1} & \bar{y}_{r,1}\bar{y}_{l,1} & \bar{y}_{r,1} & \bar{x}_{l,1} & \bar{y}_{l,1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{x}_{r,n}\bar{x}_{l,n} & \bar{x}_{r,n}\bar{y}_{l,n} & \bar{x}_{r,n} & \bar{y}_{r,n}\bar{x}_{l,n} & \bar{y}_{r,n}\bar{y}_{l,n} & \bar{y}_{r,n} & \bar{x}_{l,n} & \bar{y}_{l,n} & 1 \end{bmatrix}$$

$$\mathrm{X}_r^{\alpha}\mathrm{Y}_r^{\gamma}\mathrm{X}_l^{\beta}\mathrm{Y}_l^{\delta} \equiv \sum_{i=1}^{n} x_{r,i}^{\alpha} y_{r,i}^{\beta} x_{l,i}^{\gamma} x_{l,i}^{\delta}$$

The Eight-point Algorithm

- SVD $A=UDV^{\mathsf{T}} \Rightarrow$ the solution is the column of V corresponding to the only null singular value of A
- $V = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_9]$; $\mathbf{v}_i$ - the eigenvectors of the 9×9 matrix $A^{\mathsf{T}}A$

$$A^{\mathsf{T}}A = \begin{bmatrix}
\mathrm{X}_r^2\mathrm{X}_l^2 & \mathrm{X}_r^2\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r^2\mathrm{X}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l^2 & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l & \mathrm{X}_r\mathrm{X}_l^2 & \mathrm{X}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{X}_l \\
\mathrm{X}_r^2\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r^2\mathrm{Y}_l^2 & \mathrm{X}_r^2\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l^2 & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l & \mathrm{X}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_l^2 & \mathrm{X}_r\mathrm{Y}_l \\
\mathrm{X}_r^2\mathrm{X}_l & \mathrm{X}_r^2\mathrm{Y}_l & \mathrm{X}_r^2 & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r & \mathrm{X}_r\mathrm{X}_l & \mathrm{X}_r\mathrm{Y}_l & \mathrm{X}_r \\
\mathrm{X}_r\mathrm{X}_l^2\mathrm{Y}_r & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l & \mathrm{Y}_r^2\mathrm{X}_l^2 & \mathrm{Y}_r^2\mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_r^2\mathrm{X}_l & \mathrm{Y}_r\mathrm{X}_l^2 & \mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_r\mathrm{X}_l \\
\mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l^2 & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l & \mathrm{Y}_r^2\mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_r^2\mathrm{Y}_l^2 & \mathrm{Y}_r^2\mathrm{Y}_l & \mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_r\mathrm{Y}_l^2 & \mathrm{Y}_r\mathrm{Y}_l \\
\mathrm{X}_r\mathrm{Y}_r\mathrm{X}_l & \mathrm{X}_r\mathrm{Y}_r\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_r & \mathrm{Y}_r^2\mathrm{X}_l & \mathrm{Y}_r^2\mathrm{Y}_l & \mathrm{Y}_r^2 & \mathrm{Y}_r\mathrm{X}_l & \mathrm{Y}_r\mathrm{Y}_l & \mathrm{Y}_r \\
\mathrm{X}_r\mathrm{X}_l^2 & \mathrm{X}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{X}_l & \mathrm{Y}_r^2\mathrm{Y}_l & \mathrm{Y}_r\mathrm{X}_l\mathrm{Y} & \mathrm{Y}_r\mathrm{X}_l & \mathrm{X}_l^2 & \mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_l \\
\mathrm{X}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{X}_r\mathrm{Y}_l^2 & \mathrm{X}_r\mathrm{Y}_l & \mathrm{Y}_r\mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_r\mathrm{Y}_l^2 & \mathrm{Y}_r\mathrm{Y}_l & \mathrm{X}_l\mathrm{Y}_l & \mathrm{Y}_l^2 & \mathrm{Y}_l \\
\mathrm{X}_r\mathrm{X}_l & \mathrm{X}_r\mathrm{Y}_l & \mathrm{X}_r & \mathrm{Y}_r\mathrm{X}_l & \mathrm{Y}_r\mathrm{Y}_l & \mathrm{Y}_r & \mathrm{X}_l & \mathrm{Y}_l & n
\end{bmatrix}$$

The Eight-point Algorithm

- Due to noise, the solution is the column of V associated with **the least singular value**
- The estimated fundamental matrix F_{est} is almost always non-singular, i.e. is full rank (3) rather than the expected rank 2
  - The singularity is enforced by adjusting the entries of F_{est}:
    - The SVD $F_{\text{est}} = UDV^{\mathsf{T}}$
    - Set the smallest singular value in the diagonal matrix D to zero to obtain the corrected matrix D'
    - The corrected estimate: $F' = UD'V^{\mathsf{T}}$

To Avoid Numerical Instabilities:

- Coordinates of the corresponding points have to be normalised to make entries of A of comparable size
  - Translate the two coordinates of each point to the centroid of each data set: $m_x = \frac{1}{n}\sum_{i=1}^{n} x_i; \quad m_y = \frac{1}{n}\sum_{i=1}^{n} y_i$
  - Scale the norm of each point so that the average norm over the data set is 1: $d = \frac{1}{n\sqrt{2}}\sum_i \sqrt{(x_i - m_x)^2 + (y_i - m_y)^2}$

$$\mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \Rightarrow \mathbf{p}'_i = \begin{bmatrix} (x_i - m_x)/d \\ (y_i - m_y)/d \\ 1 \end{bmatrix} \Leftrightarrow \mathbf{p}'_i = H\mathbf{p}_i \equiv \begin{bmatrix} 1/d & 0 & -m_x/d \\ 0 & 1/d & -m_y/d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

Stable Eight-Point Algorithm

- **Input**: n pixel-to-pixel correspondences

$$\left\{\left(\mathbf{p}_{l,i} = \left[x_{l,i} \;\; y_{l,i} \;\; 1\right]^{\mathsf{T}};\;\; \mathbf{p}_{r,i} = \left[x_{r,i} \;\; y_{r,i} \;\; 1\right]^{\mathsf{T}}\right):\;\; i = 1,\ldots,n\right\}$$

- **Data normalisation**:

$$\left\{\left(\mathbf{p}'_{l,i} = H_l \mathbf{p}_{l,i};\;\; \mathbf{p}'_{r,i} = H_r \mathbf{p}_{r,i}\right):\;\; i = 1,\ldots,n\right\}$$

$$H_l = \begin{bmatrix} \frac{1}{d_l} & 0 & -\frac{m_{l,x}}{d_l} \\ 0 & \frac{1}{d_l} & -\frac{m_{l,y}}{d_l} \\ 0 & 0 & 1 \end{bmatrix};\;\; H_l^{-1} = \begin{bmatrix} d_l & 0 & m_{l,x} \\ 0 & d_l & m_{l,y} \\ 0 & 0 & 1 \end{bmatrix};\qquad H_r = \begin{bmatrix} \frac{1}{d_r} & 0 & -\frac{m_{r,x}}{d_r} \\ 0 & \frac{1}{d_r} & -\frac{m_{r,y}}{d_r} \\ 0 & 0 & 1 \end{bmatrix};\;\; H_r^{-1} = \begin{bmatrix} d_r & 0 & m_{r,x} \\ 0 & d_r & m_{r,y} \\ 0 & 0 & 1 \end{bmatrix}$$

Stable Eight-Point Algorithm

- **SVD** $A = UDV^{\mathsf{T}}$ of the $n \times 9$ matrix A for the system of n linear equations; $n \geq 8$ (over-determined for $n > 8$):

$$\mathbf{p}'^{\mathsf{T}}_{r,i} F' \mathbf{p}'_{l,i} = 0 \quad \Rightarrow \quad [x'_{r,i}, y'_{r,i}, 1] \begin{bmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \\ F_7 & F_8 & F_9 \end{bmatrix} \begin{bmatrix} x'_{l,i} \\ y'_{l,i} \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \{\mathbf{a}_i^{\mathsf{T}} \mathbf{f} = 0 : \; i = 1, 2, \ldots, n\}$$

$$A\mathbf{f} = 0 \text{ where } A = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}} \\ \mathbf{a}_2^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_n^{\mathsf{T}} \end{bmatrix}; \; \mathbf{a}_i^{\mathsf{T}} = [x'_{l,i}x'_{r,i}, y'_{l,i}x'_{r,i}, x'_{r,i}, x'_{l,i}y'_{r,i}, y'_{l,i}y'_{r,i}, y'_{r,i}, x'_{l,i}, y'_{l,i}, 1]; \; \mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_9 \end{bmatrix}$$

Stable Eight-Point Algorithm

- The entries of F' (up to an unknown, signed scale factor) are the components of the column of V corresponding to the least singular value of A

- **SVD** $F' = UDV^{\mathsf{T}}$ of F' to enforce the singularity constraint
  - Set the smallest singular value in the diagonal of D equal to 0 to obtain the corrected matrix D'
  - Compute the corrected estimate $F'' = UD'V^{\mathsf{T}}$ of the fundamental matrix
- **Renormalisation**: the output estimate $F = H_r^{-1} F'' H_l^{-1}$

Locating the Epipoles

- Accurate localisation of the epipoles:
  - To refine the locations of the conjugate epipolar lines
  - To simplify the stereo geometry
  - To recover 3D structure in the case of uncalibrated stereo
- The left epipole $\mathbf{e}_l$ lies on all the epipolar lines in the left image $\Rightarrow$ the relationship $\mathbf{p}_r^{\top} F\mathbf{e}_l = 0$ holds for every $\mathbf{p}_r$
  - F is not identically zero, so it follows that $F\mathbf{e}_l = 0$
  - F has rank 2 - the epipole $\mathbf{e}_l$ is the null space of F
    - The null space is the set of all solutions $\mathbf{s}$ to the equation $F\mathbf{s} = 0$
  - Similarly, $\mathbf{e}_r$ is the null space of $F^{\top}$

Algorithm to Locate Epipoles

- Input: the fundamental matrix F
- SVD $F = UDV^{\mathsf{T}}$
  - The epipole $\mathbf{e}_l$: the column of V corresponding to the null singular value
  - The epipole $\mathbf{e}_r$: the column of U corresponding to the null singular value

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \mathbf{1} & 0 \\ \frac{1}{\sqrt{2}} & \mathbf{0} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \mathbf{0} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \underline{\mathbf{0}} & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{V^{\mathsf{T}}} \Rightarrow \mathbf{e}_l = \mathbf{e}_r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

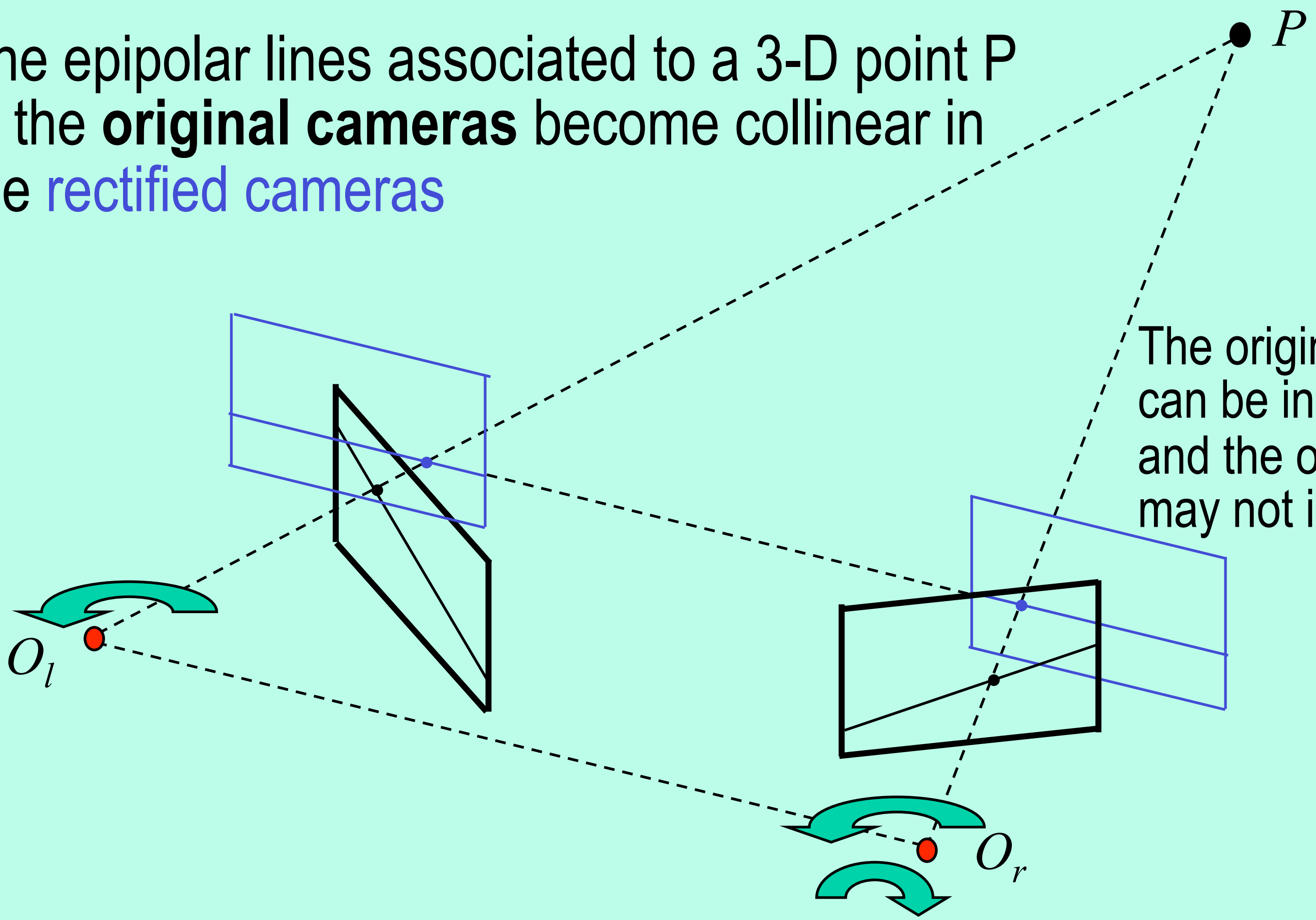
Rectification of Stereo Images

Rectification - a transformation (**warping**) of each image: pairs of conjugate epipolar lines become collinear and parallel to one of the image axes (typically, x-axis)

- The 1-D search for correspondence after rectification
- **Computation**: by using the known intrinsic parameters of the camera and extrinsic parameters of the stereo system
- The rectified images are thought of as acquired by a new stereo rig obtained by rotating the original cameras around their optical centres

Rectification of a Stereo Pair

The epipolar lines associated to a 3-D point P in the **original cameras** become collinear in the rectified cameras



The original cameras can be in any position, and the optical axes may not intersect

Rectification of a Stereo Pair

Assumptions and Basic Steps

- Assumptions for both cameras without losing generality:

  (1) The origin of the image reference frame in the principal point (the trace of the optical axis) and (2) the same focal length f

- Steps of rectification

  (1) Rotate the left camera to make its image plane parallel to the baseline of the system (the epipole goes to infinity along the x-axis)

  (2) Apply the same rotation to the right camera to recover the original geometry and then (3) rotate the right camera by R

  (4) Adjust the scale in both camera reference frames

Rotation Matrix R_{rect} for Step 1

- A triple of mutually orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$
  - An arbitrary choice due to an under-constrained problem
  - The epipole $\mathbf{e}_1$ coincides with the direction of translation

    (as the image centre is in the origin)

The direction vector of the optical axis

$$R_{\text{rect}} = \begin{bmatrix} \mathbf{e}_1^{\mathsf{T}} \\ \mathbf{e}_2^{\mathsf{T}} \\ \mathbf{e}_3^{\mathsf{T}} \end{bmatrix} \quad \text{where} \quad \mathbf{e}_1 = \frac{\mathbf{T}}{\|\mathbf{T}\|} = \frac{1}{\sqrt{T_x^2 + T_y^2 + T_z^2}} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}; \quad \mathbf{e}_2 = \frac{\mathbf{e}_1 \times [0,0,1]^{\mathsf{T}}}{\left\| \mathbf{e}_1 \times [0,0,1]^{\mathsf{T}} \right\|} = \frac{1}{\sqrt{T_x^2 + T_y^2}} \begin{bmatrix} -T_y \\ T_x \\ 0 \end{bmatrix};$$

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{\sqrt{\left(T_x^2 + T_y^2\right)\left(T_x^2 + T_y^2 + T_z^2\right)}} \begin{bmatrix} -T_x T_z \\ -T_y T_z \\ T_x^2 + T_y^2 \end{bmatrix}$$

The Rectification Algorithm

- **Input**: the intrinsic and extrinsic parameters; the images (or sets of their points) to be rectified; assumptions 1 and 2 hold
- Build the matrix R_{rect} and set $R_l = R_{\text{rect}}$ and $R_r = R_{\text{rect}}$
- For each left-camera point, $\mathbf{p}_l = [x, y, f\,]^{\mathsf{T}}$, compute the coordinates of the corresponding rectified point:

$$\mathbf{p}_l' = \left[\frac{fx'}{z'}, \frac{fy'}{z'}, f\right] \quad \text{where} \quad [x', y', z'] = R_l \mathbf{p}_l$$

- Repeat this step for the right camera using R_r and $\mathbf{p}_r$