CompSci 372 – Tutorial

Part 5

3D Geometry

Vector operations

- Length

$$|\boldsymbol{a}| = \left\| \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \right\| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

- Normal vector

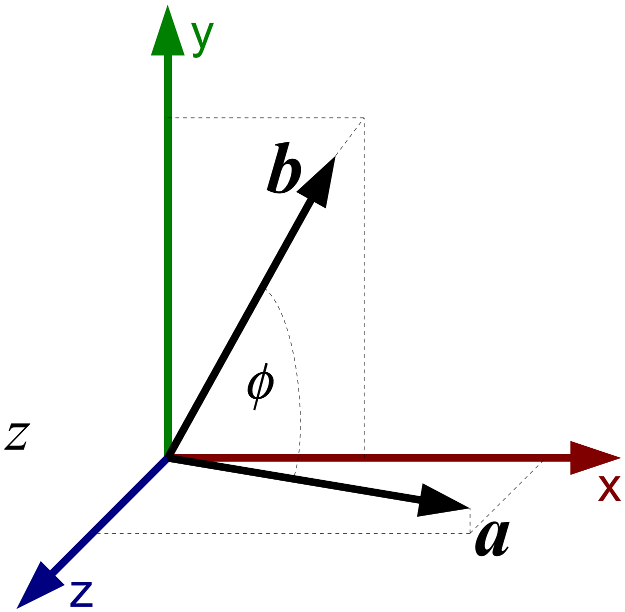
$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Vector operations

- Dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\phi$$

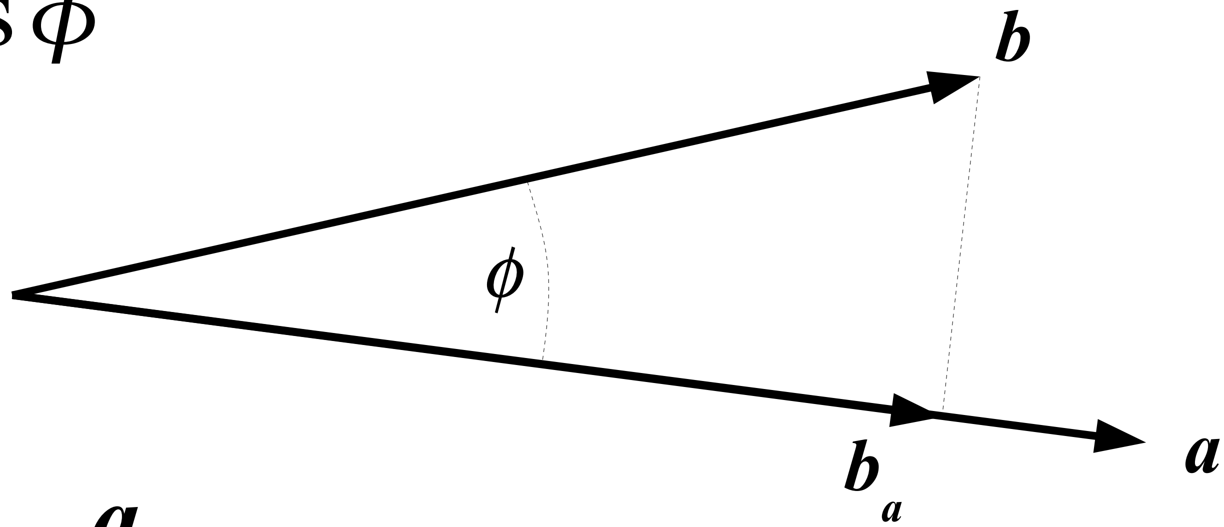
$$= a_x b_x + a_y b_y + a_z b_z$$



Projection

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\phi$$

$$\frac{|\boldsymbol{b}_{a}|}{|\boldsymbol{b}|} = \cos\phi$$



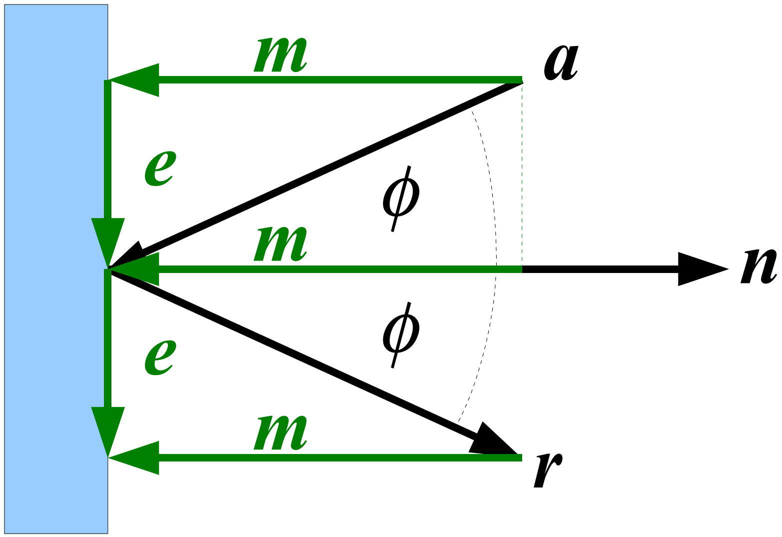
$$\boldsymbol{b}_{a} = |\boldsymbol{b}_{a}|\,\hat{\boldsymbol{a}} = |\boldsymbol{b}_{a}|\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= |\boldsymbol{b}|\cos\phi\,\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = |\boldsymbol{b}|\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$$

Reflection

$$m = \frac{a \cdot n}{n \cdot n} n$$

$$= (a \cdot \hat{n}) \hat{n}$$



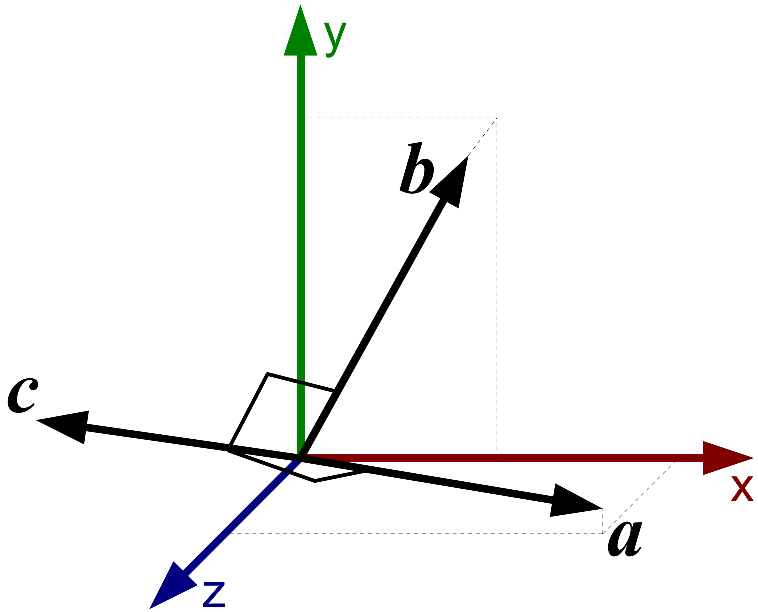
$$a = m + e$$

$$r = e - m = (a - m) - m = a - 2m$$

$$= a - 2(a \cdot \hat{n}) \hat{n}$$

Cross Product (Vector Prod.)

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

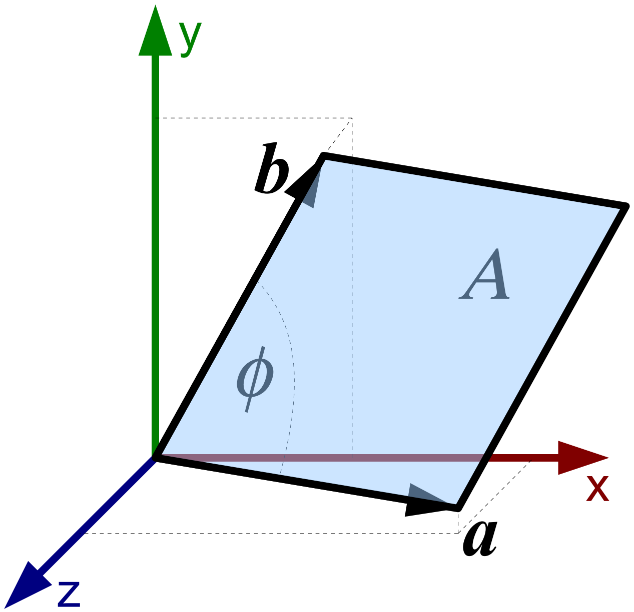


Cross Product (Vector Prod.)

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\phi$$

$$|\boldsymbol{c}| = A$$



Determinant

$$\boldsymbol{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

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$$|\boldsymbol{M}| = aei + bfg + cdh - ceg - afh - bdi$$

Inverse Matrix

$$\boldsymbol{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\boldsymbol{M}^{-1} = \frac{1}{|\boldsymbol{M}|} \begin{pmatrix} +|\boldsymbol{A}_{11}| & -|\boldsymbol{A}_{12}| & +|\boldsymbol{A}_{13}| \\ -|\boldsymbol{A}_{21}| & +|\boldsymbol{A}_{22}| & -|\boldsymbol{A}_{23}| \\ +|\boldsymbol{A}_{31}| & -|\boldsymbol{A}_{32}| & +|\boldsymbol{A}_{33}| \end{pmatrix}$$

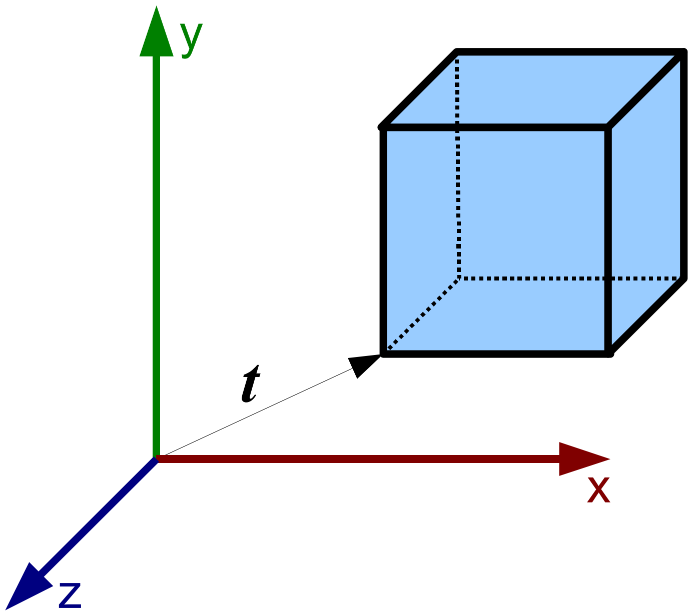
Inverse Matrix

$$\boldsymbol{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\boldsymbol{A}_{11} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \boldsymbol{A}_{23} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

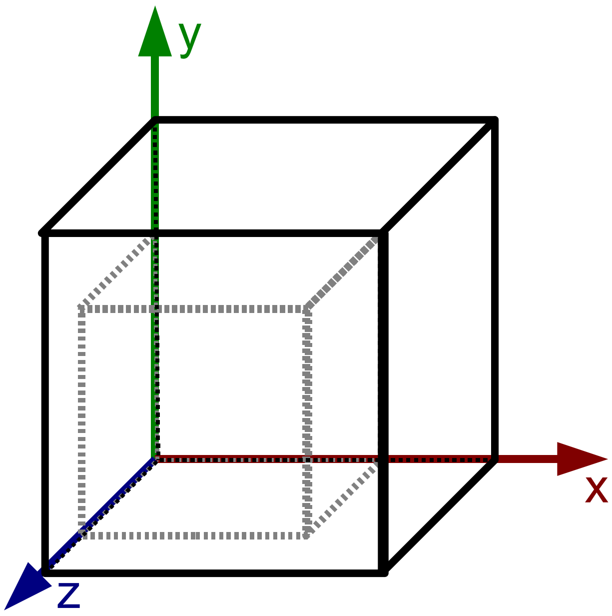
Translation

$$M_{Translate} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



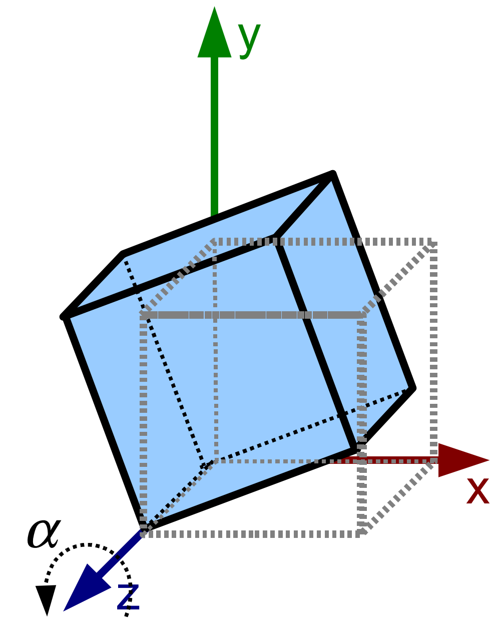
Scaling

$$M_{Scale} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



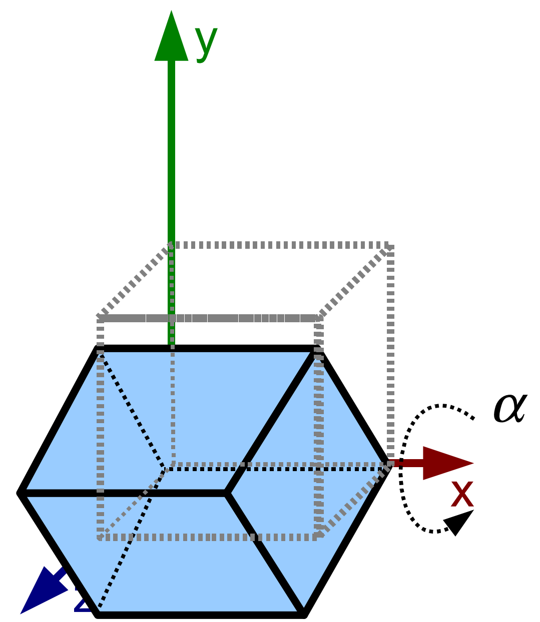
Rotation around Z

$$M_{RotateZ} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



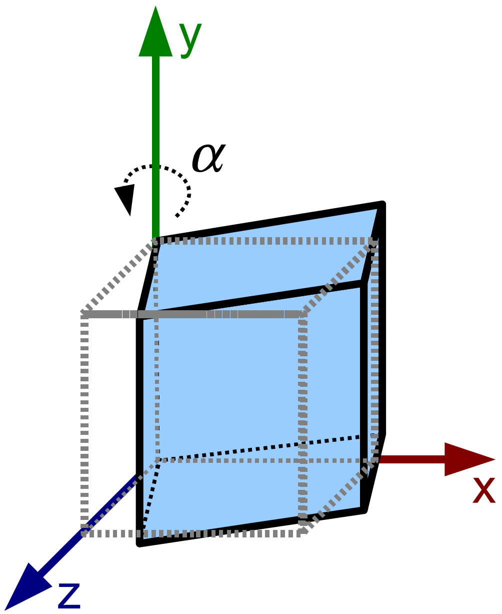
Rotation around X

$$M_{RotateX} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



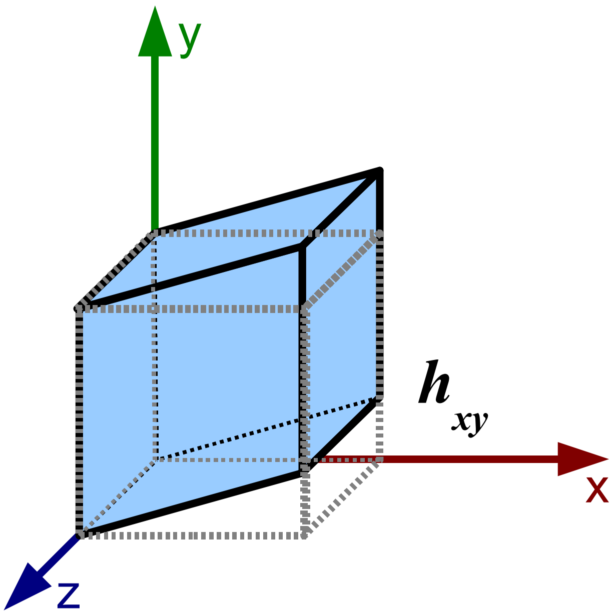
Rotation around Y

$$M_{RotateY} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

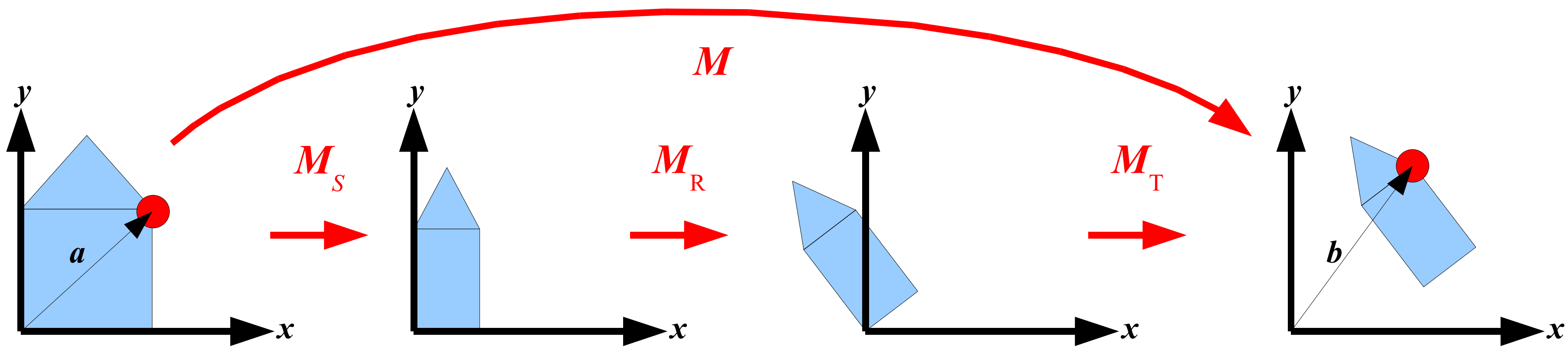


Shearing

$$M_{Shear} = \begin{pmatrix} 1 & h_{yx} & h_{zx} & 0 \\ h_{xy} & 1 & h_{zy} & 0 \\ h_{xz} & h_{yz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



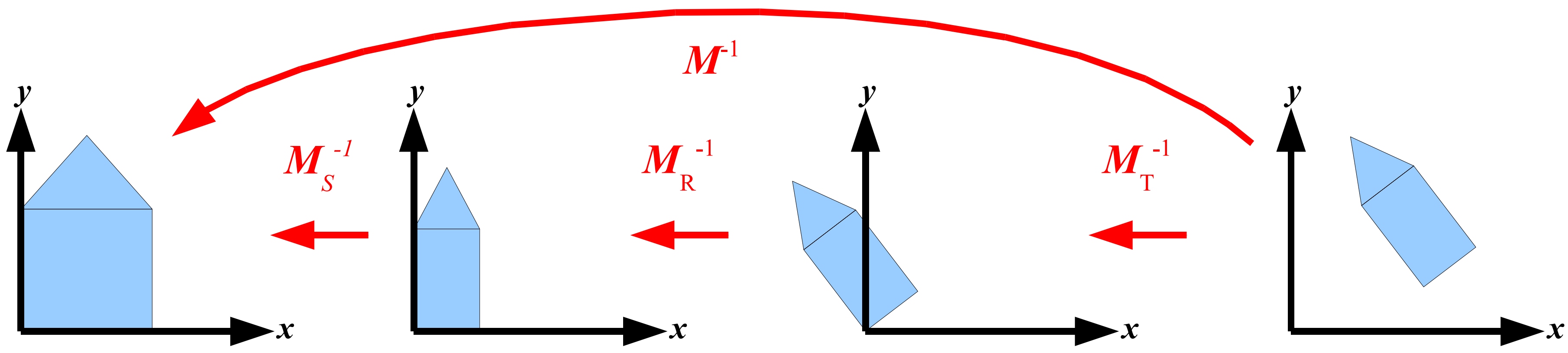
Concatenation



$$M = M_T M_R M_S$$

$$b = M a = M_T\left(M_R\left(M_S a\right)\right)$$

Concatenation



$$M^{-1} = M_S^{-1} M_R^{-1} M_T^{-1}$$

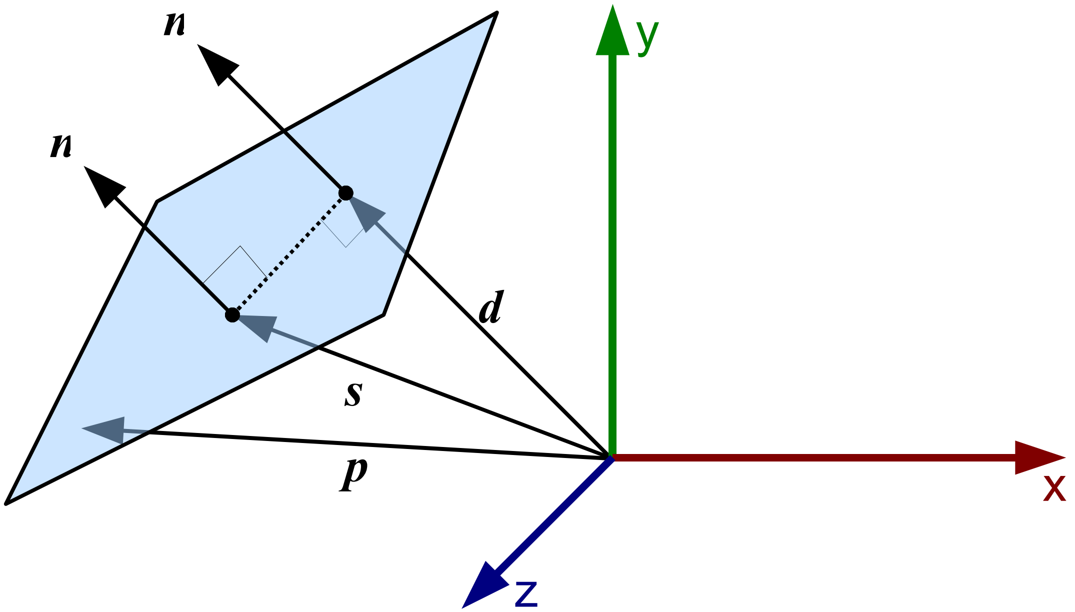
Plane

- Point-Normal-Form

$$\boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{s}) = 0$$

$$\boldsymbol{n} \cdot \boldsymbol{p} = \boldsymbol{n} \cdot \boldsymbol{s} = d$$

$$d = |\boldsymbol{d}| \qquad \boldsymbol{d} = \boldsymbol{n}\, d$$



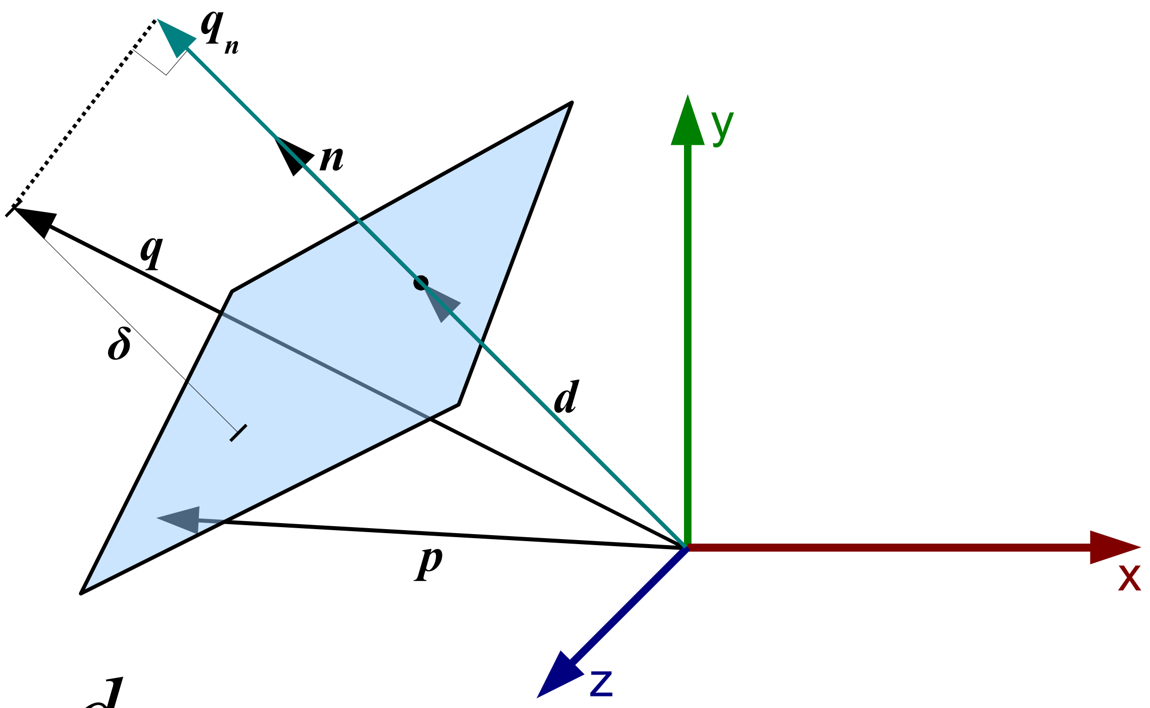
- d: Distance of plane to origin
- $\boldsymbol{d}$: Shortest “path to plane”

Distance of Point to Plane

$$\boldsymbol{n} \cdot \boldsymbol{p} = d$$

$$\boldsymbol{n} \cdot \boldsymbol{q} = q_n$$

$$\delta = q_n - d = \boldsymbol{n} \cdot \boldsymbol{q} - d$$

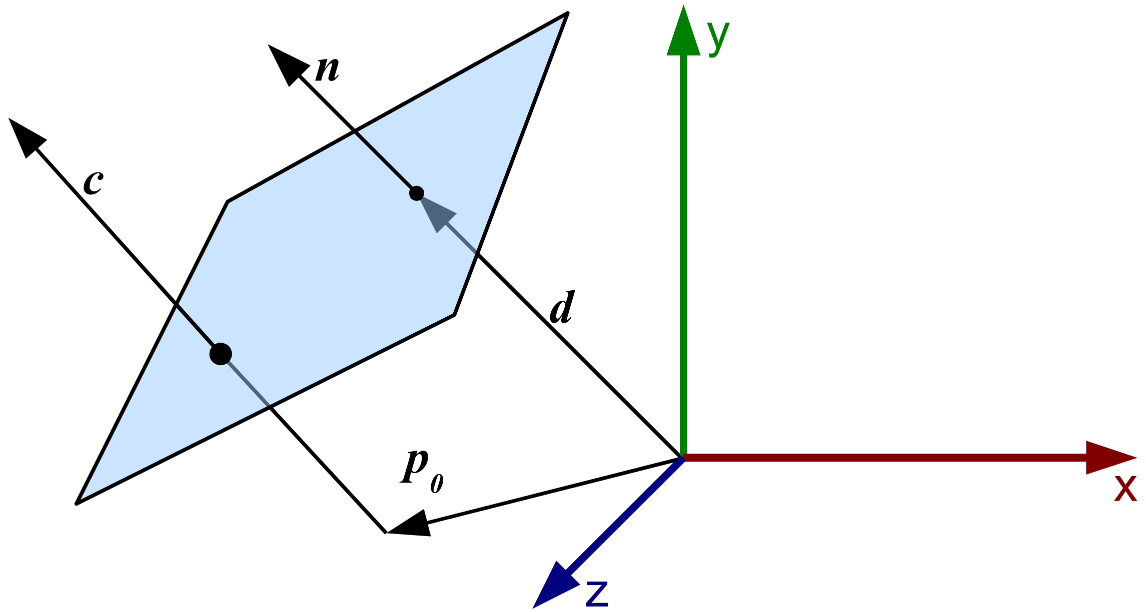


- Meaning of δ:

  =0: $\boldsymbol{q}$ is on plane   >0: “outside”   <0: “inside”

Line-Plane intersection

$$\boldsymbol{n}\cdot\boldsymbol{p}=d$$



$$\boldsymbol{p}(t)=\boldsymbol{p}_0+t\,\boldsymbol{c}$$

$$\boldsymbol{n}\cdot\boldsymbol{p}(t)=d$$

$$\boldsymbol{n}\cdot(\boldsymbol{p}_0+t\,\boldsymbol{c})=d \quad\Leftrightarrow\quad \boldsymbol{n}\cdot\boldsymbol{p}_0+\boldsymbol{n}\cdot t\,\boldsymbol{c}=d$$

$$\Leftrightarrow\quad t=\frac{d-\boldsymbol{n}\cdot\boldsymbol{p}_0}{\boldsymbol{n}\cdot\boldsymbol{c}}$$