CompSci 372 – Tutorial

Part 4

2D Geometry

Vector operations

- Length

$$|\boldsymbol{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right| = \sqrt{a_1{}^2 + a_2{}^2}$$

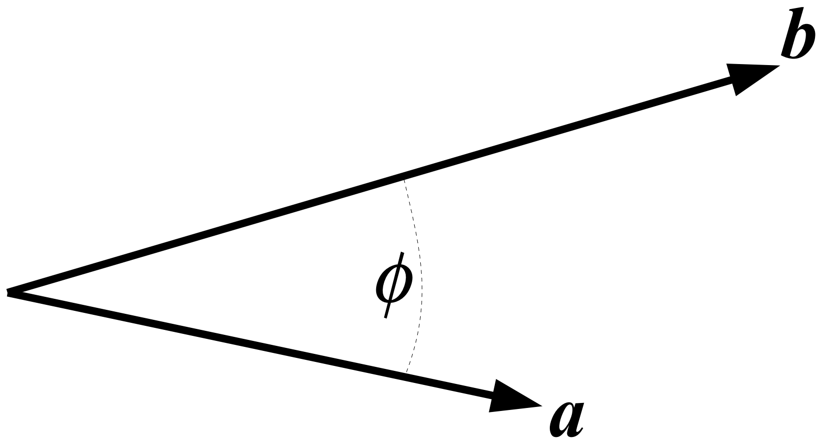
- Normal vector

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Vector operations

- Dot product

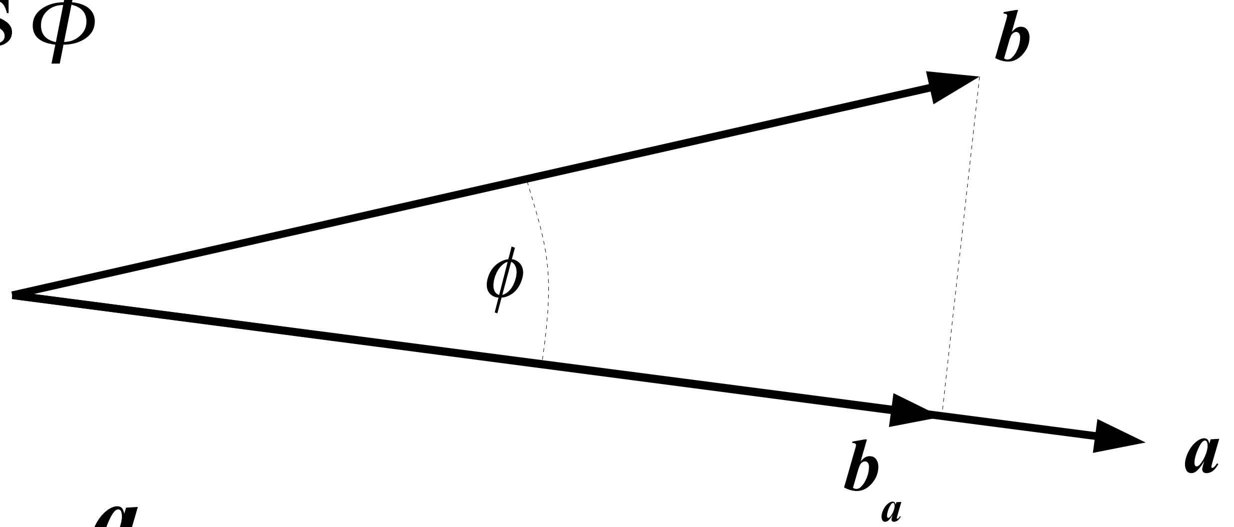
$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\phi$$
$$= a_x b_x + a_y b_y$$



Projection

$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\phi$$

$$\frac{|\boldsymbol{b}_{\boldsymbol{a}}|}{|\boldsymbol{b}|}=\cos\phi$$



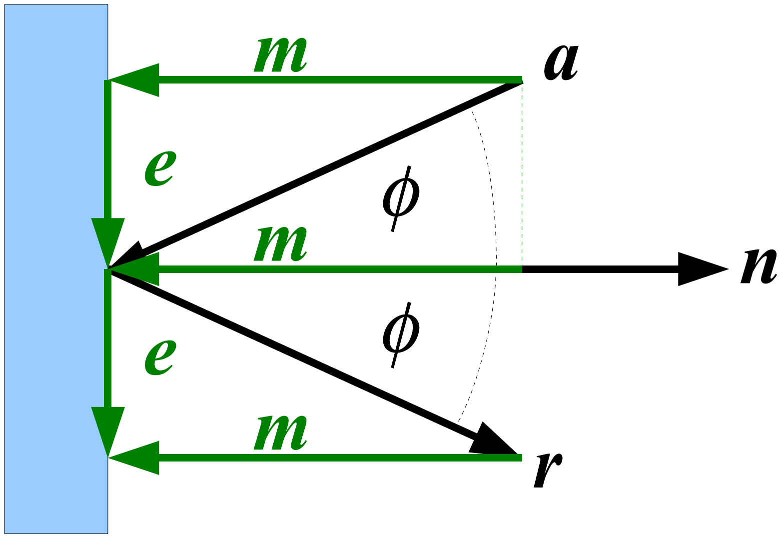
$$\boldsymbol{b}_{\boldsymbol{a}}=|\boldsymbol{b}_{\boldsymbol{a}}|\hat{\boldsymbol{a}}=|\boldsymbol{b}_{\boldsymbol{a}}|\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$=|\boldsymbol{b}|\cos\phi\frac{\boldsymbol{a}}{|\boldsymbol{a}|}=|\boldsymbol{b}|\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\frac{\boldsymbol{a}}{|\boldsymbol{a}|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$$

Reflection

$$m = \frac{a \cdot n}{n \cdot n} n$$

$$= (a \cdot \hat{n}) \hat{n}$$



$$a = m + e$$

$$r = e - m = (a - m) - m = a - 2m$$

$$= a - 2(a \cdot \hat{n}) \hat{n}$$

Matrix Dimension

- Matrix dimension: $m \times n$
  - m: Number of Rows
  - n: Number of Columns
  - Opposite to e.g. "640x480 pixel"

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Transposition

- Mirror on a 45° line

$$\boldsymbol{M} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \boldsymbol{M}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

- m×n-Matrix turns into an n×m-Matrix

Determinant

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \quad det(\boldsymbol{M}) = ad - bc$$

\- +

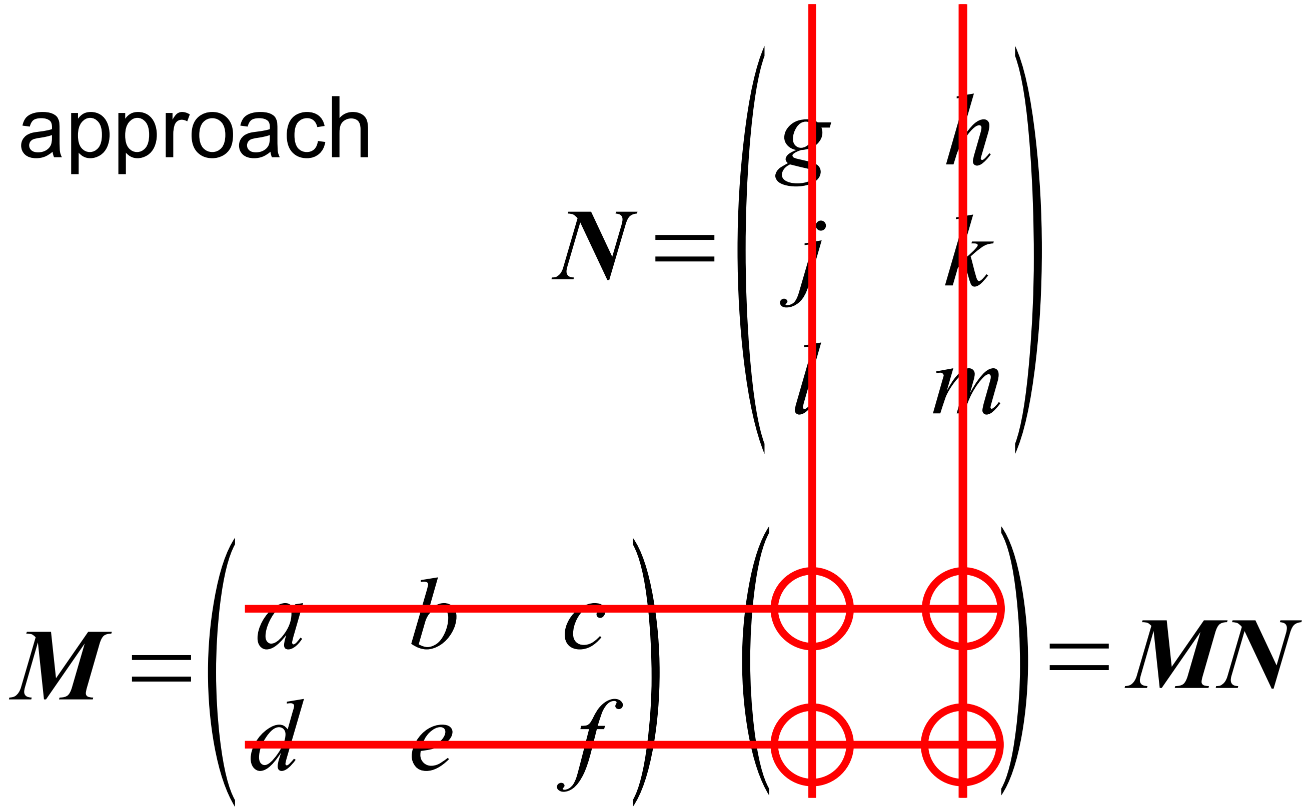
Inverse Matrix

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\boldsymbol{M}^{-1} = \frac{1}{det(\boldsymbol{M})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Matrix Multiplication

- "Crosshair" approach



$$N = \begin{pmatrix} g & h \\ j & k \\ l & m \end{pmatrix}$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$MN = \begin{pmatrix} ag + bj + cl & ah + bk + cm \\ dg + ej + fl & dh + ek + fm \end{pmatrix}$$

Inverse Matrix Proof

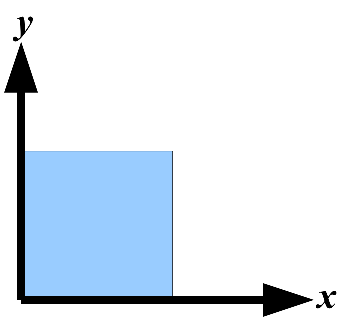
$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boldsymbol{M}^{-1} = \frac{1}{det(\boldsymbol{M})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\boldsymbol{M}\boldsymbol{M}^{-1} = \frac{1}{det(\boldsymbol{M})} \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -bc + da \end{pmatrix}$$

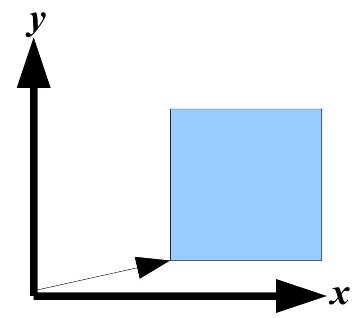
$$\boldsymbol{M}\boldsymbol{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Affine Transformations

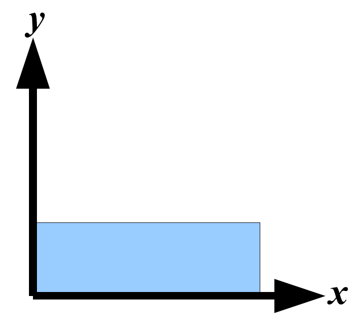
- Preserve parallel lines
- Preserve ratios



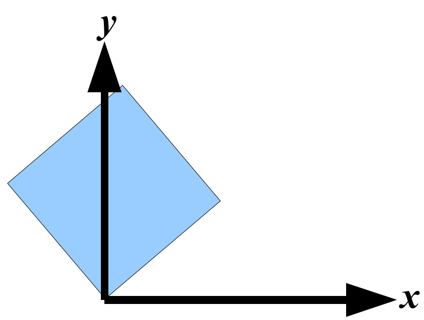
- Translation



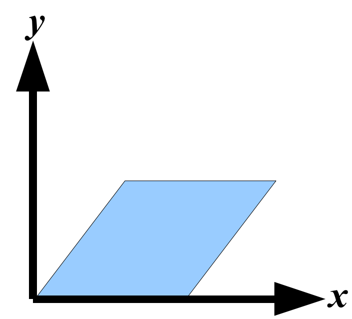
- Scaling



- Rotation



- Shearing



Homogeneous Coordinates

- Rotation, Scaling, Shearing: $\boldsymbol{M}$
- Translation: $\boldsymbol{b}$
- General Form: $\boldsymbol{y} = \boldsymbol{M}\,\boldsymbol{x} + \boldsymbol{b}$

- Complicated: Two parameters
  - How can we handle Rotation/Scaling/Shearing **and** Translation with just one matrix?

Homogeneous Coordinates

- Add a homogeneous coordinate w

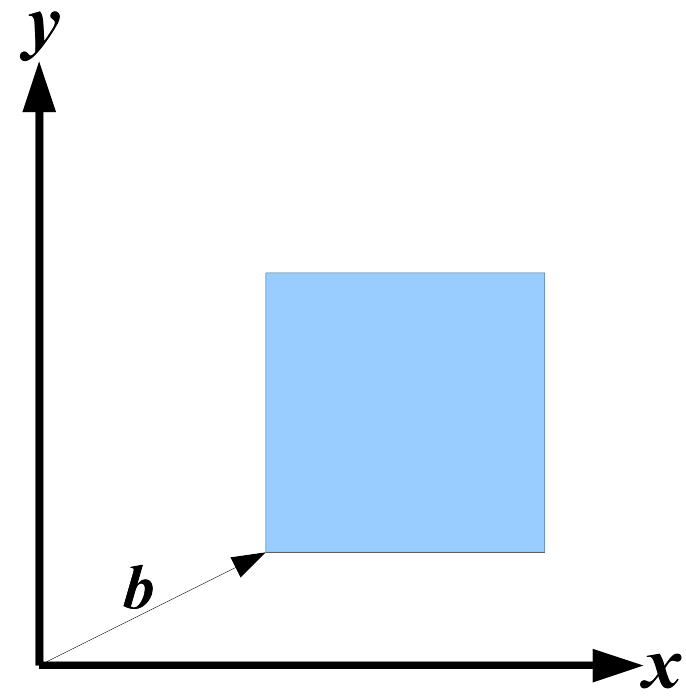
$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} \rightarrow \begin{pmatrix} w\,a_x \\ w\,a_y \\ w \end{pmatrix}$$

- Combine $\boldsymbol{M}$ and $\boldsymbol{b}$

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} \rightarrow \begin{pmatrix} m_{11} & m_{12} & b_x \\ m_{21} & m_{22} & b_y \\ 0 & 0 & 1 \end{pmatrix}$$

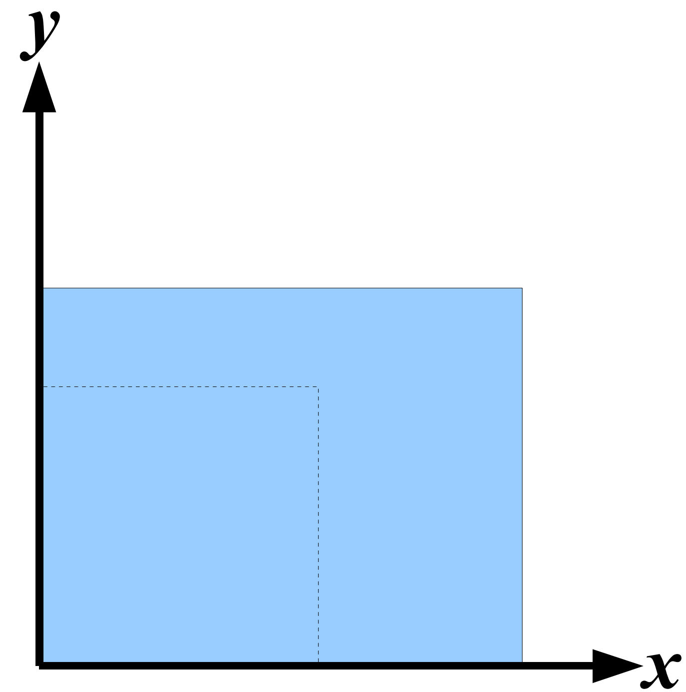
Translation

$$\boldsymbol{M}_{\boldsymbol{Translation}} = \begin{pmatrix} 1 & 0 & b_x \\ 0 & 1 & b_y \\ 0 & 0 & 1 \end{pmatrix}$$



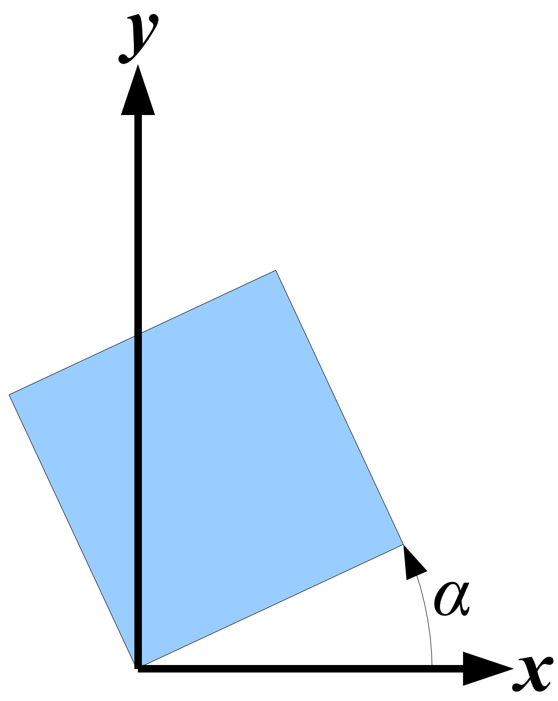
Scaling

$$M_{Scaling} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



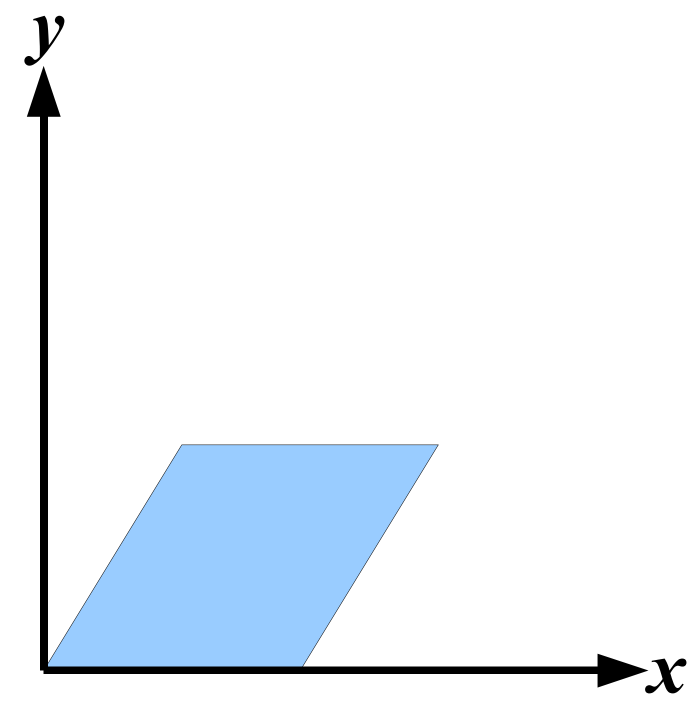
Rotation

$$M_{Rotation} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



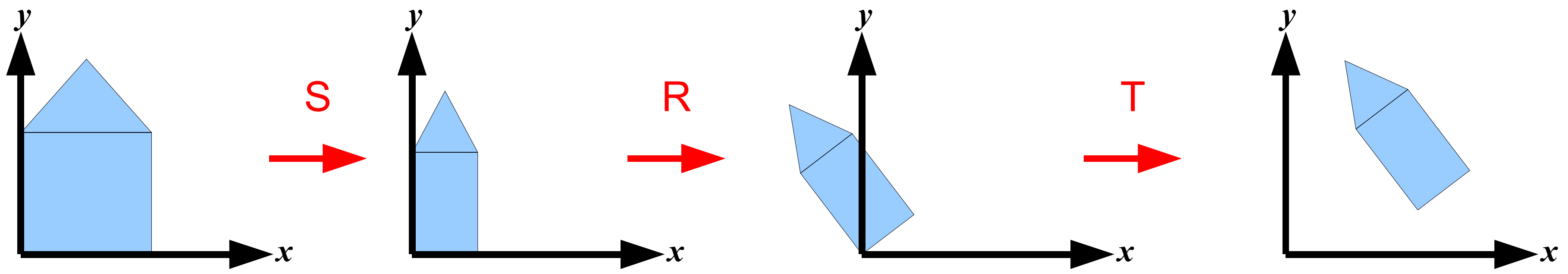
Shearing

$$M_{Shear} = \begin{pmatrix} 1 & s_x & 0 \\ s_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Combination

- First step → Rightmost Matrix
- Last step → Leftmost Matrix



$$M = M_T M_R M_S$$