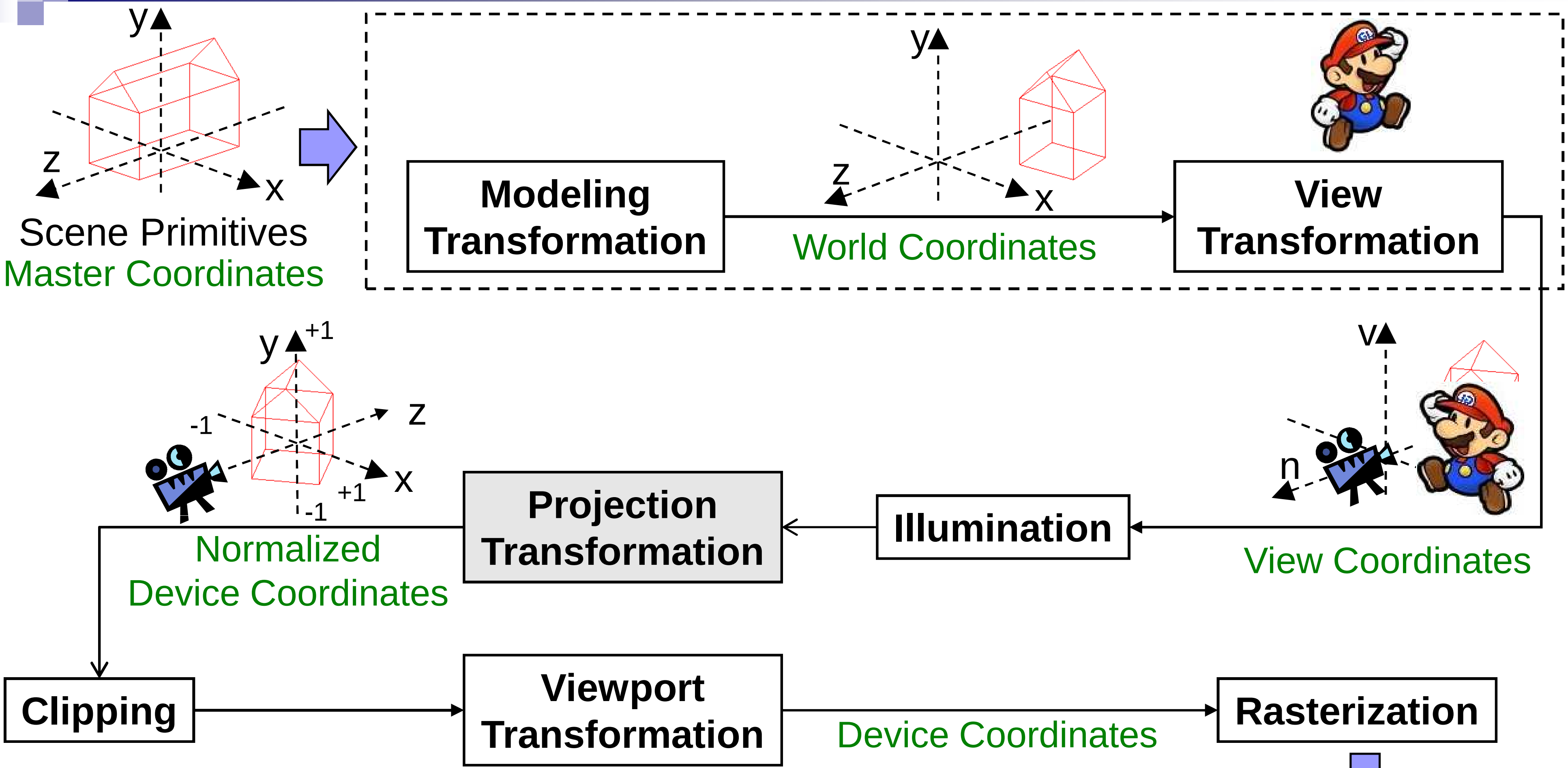
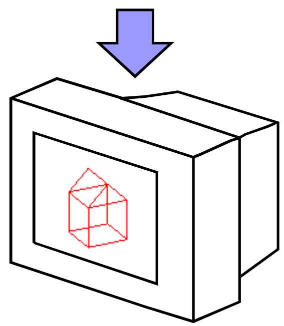
GRAPHICS GROUP

Computer Graphics: Projection Transformations

Part 2 – Lecture 2

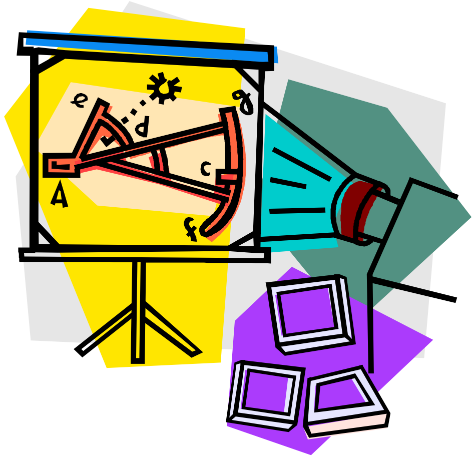




PROJECTION TRANSFORMATIONS

Principles of Geometric Projections

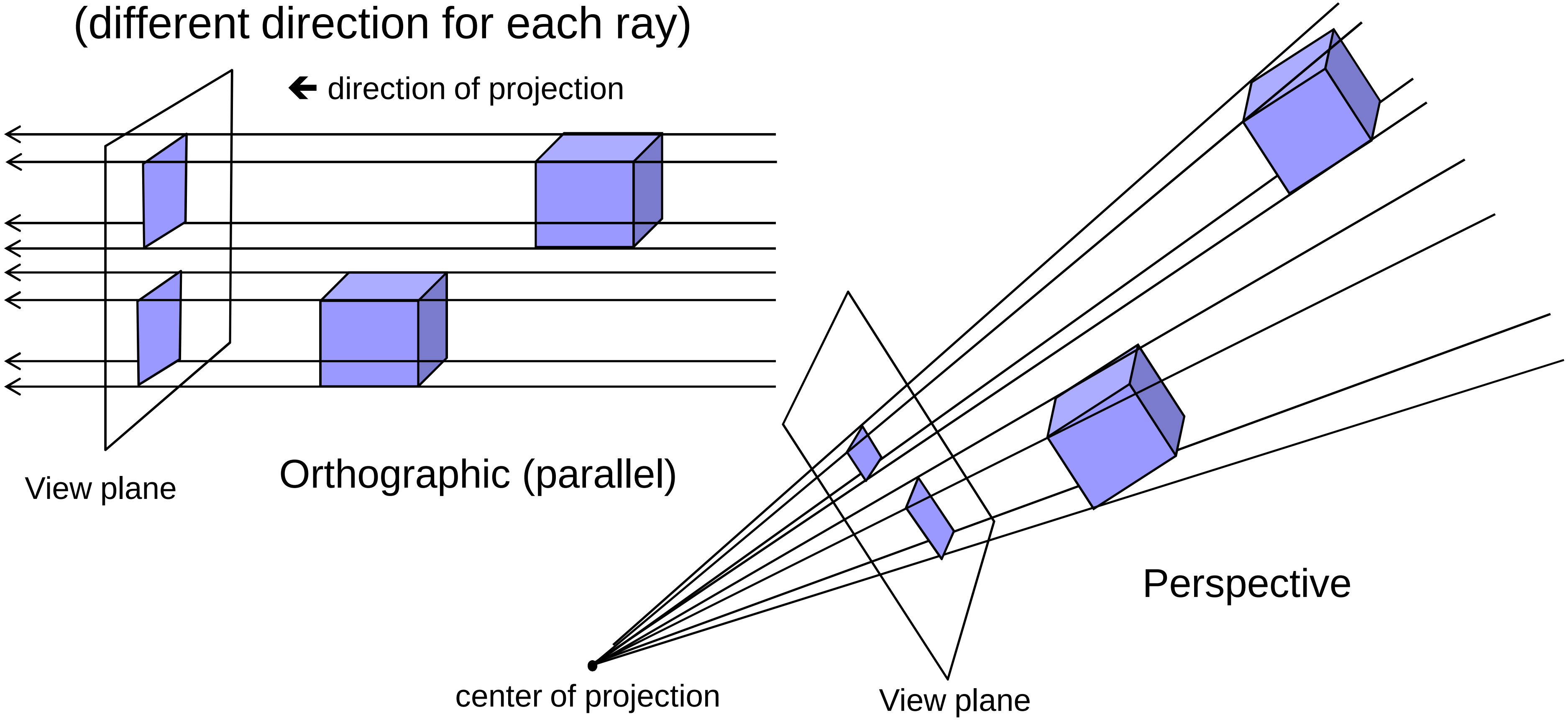
- **Projection:** a mapping of coordinate values from a higher dimension to lower dimension, usually N ⇨ N-1, e.g. 3D ⇨ 2D
- **Requirements:**



  - **Projection surface:** plane (linear projection) or surface such as a sphere or conic section (non-linear projection)
  - **Projection rays, or *projectors*:** lines from object projected towards projection surface
  - **Direction of projection:** orientation of each projector
    - Orthographic (parallel) projection: all projectors parallel to a common direction of projection.
    - Perspective projection: all projectors pass through a center of projection (3D point), but have different directions
- **How to project:**
  Intersect projection ray through object vertex with the projection surface

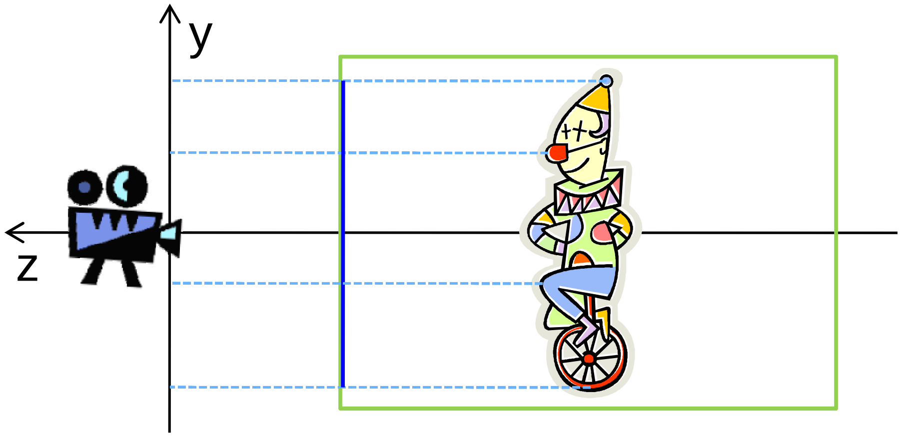
Orthographic vs. Perspective Projection

- **<u>Orthographic projection</u>:** ray through object vertex in the projection direction (same direction for all rays, orthogonal to projection plane)
- **<u>Perspective projection</u>:** ray through object vertex and center of projection (different direction for each ray)

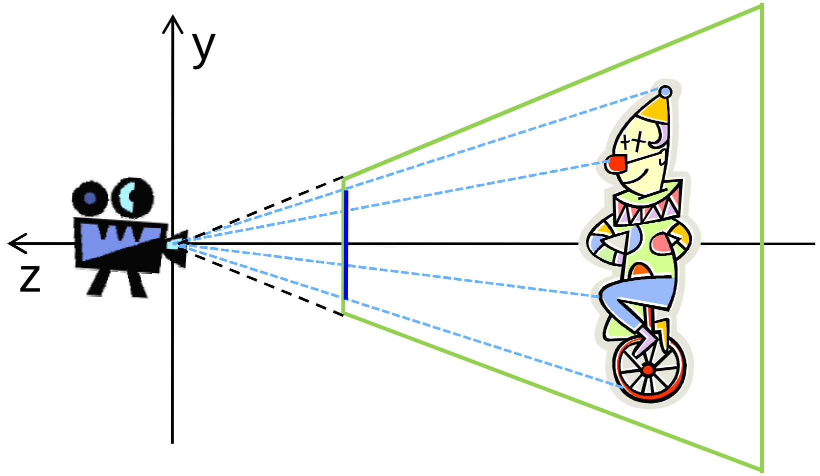


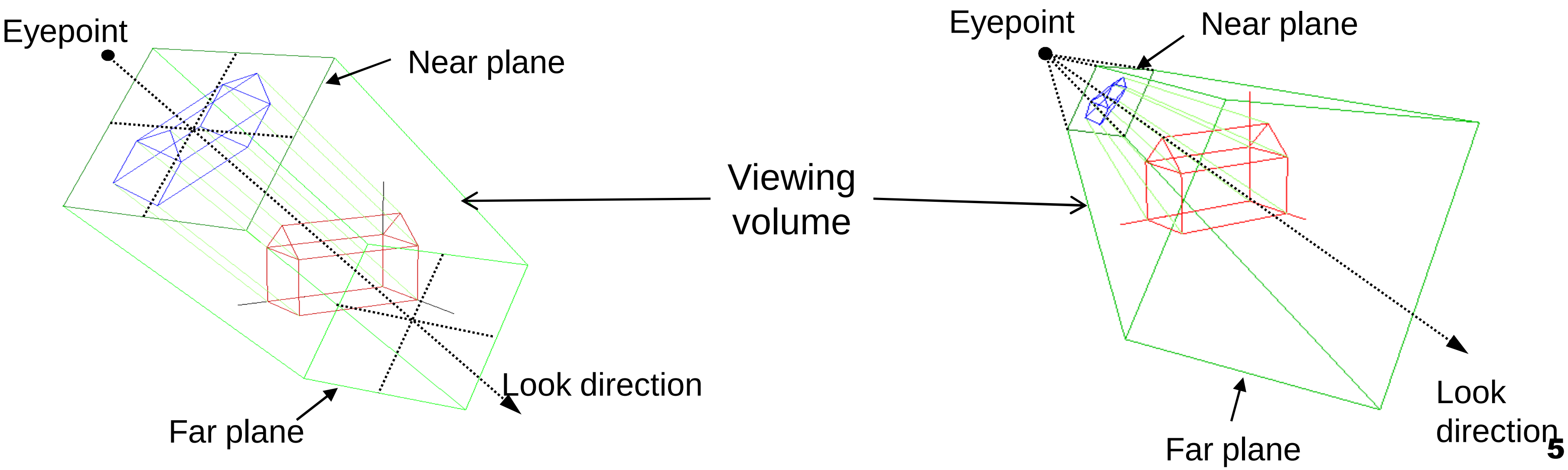
Orthographic vs. Perspective Projection

Orthographic Projection



Perspective Projection

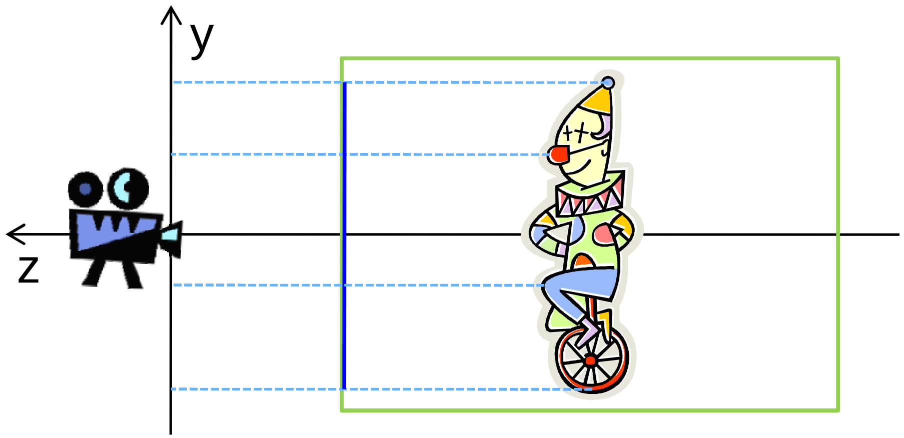




Ortho / Perspective Cameras in OpenGL

- **Orthographic**
  - `void glOrtho( GLdouble left, GLdouble right,`
    `GLdouble bottom, GLdouble top,`
    `GLdouble zNear, GLdouble zFar )`
  - View volume boundaries in World Coord units, relative to eyepoint in the look direction. Z is positive distance from eye (along negative Z axis)
  - View volume ***may be* symmetric** about look direction vector (typical)
- **Perspective**
  - `void gluPerspective( GLdouble fovy, GLdouble aspect,`
    `GLdouble zNear, GLdouble zFar )`
  - Vertical field of view angle `fovy` specified in degrees.
  - Horizontal fov determined by `aspect` ratio = width/height

    ```
    fovx = aspect * fovy;
    ```
  - View volume (frustum, or truncated pyramid) ***always* symmetric** about eyepoint towards the look direction.

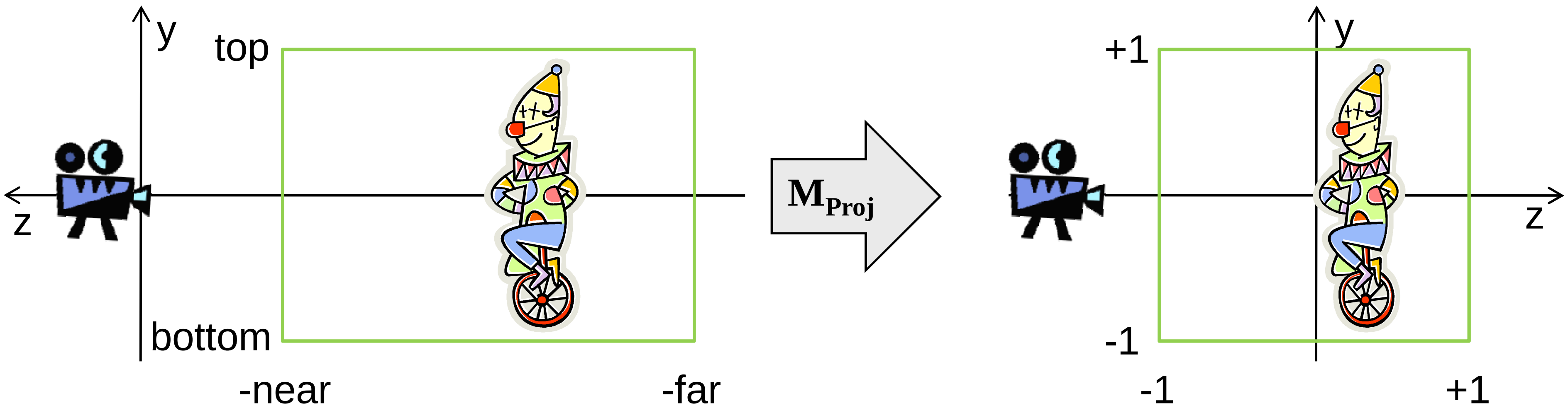


ORTHOGRAPHIC PROJECTION

Projection Transformation Matrix $\mathbf{M}_{\mathbf{proj}}$

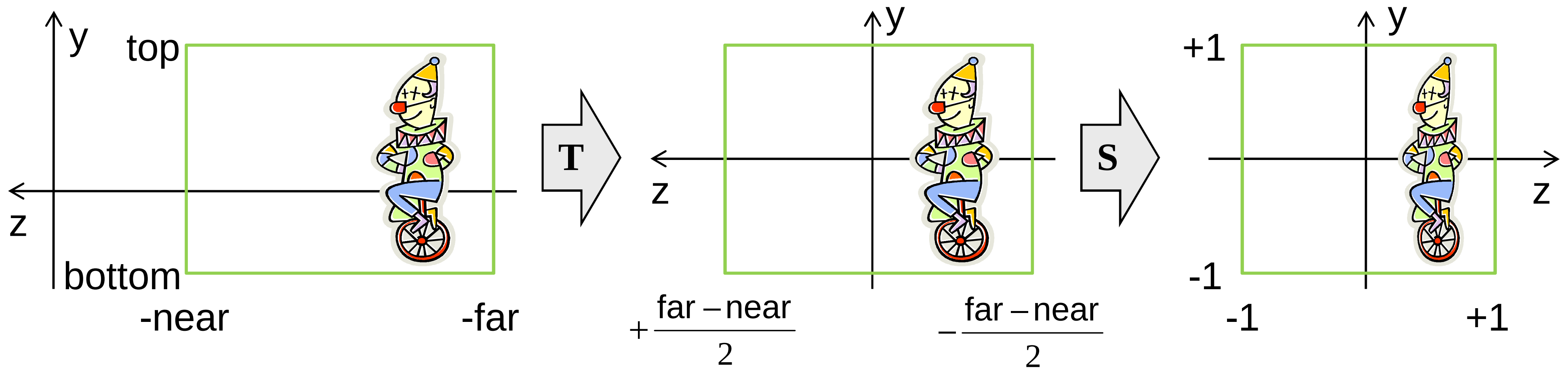
- Maps **View Coords.** to **Normalized Device Coords.** (**NDC**)
- View volume boundaries are mapped to (-1,+1) cube in X, Y, Z
- View Coordinates are **RHS** and NDC are **LHS**, so $\mathbf{M}_{\mathbf{proj}}$ inverts Z values

- For orthographic projection:



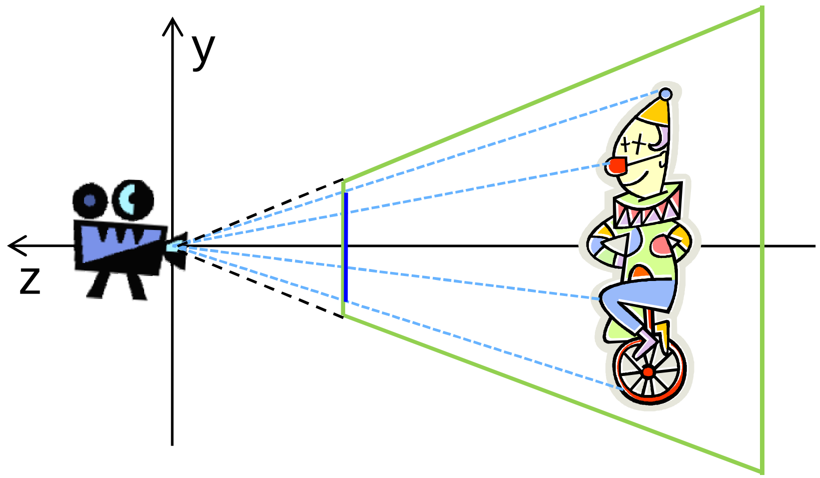
To get a 2D image: take only x and y components

$\mathbf{M}_{\mathbf{proj}}$ for Orthographic Projections



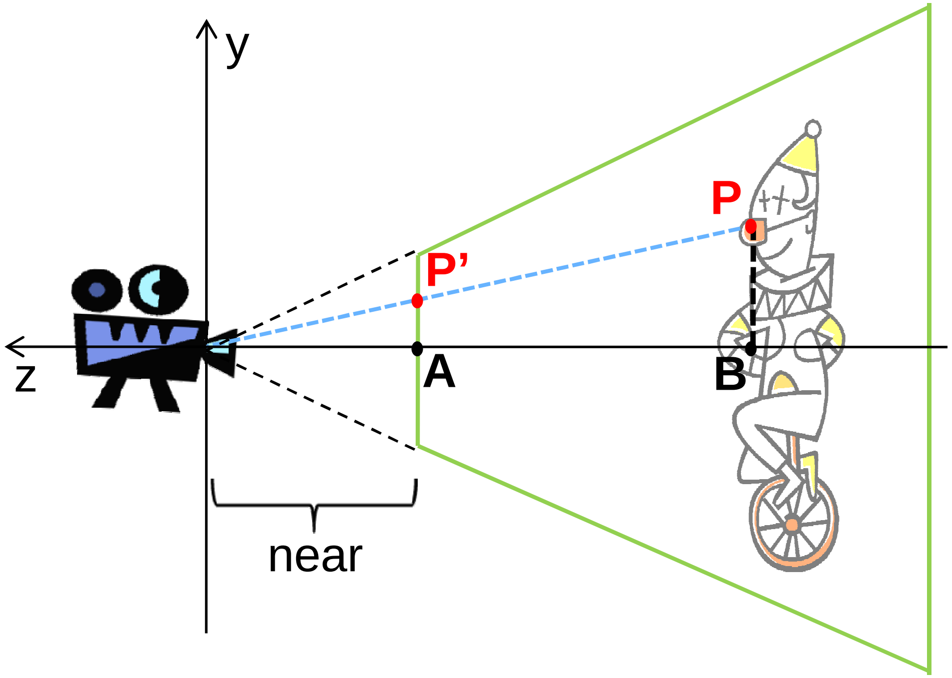
$$\mathbf{M}_{\mathbf{proj}} = \mathbf{S}\ \ \mathbf{T}$$

$$= \begin{pmatrix} \frac{2}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2}{top-bottom} & 0 & 0 \\ 0 & 0 & -\frac{2}{far-near} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -(left+right)/2 \\ 0 & 1 & 0 & -(top+bottom)/2 \\ 0 & 0 & 1 & +(near+far)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



PERSPECTIVE PROJECTION

Perspective Projection of a Vertex



- What are the coordinates of **P'** ?
- Camera-**A**-**P'** and Camera-**B**-**P** are similar triangles
- Ratios of similar sides are equal:

$$\frac{P_y'}{near} = \frac{P_y}{-P_z} \Leftrightarrow P_y' = \frac{near}{-P_z} P_y$$

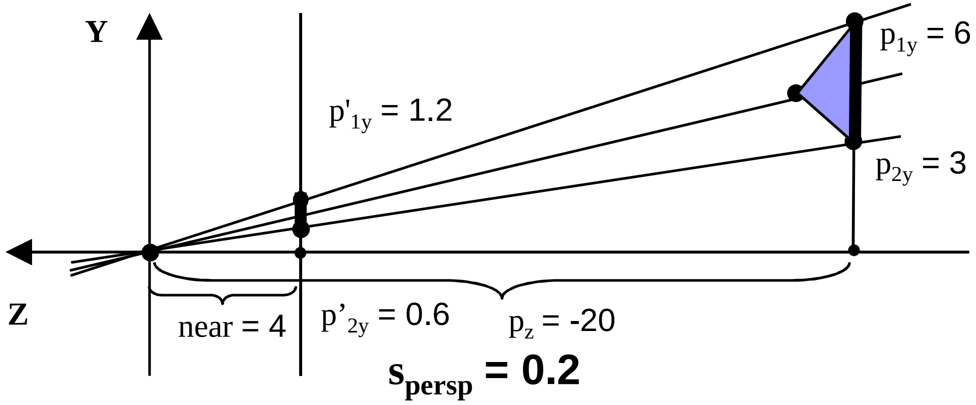
- When looking from the bottom, we get analogous calculations for the x-coordinate of **P'**:

$$\frac{P_x'}{near} = \frac{P_x}{-P_z} \Leftrightarrow P_x' = \frac{near}{-P_z} P_x$$

- Perspective scaling factor

$$s_{persp} = \frac{near}{-P_z}$$

Perspective Foreshortening

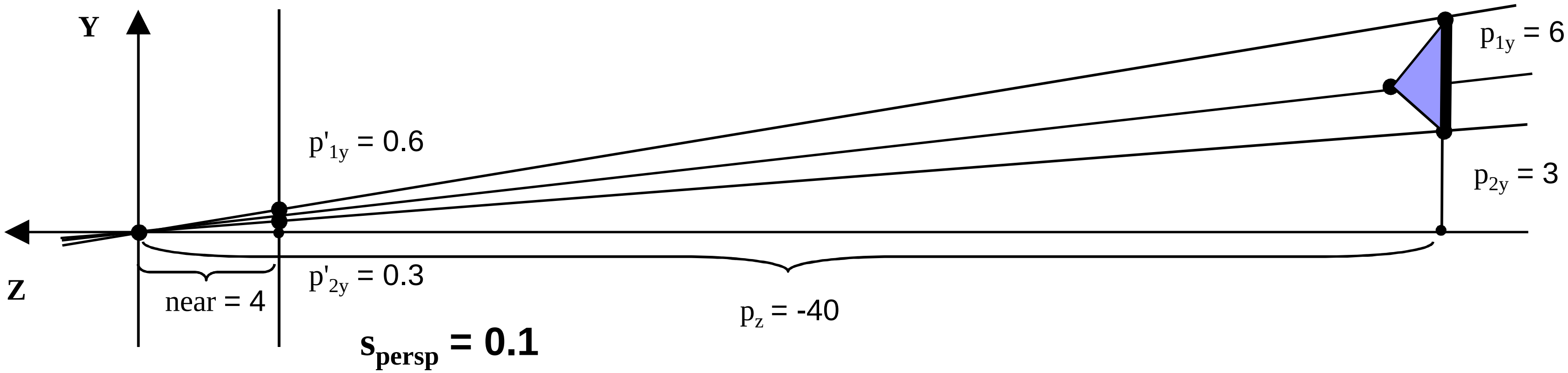


$d_y = (p_{1y} - p_{2y}) = 3$

$d_y' = (p'_{1y} - p'_{2y})$

$= s_{persp}\, d_y$

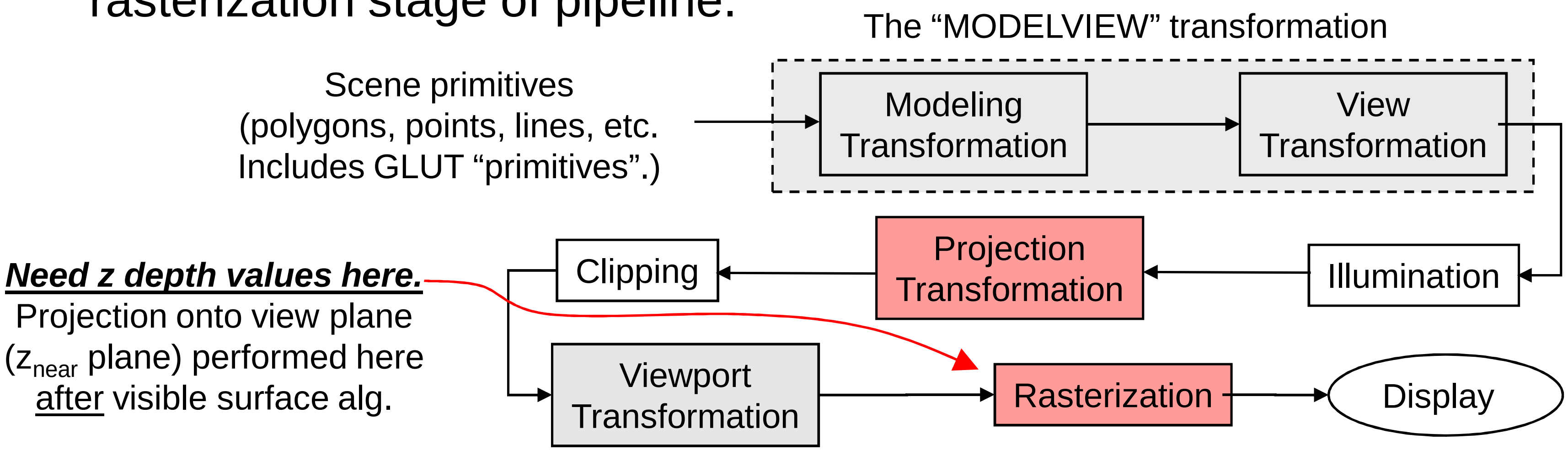
$= (0.2)\ 3 = 0.6$



$d_y = (p_{1y} - p_{2y}) = 3$ $\quad$ $d_y' = (p'_{1y} - p'_{2y}) = \mathbf{s_{persp}}\, d_y = (0.1)\ 3 = 0.3$

Perspective Transformation

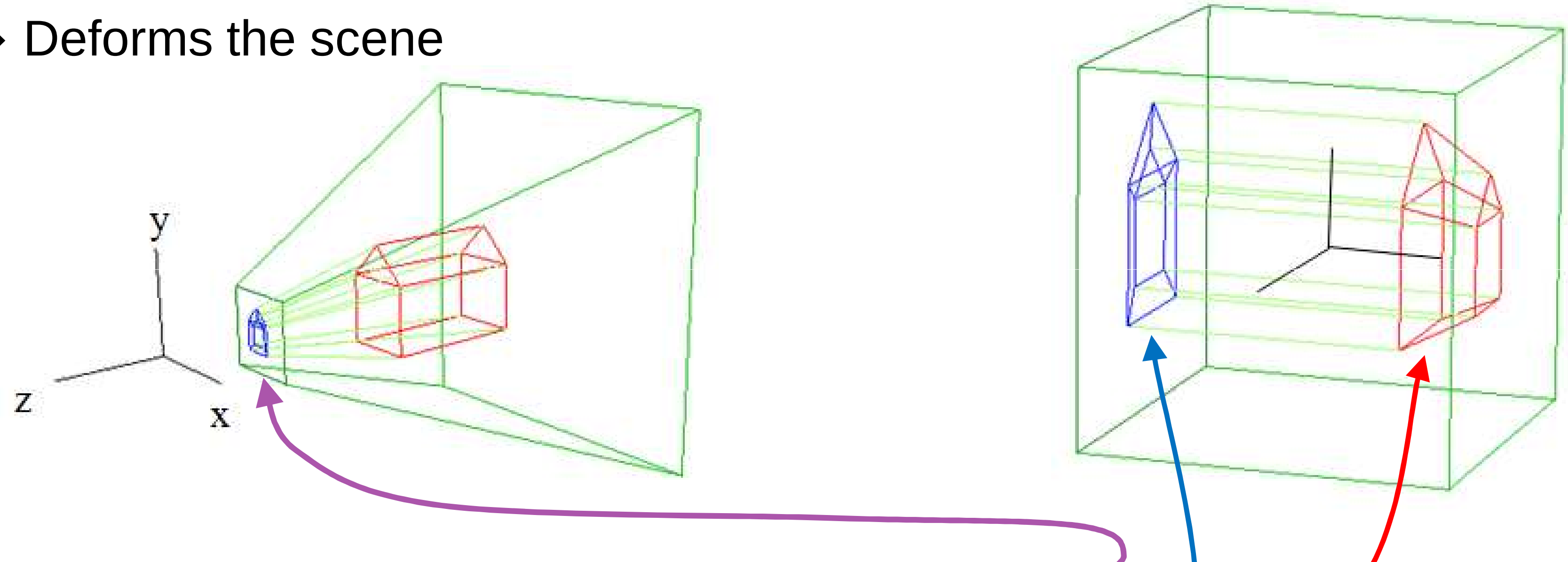
- <u>Perspective projection CANNOT be used in 3D graphics pipeline!</u>
  - Why not? Because it sets all projected z coordinates to <u>same value</u>, z_{near} But, visible surface algorithm (Z buffer alg.) needs <u>z depth values</u> during rasterization stage of pipeline.



  - Therefore, pipeline uses **perspective *<u>transformation</u>***, not **perspective *<u>projection</u>***
  - Scales x, y, **<u>and z</u>** coordinates by a scale factor dependent upon 1/z
  - Projection is performed during rasterization stage after hidden surface removal

Perspective Transformation and Projection

- **Perspective transformation** converts 3D coordinates to perspective corrected 3D coordinates
  → Deforms the scene



  - We want perspective projection to look like this
  - But, we actually perform it in a 2 step process:
    - Perspective **transformation**: 3D → 3D
    - Orthographic **projection**: 3D → 2D

Perspective Transformation (cont'd)

- Perspective transformation requirements:
  1. x and y values must be scaled by same factor as derived in perspective projection equations.
  2. z values must maintain depth ordering (monotonic increasing)
  3. z values must map: $-\mathbf{z}_{\mathbf{near}}$ → -1 and $-\mathbf{z}_{\mathbf{far}}$ → +1, view volume → NDC cube.
- In other words, we need a transformation that given a point *P* results in a transformed point *P'* such that P'_x and P'_y meet requirement 1 and $f(p_z)$ meets requirements 2 and 3.

$$P' = \left(\frac{-near}{p_z} p_x, \quad \frac{-near}{p_z} p_y, \quad f(p_z) \right)$$

- **Question:** Is there any matrix, $\mathbf{P}$, such that $\mathbf{P}\, P = P'$?
- **Answer:** **Not possible** because no linear combination of p_x, p_y, p_z, can result in a term with p_z in the denominator!

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-near\, p_x}{p_z} \\ \frac{-near\, p_y}{p_z} \\ f(p_z) \\ 1 \end{pmatrix}$$

Perspective Transformation (cont'd)

- But, there is a matrix **P** that can produce this result:

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} near\ p_x \\ near\ p_y \\ -f(p_z)\ p_z \\ -p_z \end{pmatrix}$$

- After conversion to ordinary coordinates: $P' = \left(\frac{near\ p_x}{-p_z}, \ \frac{near\ p_y}{-p_z}, \ f(p_z) \right)$

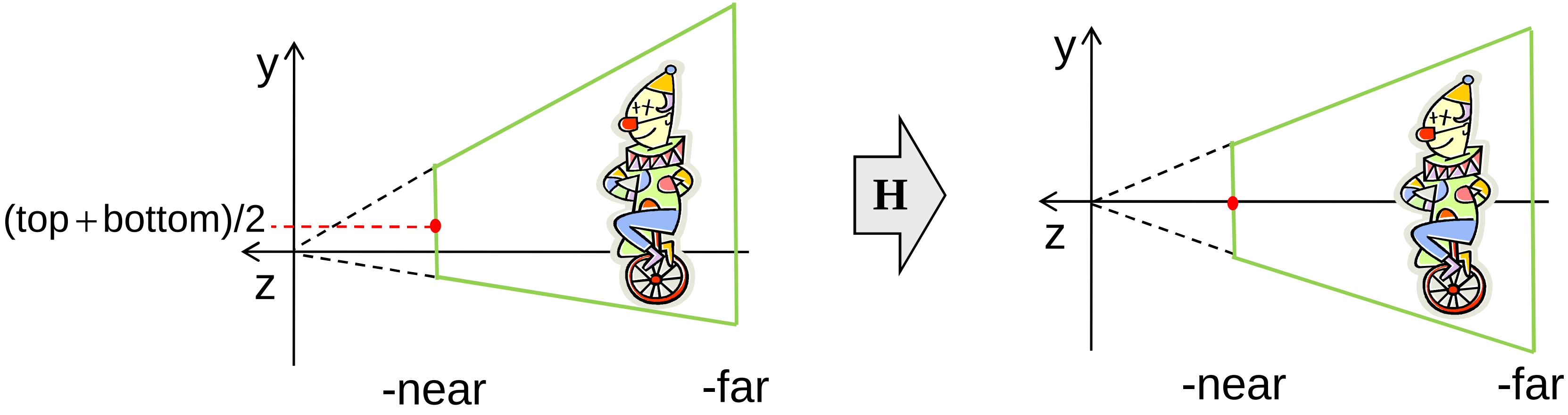
$$\mathbf{P} = \begin{pmatrix} near & 0 & 0 & 0 \\ 0 & near & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with $a = -\frac{far + near}{far - near}, \quad b = \frac{-2\ far\ near}{far - near}$

so that $P' = \left(\frac{near\, p_x}{-p_z}, \ \frac{near\, p_y}{-p_z}, \ \frac{a\, p_z + b}{-p_z} \right)$

- Result: perspective transformation can be done with matrix multiplication in the rendering pipeline (using hardware!)

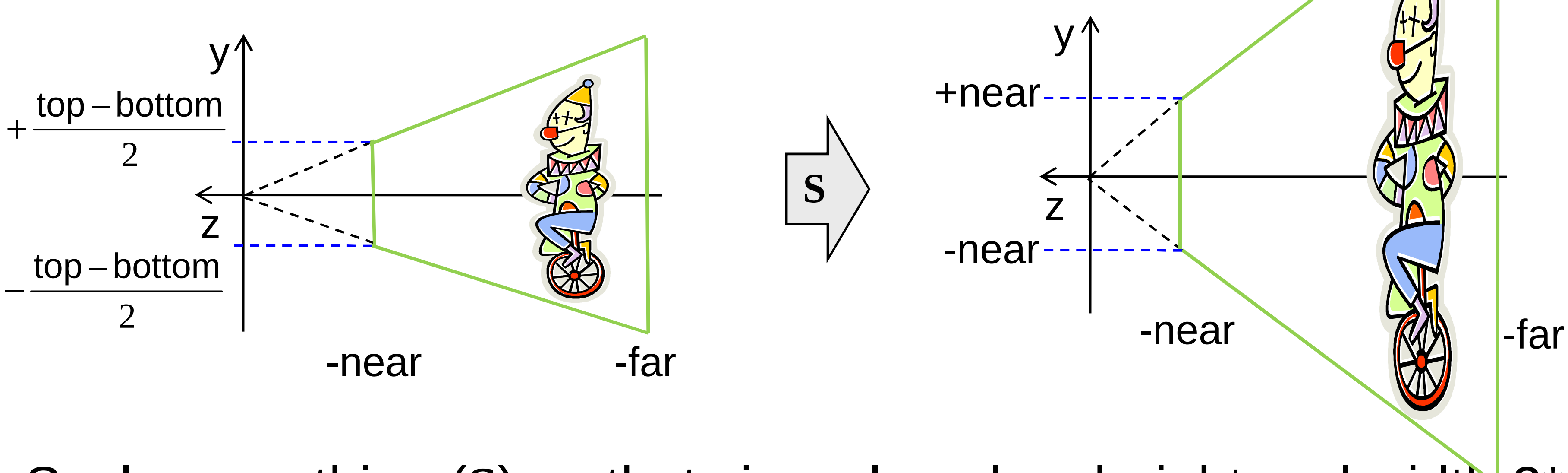
$\mathbf{M}_{\mathbf{proj}}$ for Perspective Projections 1



Shear everything (**H**) so that z-axis is in the middle of the frustum

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & \dfrac{-(left+right)/2}{-near} & 0 \\ 0 & 1 & \dfrac{-(top+bottom)/2}{-near} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dfrac{left+right}{2\ near} & 0 \\ 0 & 1 & \dfrac{top+bottom}{2\ near} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

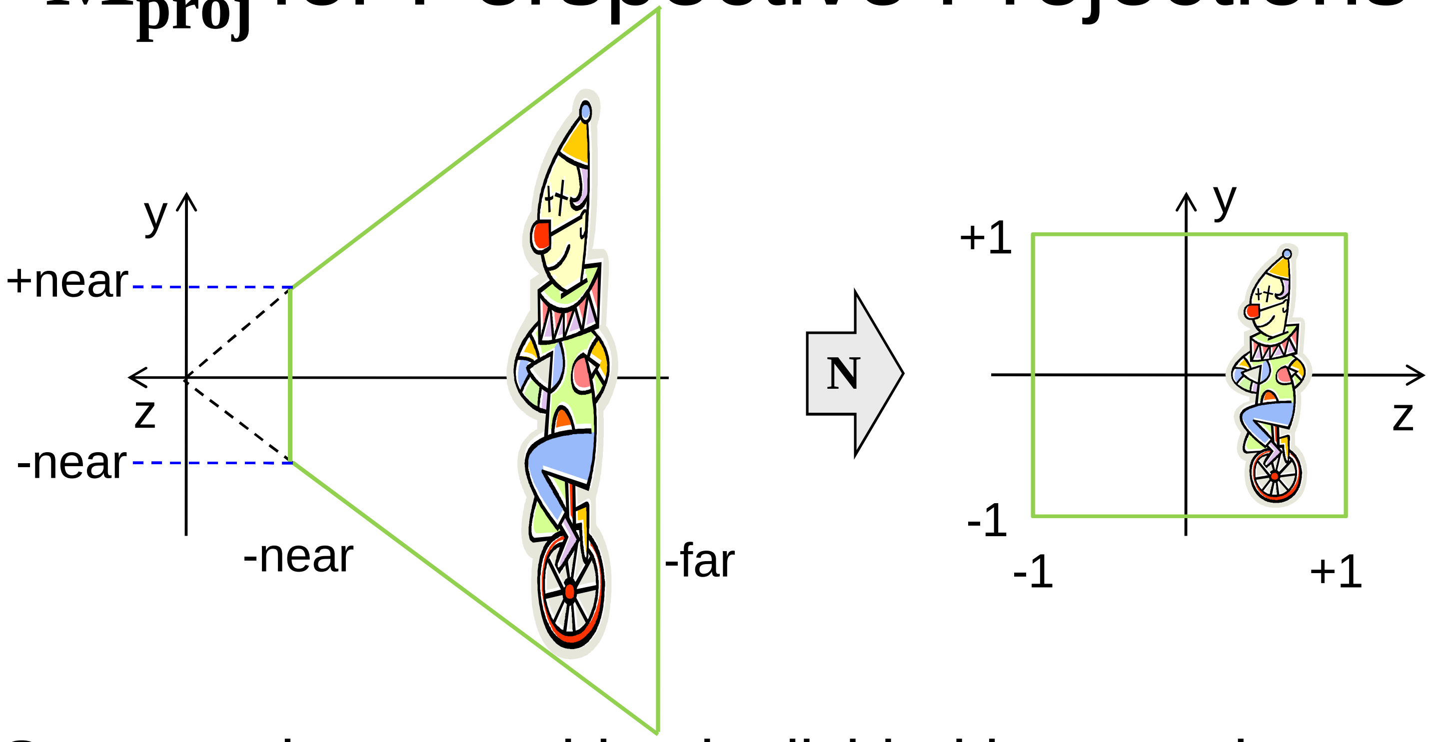
$\mathbf{M_{proj}}$ for Perspective Projections 2



Scale everything ($\mathbf{S}$) so that view plane has height and width 2*near

$$\mathbf{S} = \begin{pmatrix} \frac{near}{(right-left)/2} & 0 & 0 & 0 \\ 0 & \frac{near}{(top-bottom)/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2\ near}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2\ near}{top-bottom} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{M}_{\mathbf{proj}}$ for Perspective Projections 3



Set w so that everything is divided by -z and normalize z to (-1, +1)

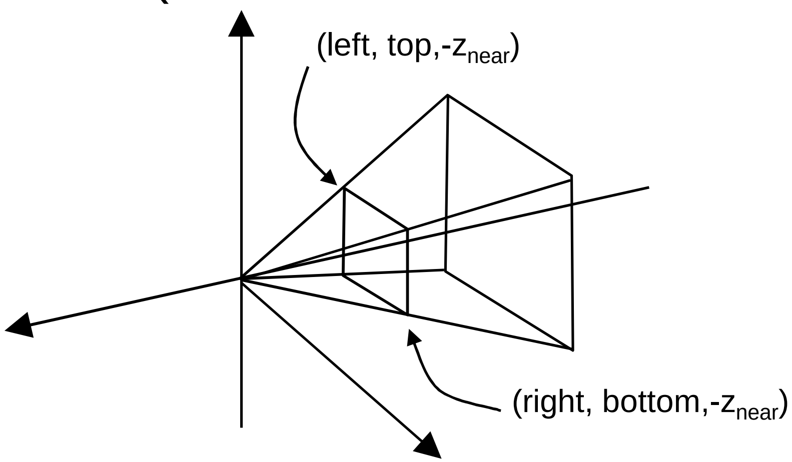
$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

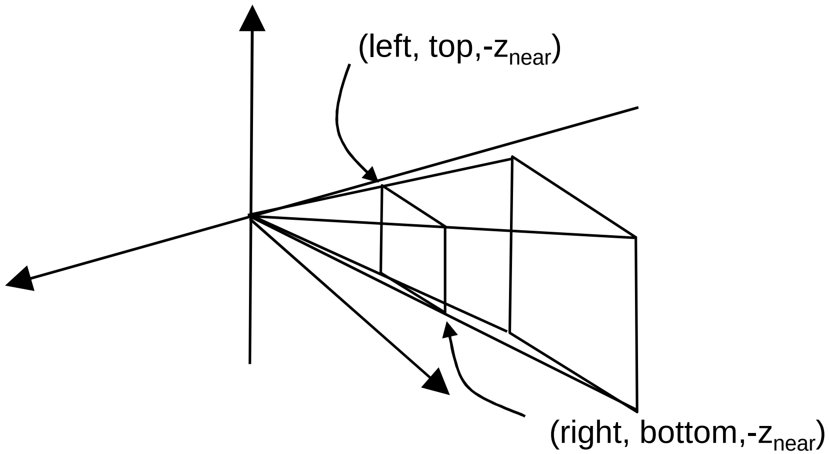
$$a = -\frac{far + near}{far - near}, \quad b = \frac{-2 \; far \; near}{far - near}$$

The final result is: $\mathbf{M}_{\mathbf{proj}} = \mathbf{N} \; \mathbf{S} \; \mathbf{H}$

Perspective Transformation in OpenGL

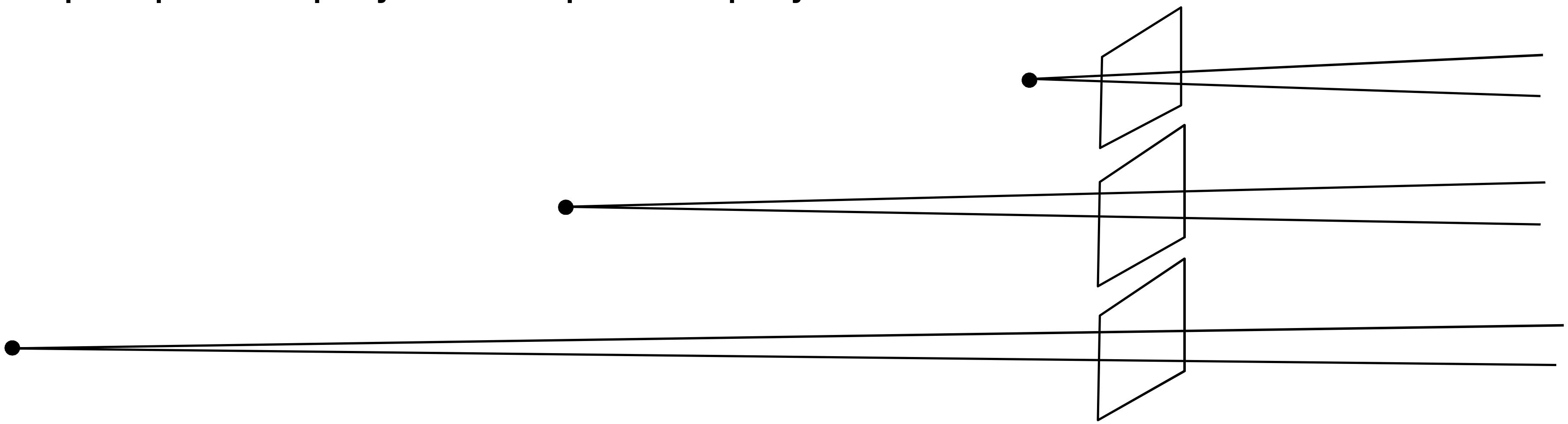
- View volume given by ***frustum*** (truncated pyramid):
  `glFrustum(left, right, bottom, top, znear, zfar)`
- `gluPerspective` computes these terms from its parameters:
```
top = zNear * tan((π/180)viewAngle/2);
bottom = -top;
right = top * aspect;   left = -right;
```
- **Note:** with `gluPerspective` the view volume is symmetric about the view direction vector. With `glFrustum` you can specify a non-symmetric view volume (useful for some stereo viewers)





Principles Geometric Projections (cont'd)

- **Observation about perspective projection:** as center of projection moves farther and farther away, lines of projection become more nearly parallel. In the limit, when center of projection is at an infinite distance, perspective projection ≡ parallel projection.



- Rays of light from a point source shining on an opaque object forming a shadow on a projection plane are similar to perspective projection rays.
- Rays of light from a point source at "infinite distance" (e.g., the Sun 93×10^6 miles from the Earth) forming a shadow are similar to parallel projection.

SUMMARY

Summary

- Projection transformation matrix $\mathbf{M_{proj}}$:
  maps World Coordinate values in view volume to Normalized Device Coordinates (NDC) in the range (-1, +1)
- Orthographic projection:
  - Objects keep their original size, no matter how far away
  - $\mathbf{M_{proj}} = \mathbf{S}\ \mathbf{T}$ (translate and scale)
- Perspective projection:
  - The further away an object, the smaller it appears
  - $\mathbf{M_{proj}} = \mathbf{N}\ \mathbf{S}\ \mathbf{H}$ (shear, scale, normalize z & set w for division by z)

References:

Perspective Projections: Hill, Chapter 7.4

Quiz

1. What are normalized device coordinates (NDCs)?
2. What is the difference between orthographic and perspective projection?
3. For given left, right, top, bottom, near and far, derive the S and T in the transformation matrix $\mathbf{M}_{\mathbf{proj}}$ = S T for orthographic projections.
4. In the diagram below, how do you calculate P' for a given P and near?