



Computer
Science

COMPSCI 372 S2 C – Exercise Sheet 5
Sample Solution

Q1: Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$

Answers:

1. $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot 2 + 3 \cdot 0 + (-2) \cdot 1 = 0$ (the two vectors are orthogonal)

2. $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 - (-2) \cdot 0 \\ (-2) \cdot 2 - 1 \cdot 1 \\ 1 \cdot 0 - 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix}$

Test whether your result is correct by checking whether the resulting vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} .

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 3 - 15 + 12 = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 6 - 6 = 0$$

3. the angle between \mathbf{u} and \mathbf{w}

$$\cos \varphi = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| |\mathbf{w}|} = \frac{-2}{\sqrt{14} \cdot 2} = -\frac{1}{\sqrt{14}} \quad \Rightarrow \quad \varphi = \cos^{-1} \left(-\frac{1}{\sqrt{14}} \right) = 105.5^\circ$$

Q2: Compute the area of the triangle defined by the vertices $P_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

$$A = \frac{1}{2} |(P_2 - P_1) \times (P_3 - P_1)| = \frac{1}{2} \left| \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right| = \frac{1}{2} \left| \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix} \right| = \frac{1}{2} \sqrt{9 + 36 + 4} = 3.5$$

Q3: Find all vectors which are orthogonal to the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

A vector \mathbf{v} is orthogonal to \mathbf{u} if $\mathbf{u} \cdot \mathbf{v} = 0$

$$\Leftrightarrow \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = 0 \Leftrightarrow 2v_x + v_z = 0 \Leftrightarrow v_z = -2v_x$$

Hence all vectors of the form

$$\mathbf{v} = \begin{pmatrix} k \\ m \\ -2k \end{pmatrix} \text{ where } k, m \in \mathfrak{R} \text{ are orthogonal to } \mathbf{u}.$$

Q4: Let $\mathbf{M} = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 4 & 1 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & -3 \end{pmatrix}$

Answers:

$$1. \quad \mathbf{M}^T + \mathbf{N} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 8 \\ -2 & -2 \end{pmatrix}$$

$$2. \quad \mathbf{M} \cdot \mathbf{N} = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 3*1 + (-1)*0 + 0*(-2) & 3*0 + (-1)*4 + 0*(-3) \\ 2*1 + 4*0 + 1*(-2) & 2*0 + 4*4 + 1*(-3) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 0 & 13 \end{pmatrix}$$

$$3. \quad \mathbf{N} \cdot \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1*3+0*2 & 1*(-1)+0*4 & 1*0+0*1 \\ 0*3+4*2 & 0*(-1)+4*4 & 0*0+4*1 \\ (-2)*3+(-3)*2 & (-2)*(-1)+(-3)*4 & (-2)*0+(-3)*1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ 8 & 16 & 4 \\ -12 & -10 & -3 \end{pmatrix}$$

Q5: Determine whether the four points $P_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $P_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $P_4 = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ lie on the same plane.

Form a plane equation from the first three points :

$$\mathbf{a} = P_2 - P_1 = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{b} = P_3 - P_1 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix}$$

$$d = \mathbf{n} \cdot P_1 = -12 \Rightarrow \text{The plane equation is } \begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix} \cdot \mathbf{p} = -12$$

Inserting the 4th point into the plane equation gives $\begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix} \cdot P_4 = -12 \Rightarrow$ The points lie on a plane

Q6: Compute the distance of the point $Q = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \mathbf{p} = 2$

Normalising the plane equation gives: $\frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \mathbf{p} = \frac{2}{\sqrt{11}}$

The distance is therefore: $\frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \mathbf{Q} - \frac{2}{\sqrt{11}} = \frac{2}{\sqrt{11}}$

Q7: Let $\mathbf{M} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 2 \end{pmatrix}$

a) Compute the inverse of \mathbf{M} .

$$\mathbf{M}^{-1} = \frac{1}{-2} \begin{pmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & - \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ - \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} & + \begin{vmatrix} 0 & 0 \\ 5 & 2 \end{vmatrix} & - \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} \\ + \begin{vmatrix} 3 & 0 \\ 5 & 0 \end{vmatrix} & - \begin{vmatrix} 0 & 2 \\ 5 & 0 \end{vmatrix} & + \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 0 & -4 & 2 \\ -1 & 0 & 0 \\ 0 & 10 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 0.5 & 0 & 0 \\ 0 & -5 & 3 \end{pmatrix}$$

b) Test whether your result is correct by computing $\mathbf{M} \mathbf{M}^{-1}$.

$$\mathbf{M} \mathbf{M}^{-1} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ 0.5 & 0 & 0 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q8: Compute the intersection point (if any)

a) of the line $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with the plane $\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \bullet \mathbf{p} = 4$

The direction vector of the line is orthogonal to the plane's normal, hence the line is parallel to the plane. Since furthermore the start point of the line does not fulfil the plane equation the line does not lie in the plane and hence the line will never intersect the plane.

b) of the line $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ with the plane $\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \bullet \mathbf{p} = 4$